Swarms of Interacting Agents in Random Environments

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Basic dynamics - stochastic interacting agents, (self-propagating particles)

Basic dynamics stylized by sets of coupled stochastic differential equations:

$$\dot{X}_k(t) = \underbrace{f_k\left[X_k(t)\right]}_{\text{self}-\text{propagation}} + \underbrace{\mathcal{J}_k\left[X_k(t), \vec{X}(t)\right]}_{\text{mutual interactions kernels}} + \underbrace{\xi_k(t)}_{\text{noise}}, \qquad X_k(t) \in \Omega,$$

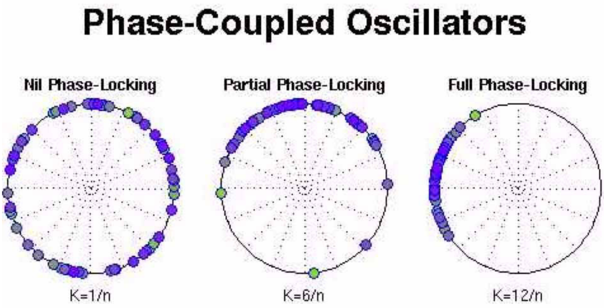
$$\vec{X}(t) = \left(X_1(t), X_2(t), \cdots, X_N(t)\right) \qquad k = 1, 2, \cdots, N.$$

Mutual interactions kernels, (for example = "imitation" of neighbours behaviours).

How from mutual interactions emerges a global dynamical order ?

Exhibit some explicitly tractable models of such collective evolutions

Pioneering model ($\sim$ 1975)
Kuramoto's phase oscillators



Nil, partial and full phase-locking behavior in a network of phase-coupled oscillators with all-to-all connectivity. The natural frequencies of the oscillators are normally distributed SD=±0.5Hz. The phase-locking behaviour is dictated by the strength of the global coupling constant K.

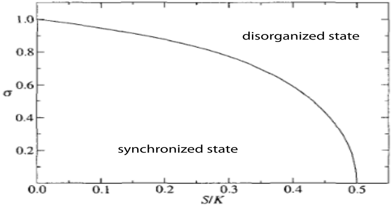
Agents with scalar dynamics on the circle $\mathbb{S}$

Kuramoto-Sakaguchi's (K-S) coupled phase oscillators - (1975)

$$d\theta_m(t) = \omega_m\, dt + \underbrace{\frac{K}{N}\left[\sum_{j\neq m} \sin\left[\theta_j(t) - \theta_m(t)\right]\right] dt}_{vanishes\ when\ \theta_j \equiv \theta_m, \forall j\ \Rightarrow\ synchronizing\ \ effect} + \sqrt{S}\ \underbrace{dW_{m,t}}_{indep.\ WGN}, \quad \begin{cases} \theta_m \in \Omega := [0, 2\pi[, \\ m = 1, 2, \cdots, N. \end{cases}$$

$$\sigma(t)e^{i\Phi(t)} := \frac{1}{N}\sum_{m=1}^{N} e^{i\theta_m(t)}, \qquad \sigma(t) := \text{order parameter} = \begin{cases} 1, & \text{full synchronization,} \\ 0, & \text{pure randomness.} \end{cases}$$

$\overbrace{\omega_m \equiv \omega}^{\text{homog.}}$ and $N \to \infty \Rightarrow$ mean-field, Fokker-Planck Eq. $\Rightarrow$ phase transition diagram



Synchronization arises for $K \geq K_c = 2S$

Long-range interactions $\Rightarrow$ phase transitions exists for 1D.

Sketch of the K-S' s derivation

- Order parameter $\sigma(t)$: $\sigma(t)e^{i\Phi(t)} := \frac{1}{N}\sum_{m=1}^{N} e^{i\theta_m(t)}$

$$\Downarrow$$

$$d\theta_m(t) = K\sigma(t)\left[\sin(\Phi(t) - \theta_m(t))\right] + \sqrt{S}dW_{m,t}$$

- Fokker-Planck: $\boxed{\partial_t\left[n(\theta,t)\right] = -K\sigma\partial_\theta\left[\sin\left(\Phi-\theta\right)n(\theta,t)\right] + S\partial^2_{\theta\theta}\left[n(\theta,t)\right]}$

- stationary measure: $n_s(\theta) = \frac{1}{2\pi\,\mathcal{I}_0\left[\frac{K\sigma}{S}\right]}e^{\left\{\left[\frac{K\sigma}{S}\right]\cos(\Phi-\theta)\right\}}$, ($\mathcal{I}_0(\cdot)$ Bessel funct.)

- contin. limit - mean-field: $\frac{1}{N}\sum_{m=1}^{N} e^{i\theta_m(t)} \cong \int_0^{2\pi} n(\theta,t)e^{i\theta}d\theta$, (for $N \to \infty$)

- self-consistency $\Rightarrow$ $\sigma e^{i\Phi} = \int_0^{2\pi} n_s(\theta)e^{i\theta}d\theta$ $\Rightarrow$ $\boxed{\sigma = \frac{\mathcal{I}_0^1\left[\frac{K\sigma}{S}\right]}{\mathcal{I}_0\left[\frac{K\sigma}{S}\right]} \Rightarrow K_c = 2S}$

- solving for σ enables to draw the K-S phase diagram.

Von Mises angle statistics

Observe the specific expression for the stationary probability measure $n_s(\theta)$!

$$n_s(\theta) \equiv \mathrm{VM}_\kappa(\theta) := \frac{1}{2\pi\, \mathcal{I}_0[\kappa]} e^{\{\kappa \cos(\theta)\}} = \frac{1}{2\pi} + \frac{1}{\pi} \sum_{m=1}^{\infty} \frac{\mathcal{I}_m(\kappa)}{\mathcal{I}_0(\kappa)} \cos(m\theta), \quad \kappa \in \mathbb{R}^+,$$

density known as the Von Mises angle statistics

Proposition, (K. V. Mardia (1972), G. Watson (1983)).

The probability density $\mathrm{VM}_\kappa(\theta)$ describes the exit-law outside the unit (Euclidean) circle $x^2 + y^2 = 1$ of a complex Brownian Motion with constant drift $(Z_t + t\,\vec{u},\ \ t \in \mathbb{R}^+)$ starting at 0 and κ is the Euclidean norm of the constant drift vector $\vec{u}$.

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"generalize K-S dynamics by introducing a curvature on the probability state space"

$\Downarrow$

"introduce multiplicative noise sources into the dynamics"

Multiplicative noise K-S dynamics:

$$\begin{cases} d\theta_m(t) = \frac{K}{N}\left\{\sum_{j\neq m}\sin\left[\theta_j - \theta_m\right]\right\}dt + \frac{1}{N}\sum_{j\neq m}\left\{\sqrt{1+C\cos(\theta_j-\theta_m)}\right\}\sqrt{S}\,dW_{m,t}, \\ C\in[0,1] \quad \text{and} \quad K>0, \quad \text{(two control parameters)}, \end{cases}$$

$$\boxed{\partial_t\left[n(\theta,t)\right] = -K\sigma\partial_\theta\left[\sin\left(\Phi-\theta\right)n(\theta,t)\right] + S\partial^2_{\theta\theta}\left[(1+\sigma C\cos(\Phi-\theta))n(\theta,t)\right]}$$

stationary measure

$$\begin{cases} n_s(\theta) = \mathcal{Z}^{-1}\left[1+\tanh(\eta)\cos(\Phi-\theta)\right]^{-\alpha}, \\ \mathcal{Z}^{-1} = 2\pi\left\{P^{(0)}_{-\alpha}\left[\cosh(\eta)\right]\right\}\cosh(\eta)^{\alpha}, \quad (P^{(0)}_{-\alpha}(x)\ \text{Legendre funct}), \\ \alpha = 1-\frac{K}{SC}, \quad \text{and} \quad \eta=\sigma C. \end{cases}$$

self-consitency Eq. $\Rightarrow$ $\boxed{\sigma = \frac{1}{1-\alpha}\frac{P^{(1)}_{-\alpha}\left[\cosh(\eta)\right]}{P^{(0)}_{-\alpha}\left[\cosh(\eta)\right]} \Rightarrow K_c = S(2+C)}$

Exit law of hyperbolic Brownian motion from hyperbolic disk

Proposition, (J.-Cl. Gruet, (2000), see also M. C. Jones & A. Pewsey, (2005)).

Let T_η be the first hitting time of the hyperbolic disk $\mathcal{D}$ of radius η centred at 0. The exit probability distribution of $\mathcal{D}$ under the law of the α-drifted hyperbolic Brownian motion starting at 0 is given by the two generalized two parameters hyperbolic von Mises law:

$$HVM(\theta) = \mathcal{Z}^{-1}\left[1 + \tanh(\eta)\cos(\theta)\right]^{-\alpha} \qquad (\theta \in [-\pi, +\pi] \;\; \eta \in \mathbb{R}^+, \;\; \alpha \in \mathbb{R}).$$

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Kuramoto's dynamics with multiplicative noise

Proposition, (R. Filliger, Ph. Blanchard and MOH (2010)).

Two parameters generalized Kuramoto-Sakagushi phase oscillators model:

$$\begin{cases} d\theta_m(t) = \frac{K}{N}\left\{\sum_{j\neq m} \sin\left[\theta_j - \theta_m\right]\right\} dt + \frac{1}{N}\sum_{j\neq m}\left\{\sqrt{1 + C\cos(\theta_j - \theta_m)}\right\}\sqrt{S}\, dW_{m,t}, \\ C \in [0,1] \qquad \text{and} \qquad K > 0, \qquad \text{(two control parameters)} \end{cases}$$

admits the associated angle probability measure in the mean-field limit is given by law $HVM(\theta)$. The onset of synchronisation is given by $K_c = S(2 + C)$.

Inhomogeneous K-S dynamics, (i.e. $\omega_k \neq \omega_j$)

Individual ω's are randomly drawn from a prob. density: $\omega \sim g(\omega)d\omega$.

- S. Strogatz & R. Mirollo
  "Stability of incoherence in a population of coupled oscillators".
  J. Stat. Phys. **63**, (1991).

- J. Acebron, L. Bonilla, C. Perez Vicente, F. Ritort & R. Spiegler.
  "The Kuramoto model: A simple paradigm for synchronization phenomena".
  Rev. Mod. Phys. **77**, (2005).

- R. Filliger, Ph. Blanchard, J. Rodriguez and MOH
  "Noise induced temporal patterns in population of globally coupled oscillators".
  IEEE Trans. (2009).

"Unwrap" the circular probability space $\mathbb{S}$

$\Downarrow$

Agents with scalar dynamics on $\mathbb{R}$

Homogenous - (Part I)

Heterogeneous - (Part II)

Part I

Homogeneous swarms on $\mathbb{R}$

Homogeneity: *agents are indistinguishable*

Exogenous mutual interaction rule: *"Avoid being the laggard"*

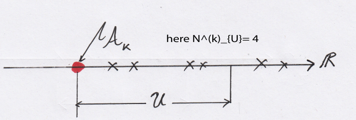
External enviroment: *White Gaussian Noise (WGN)*

Imitation dynamics for an homogenous population

$$\begin{cases} \dot{X}_k(t) = \underbrace{f\left[X_k(t)\right]}_{\text{individual dynamics}} + \underbrace{\mathcal{J}\left[X_k(t);\vec{X}(t)\right]}_{\text{interaction kernel}} + \underbrace{\sigma\, dW_k(t)}_{\text{WGN}}, \\ X_k(t=0) = x_{k,0}, \qquad X_k(t) \in \mathbb{R}, \qquad k = 1,2,\cdots,N \end{cases}$$

- Observation capability: $\mathcal{A}_k$ observes $\sharp$ of leaders within a range U.
- Avoid being the laggard:

$$\mathcal{J}\left[X_k(t),\vec{X}(t)\right] = \begin{cases} \gamma \frac{N_{\{U\}}^{(k)}}{N}, & \text{with } N_{\{U\}}^{(k)} := \sharp \text{ of } \mathcal{A}_{(j\neq k)} \in U \text{ ahead of } \mathcal{A}_k, \\ 0, & \text{if } N_{\{U\}}^{(k)} = 0. \end{cases}$$



Snapshot of the agents' positions

Nonlinear Fokker-Planck equation.

Consider large swarms, ($N \to \infty$)

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"Hydrodynamic picture": density of agents $\rho(x,t) \in [0,1]$ $\Rightarrow$ Mean-field dynamics

$\rho(x,t)$:= density of agents at position x at time t.

$$\begin{cases} N^{(k)}_{\{U\}}(t) = \frac{1}{N}\sum_{j\neq k} \mathbb{I}_{(0\leq X_j(t)-X_k(t)\leq U)} \\ \mathbb{I}_{(0\leq X_j(t)-X_k(t)\leq U)} = \begin{cases} 1, & \text{if} \quad X_j(t) > X_k(t), \qquad (X_j(t) \text{ is a } X_k(t) \text{ leader}), \\ 0, & \text{otherwise.} \end{cases} \end{cases}$$

$$\text{Meanfield description} \quad \Rightarrow \quad N_{\{U\}}(x,t) := \lim_{N\to\infty} \frac{1}{N} \sum_j \mathbb{I}_{(0\leq X_j(t)-x\leq U)} \simeq \int_x^{x+U} \rho(y,t)dy$$

Nonlinear and nonlocal Fokker-Planck equation for the density $\rho(x,t)$

$$\boxed{\partial_t\rho(x,t) = -\partial_x\left\{\left[f(x,t) + \int_x^{x+U}\rho(y,t)dy\right]\rho(x,t)\right\} + \frac{\sigma^2}{2}\partial_{xx}\left\{\rho(x,t)\right\}}$$

Simple case: $f(x,t) = \mathcal{C}$ and very short range interactions

Infinitesimal imitation range $U << 1$, $\rightarrow$ Taylor exp. 1^{st} order in U:

$$\begin{cases} \partial_t\left[\rho(x,t)\right] = \underbrace{-\partial_x\left\{\left[\mathcal{C} + \gamma U \rho(x,t)\right]\rho(x,t)\right\}}_{\text{interaction } \Rightarrow \text{ nonlinear conribution}} + \frac{\sigma^2}{2}\partial^2_{xx}\rho(x,t), \\ \\ \lim_{x\rightarrow\pm\infty}\rho(x,t) = 0. \end{cases}$$

Burgers' nonlinear field equation

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Hopf-Cole logarithmic transform

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Heat equation

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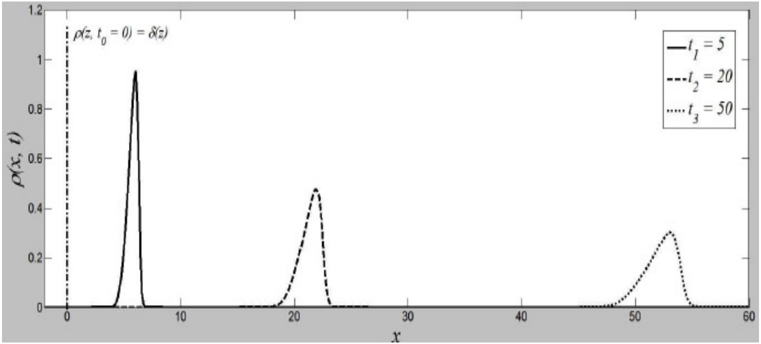
$\rho(x,t)$ is fully explicit for arbitrary times and initial conditions

Exact transient behavior for short range imitation



transient solution of the Burgers' equation - evanescent traveling wave

Asymptotically with time agents are fully dispersed

$\Downarrow$

short range interaction $\Rightarrow$ swarm cohesion is not sustained

Cooperative behavior generated by long range interactions

Infinite imitation range - ($U = \infty$)

$$\begin{cases} \partial_{xt}\left[G(x,t)\right] = \underbrace{-\partial_x\left\{\left[\mathcal{C} + \gamma G(x,t)\right]\partial_x G(x,t)\right\}}_{\text{interaction nonlinear conribution}} + \frac{\sigma^2}{2}\partial^2_{xxx}G(x,t), \\ G(x,t) = \int_x^{\infty}\rho(\zeta,t)d\zeta, \\ \lim_{x\to-\infty} G(x,t) = 1 \qquad \text{and} \qquad \lim_{x\to+\infty} G(x,t) = 0. \end{cases}$$

again Burgers' nonlinear dynamics but with new boundary conditions

$\Downarrow$

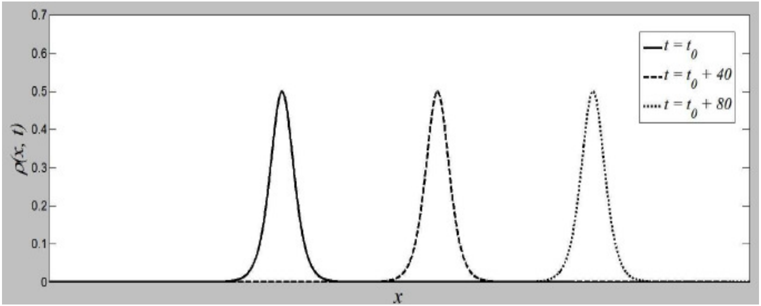
Explicit stationary behavior

$\Downarrow$

$$\boxed{\rho(x,t) = \frac{\Gamma}{\sigma^2\cosh^2\left[\frac{1}{\sigma^2}\Gamma(x-\Gamma t)\right]}} \qquad \Gamma = \Gamma(\mathcal{C},\gamma).$$

soliton like propagation

Exact stationary behavior for infinite range imitation)



transient solution of the Burgers' equation - soliton like traveling wave

Asymptotically a coherent and stable spatio-temporal pattern emerges

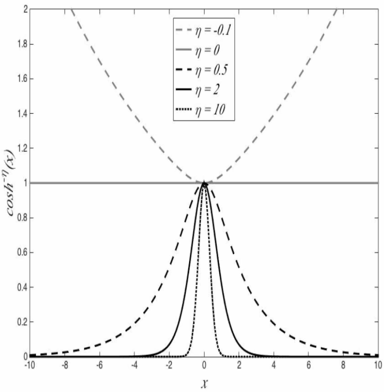
$\Downarrow$

infinite range interaction $\Rightarrow$ swarm cohesion is sustained

Barycentric modulated interactions

$$\partial_t\left[\rho(x,t)\right] = -\partial_x\left\{\left[\mathcal{C} + \int_x^{\infty} \overbrace{g\left(|\zeta - \langle X(t)|\rangle\right)}^{\text{barycentric interaction}} \rho(\zeta,t)d\zeta\right]\rho(x,t)\right\} + \frac{\sigma^2}{2}\partial^2_{xx}\rho(x,t)$$

$$\langle X(t)\rangle := \int_{\mathbb{R}} x\,\rho(x,t)\,dx.$$



Assume: $g(x) = [\cosh(x)]^{-\eta}$.

$\eta > 0 \Rightarrow$ *"conformism", (i.e. barycentre weight more),*

$\eta < 0 \Rightarrow$ *"non-conformism", (i.e. outliers weight more),* Exact (normalizable) stationary measure exists only for $\eta \in [-\infty, 2[$

$$\begin{cases} \rho(x) = \mathcal{Z}^{-1} \cosh\left(x - Vt\right)^{\eta - 2}, & \\ V = \mathcal{C} + (2-\eta)\frac{\sigma^2}{2}, & (\eta \in [-\infty, 2[). \end{cases}$$

$$\Downarrow$$

Flocking bifurcation : $\begin{cases} \eta \in [-\infty, 2], & \text{soliton like imitation wave.} \\ \eta \in [2, \infty], & \text{dispersive imitation wave.} \end{cases}$

Cooperative soliton like regime for $\eta \in [-2, \infty]$

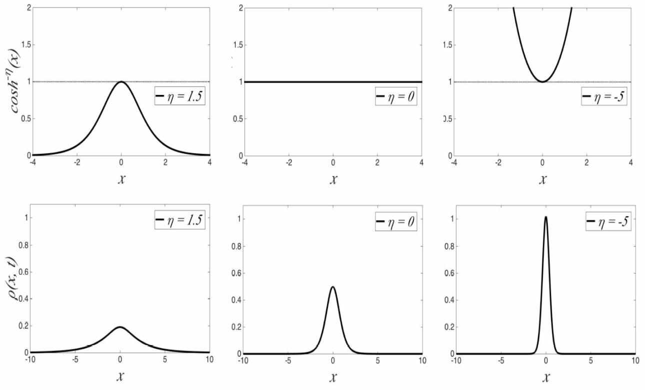
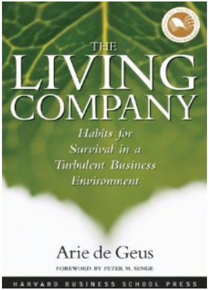


Figure 3: Barycentric modulation functions $\cosh^{-\eta}(x)$, for different values of η, and corresponding collective productivity long-wave $\rho x, t$.

Titmice (non-territorial) versus robins (territorial birds)





Basic goal: Goal here: obtain the previous soliton behavior via a MFG dynamics.

$$
\text{MFG}\quad \begin{cases}
dX_m(t) = a(X_m(t), t)dt + \sigma dW_{m,t}, \qquad m = 1, 2, \cdots, N, \\[2ex]
J\left(a(\cdot), X_m(t)\right) = \mathbb{E}\left\{ \int_0^T \underbrace{\left[\overbrace{c(a(X_m(s), s)}^{\text{driving cost}} + \overbrace{V\left(\rho(\cdot, s); X_m(s)\right)}^{\text{interaction cost}} \right]}_{\mathcal{L}[a(s), X_m(s), \rho(\cdot, s)]} ds + \overbrace{C_T\left[X_m(T)\right]}^{\text{final cost}} \right\}, \\[2ex]
\rho(x, t) = \frac{1}{N} \sum_{j=1}^{N} \delta(x - X_j(t)), \qquad \text{(empirical distribution)}.
\end{cases}
$$

MFG dynamics - coupled forward-backward pde's

Value function : $$u(X(t),t) := \min_{a(\cdot)} \int_t^T \mathcal{L}\left[a(s), X_m(s), \rho(\cdot, s)\right] ds + C_T(X_T)$$

Special choice of $\mathcal{L} = \frac{1}{2}\left[a(X_m(t),t) - b\right]^2 + \left[\rho(x,t)\right)]^p$

dynamic progr. $\Downarrow$ HBJ Eq.

$$\begin{cases} \partial_t u(x,t) + \frac{\sigma^2}{2}\partial_{xx}\left(u(x,t)\right) - \frac{1}{2}|\nabla u(x,t)|^2 = -\left[\rho(x,t)\right)]^p & \text{(HBJ)} \\ \partial_t \rho(x,t) = -\partial_x\left\{\partial_x u(x,t)\rho(x,t)\right\} + \frac{\sigma^2}{2}\partial_{xx}\left(\rho(x,t)\right) & \text{(FP)} \end{cases}$$

Connection with nonlinear Schrödinger eq. (I. Swiecki, et al. PRL (2016))

$$\begin{cases} \Phi(x,t) := e^{-\frac{u(x,t)}{\sigma^2}} \qquad\qquad \Psi(x,t) := \rho(x,t) e^{+\frac{u(x,t)}{\sigma^2}}, \\ -\sigma^2 \partial_t \Phi(x,t) = \frac{\sigma^4}{2} \partial_{xx} \Phi(x,t) + \rho(x,t)^p, \\ +\sigma^2 \partial_t \Psi(x,t) = \frac{\sigma^4}{2} \partial_{xx} \Psi(x,t) + \rho(x,t)^p. \end{cases}$$

Stationary solution

$$\Downarrow$$

$$\Psi(x)\Phi(x) = \rho(x) \propto \cosh^{-\frac{2}{p}}.$$

exponent identification $\Rightarrow$ $\boxed{\frac{2}{p} = 2 - \eta.}$

I. Swiecki, T. Gobron and D. Ullmo. *" Schrödinger appraoch to mean-field games".* Phys. Rev. Lett. **116**, 2016.

Part I - (continued)

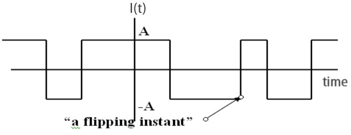
Homogeneous swarms on $\mathbb{R}$

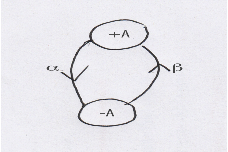
Dynamics driven by non-Gaussian noise sources

Piecewise deterministic (i.e. random telegraph) and discontinuous noise sources

$$\dot{X}_k(t) = f(X_k(t)) + \begin{cases} \mathcal{I}_k\left[\left(X_k(t); \vec{X}(t)\right), t\right] & \text{RT noise,} \\ q_k\left[\left(X_k(t); \vec{X}(t)\right), t\right] & \text{Shot noise.} \end{cases}$$

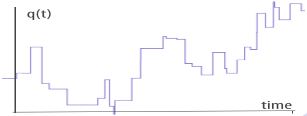
- Random telegraphic (RT) noise - (i.e two states Markov chain in continuous time)





Random telegraphic noise

- Shot noise - (i.e Compound Poisson jump process)



Shot noise

Swarms driven by piecewise deterministic stochastic processes

$$\dot{X}_k(t) = \underbrace{\mathcal{C}}_{\text{const. drift}} dt + \mathcal{I}\left[\left(X_k(t); \vec{X}(t)\right), t\right], \quad \text{switching rate depends on local agents' density}$$

$$\mathcal{I}(t) \in \{-A, +A\} \;\; \text{switching rates} \begin{cases} \alpha\left[\left(X_k(t); \vec{X}(t)\right), t\right], \\ \beta\left[\left(X_k(t); \vec{X}(t)\right), t\right]. \end{cases}$$

$(X(t), \mathcal{I}(t))$ Markov process, ($X(t)$ alone is not Markov !)

$$\text{transition probability densities} \begin{cases} P(x, +A, t \mid i.c.) := P_+, \\ P(x, -A, t \mid i.c.) := P_-. \end{cases}$$

$$\text{Fokker} - \text{Planck Eq.} \begin{cases} \partial_t P_+ - \partial_x\left[(\mathcal{C} - A)P_+\right] = -\alpha(x,t)P_+ + \beta(x,t)P_+, \\ \partial_t P_- - \partial_x\left[(\mathcal{C} + A)P_-\right] = +\alpha(x,t)P_+ - \beta(x,t)P_+. \end{cases}$$

Large swarms $\rightarrow$ mean-field approach

$$\text{follow the leaders rule} \Rightarrow \begin{cases} \alpha\left[\left(X_k(t);\vec{X}(t)\right),t\right] \longrightarrow \alpha(x,t) := \alpha - \int_x^U P_-(y,t)dy, \\ \beta\left[\left(X_k(t);\vec{X}(t)\right),t\right] \longrightarrow \beta(x,t) := \beta + \int_x^U P_+(y,t)dy. \end{cases}$$

- "myopic" (i.e short range) interactions, $U << 1$.

$$\begin{cases} \partial_t P^+(x,t) - (\mathcal{C} - A)\partial_x P^+(x,t) = -P_+P_- - \alpha P^+(x,t) + \beta P^-(x,t), \\ \partial_t P^-(x,t) - (\mathcal{C} + A)\partial_x P^+(x,t) = +P_+P_- + \alpha P^+(x,t) - \beta P^-(x,t). \end{cases}$$

Exactly solvable discrete velocity Boltzman Eq. (T. W. Ruijgrok T.T. Wu (1981)

Genralized Hopf-Cole transformation $\Rightarrow$ can be linearized in the Telegrapher's Eq.

- Long range interactions, $U \to \infty$.

Define $F_{\pm}(x,t) := \int_x^{\infty} P_{\pm}(y,t)dy$.

$$\begin{cases} \partial_t F^+(x,t) - (\mathcal{C} - A)\partial_x F^+(x,t) = -UF_+F_- - \alpha F^+(x,t) + \beta F^-(x,t), \\ \partial_t F^-(x,t) - (\mathcal{C} + A)\partial_x F^+(x,t) = +UF_+P_- + \alpha F^+(x,t) - \beta F^-(x,t). \end{cases}$$

Exactly solvable discrete velocity Boltzman Eq. T. W. Ruijgrok T.T. Wu (1981)

Genralized Hopf-Cole transformation $\Rightarrow$ be linearized to the Telegrapher's Eq.

White Gaussian noise versus Telegraphic noise - "in a nutshell view"

White Gaussian noise $\leftrightarrow$ Telegraphic noise

$\Downarrow$

Burgers Eq. $\leftrightarrow$ discrete Boltzmann eq.

$\Downarrow$

Hopf-Cole logarithmic transformation

$\Downarrow$

Heat eq. - $\boxed{\partial_t P = D\partial_{xx} P}$ $\leftrightarrow$ Telegraphers eq. $\boxed{\partial_{tt} P + \nu\, \partial_t P = D\partial_{xx} P}$

$\Downarrow$

linearirzation $\rightarrow$ explicit analytic approach

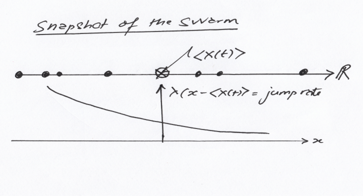
$\Downarrow$

Bifurcation from non-cooperative to cooperative evolution

Swarms driven by compound Poisson (directly inspired from: M. Balazs, M. Racz and B. Toth, (2014)).

Coupled SDE's driven by jump processes $X_k(t) \in \mathbb{R}$ and $k = 1, 2, \cdots, N$.

$$\begin{cases} \dot{X}_k(t) = f[X_k(t)] + q_k \left[\left(X_k(t); \vec{X}(t) \right), t \right] & \text{Compound Poisson process,} \\ \lambda \mapsto \lambda \left[\left(X_k(t); \vec{X}(t) \right), t \right] : \mathbb{R} \to \mathbb{R}^+ & \text{Poissonian jump rate,} \\ \varphi(x) : \mathbb{R}^+ \to \mathbb{R}^+ & \text{jumps size distribution, (here purely positive jumps).} \end{cases}$$



Interaction rule

$$\begin{cases} \text{Avoid being the laggard} \quad \Rightarrow \quad \lambda \left[\left(X_k(t); \vec{X}(t) \right), t \right] = \lambda \left[X_k(t) - \mathbb{E}\left\{ X_t \right\} \right] > 0, \\ \mathbb{E}\left\{ X(t) \right\} := \frac{1}{N} \sum_{j=1}^{N} X_j(t) \quad \text{and} \quad \lambda \left[X_k(t) - \mathbb{E}\left\{ X_t \right\} \right] : \mathbb{R} \to \mathbb{R}^+ \text{ monot. decreas.} \end{cases}$$

Master Equation - General formalism for compound Poisson processes

- Markovian dynamics
- large swarms' populations $\Rightarrow$ Mean-field approach
- Fokker-Planck $\Rightarrow$ Master equation

$$\partial_t P(x,t|x_0,0) = \partial_x \left[f(x)P(x,t|x_0,0)\right] - \lambda(x,t)P(x,t|x_0,0) + \int_{-\infty}^{x} \varphi(x-z)\lambda(z,t)P(z,t|x_0,0)dz$$

$P(x,t|x_0,0)$ transition pdf of the jump Markov process $X(t)$

Solve the master equation for $P(x,t|x_0,0)$ $\Rightarrow$ characterizes the swarm propagation

Focus on the jumps class: $\varphi(x) = \frac{\gamma^m x^{m-1} e^{-\gamma x}}{\Gamma(m)} \chi_{x\geq 0}, \qquad m = 1, 2, \cdots$ (Erlang law)

Erlang jumps' distribution $\Rightarrow$ high-order differential Master Equation

Proposition (R. Filliger and MOH - 2016). *For jumps drawn from an m^{th}-order Erlang law:*

$$\varphi(x) = \frac{\gamma^m x^{m-1} e^{-\gamma x}}{\Gamma(m)} \chi_{x \geq 0}, \qquad m = 1, 2, \cdots ,$$

the associated transition probability density $P_m(x, t \mid x_0, 0) := P_m$ solves the high order pde:

$$\boxed{[\partial_x + \gamma]^m \left(\partial_t P_m - \partial_x [f \cdot P_m]\right) = \left[\gamma^m - [\partial_x + \gamma]^m\right] \left(\lambda(x, t) \cdot P_m\right)}$$

Cooperative propagation of the swarm

Assume existence of a stationary co-operative behavior:

$$\begin{cases} P_{s,m}(x - C_m t) := P_{s,m}(\xi), & \int_{-\infty}^{+\infty} P_{s,m}(\xi) d\xi = 1, \\ \lambda\left[X_k(t) - \mathbb{E}\left\{X_t\right\}\right] = \lambda(x - C_m t) = \lambda(\xi) & \\ \int_{-\infty}^{+\infty} \xi P_m(\xi) d\xi = 0, & \text{soliton like propagation with constant velocity } C_m. \end{cases}$$

- $m = 1 \quad \Rightarrow \quad P_1(\xi) = \mathcal{N} e^{-\gamma\xi + \int^{\xi} \frac{\lambda(z) dz}{C_1}}.$
- $m = 2 \quad \Rightarrow$

$$P_2(\xi) = \exp\left\{-\gamma\xi + \int^{\xi} \frac{\lambda(z)}{2C_2} dz\right\} \Psi(\xi)$$

$$\partial_{\xi\xi}\Psi(\xi) + \underbrace{\left[-\frac{\partial_\xi \lambda(\xi)}{2C_2} - \frac{\lambda^2(\xi)}{4C_2^2} - \frac{\gamma\lambda(\xi)}{C_2}\right]}_{:=W(\xi) \quad \text{Schroedinger's quantum mechanical potential}} \Psi(\xi) = 0.$$

M. Balazs, M. Racz and B. Toth. *"Modeling flocks and prices: jumping particles with an attractive interaction "* . Ann. Inst. Poincaré, (2014).

Explicitly soluble illustration - *some "baroque" analysis implies:*

- $m = 1 \Rightarrow \begin{cases} P_{s,1}(\xi) = \mathcal{N}_1(\beta, \gamma, C_1) e^{-\gamma\xi - \frac{1}{\beta C_1} e^{-\beta\xi}}, & \text{(Gumbel probab. law.)} \\ \\ \text{Swarm velocity}: \; C_1 = \frac{1}{\beta} e^{-\psi(\gamma/\beta)}. \end{cases}$

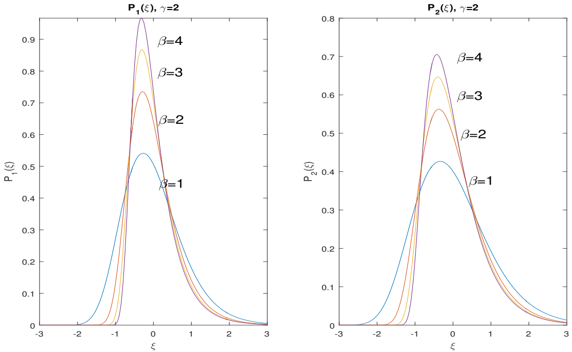
(M. Balazs, M. Racz and B. Toth. *Ann. Inst. Poincaré*, (2014)).

- $m = 2 \qquad \lambda(\xi) = e^{-\beta\xi} \quad \Rightarrow \quad W(\xi) = \left[\frac{(\beta - 2\gamma)}{2C_2} e^{-\beta\xi} - \frac{1}{4C_2^2} e^{-2\beta\xi} \right],$ (Morse potential)

- $m = 2 \Rightarrow \begin{cases} P_2(\xi) = \mathcal{N}(\beta, \gamma, C_2) e^{\left[\frac{\beta}{2} - \gamma\right]\xi - \frac{e^{-\beta\xi}}{2\beta C_2}} \underbrace{W_{\frac{\beta - 2\gamma}{2\beta}, 0}\left(\frac{e^{-\beta\xi}}{\beta C_2}\right)}_{\text{Whittaker function, second kind}}, \\ \\ \text{Swarm velocity}: \; C_2 = \frac{1}{\beta} e^{\psi(2\gamma/\beta) - 2\psi(\gamma/\beta)}, \qquad (\psi(z) := \text{Digamma funct.}). \end{cases}$

Explicit shapes of swarms densities

Probability densities $P_1(\xi)$ and $P_2(\xi)$



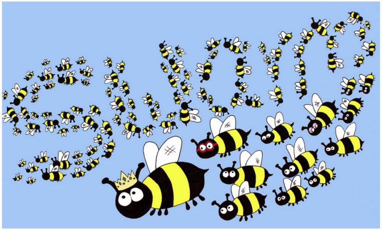
Swarm's velocities ratio: $C_2/C_1 = exp(e^{\psi(2\gamma/\beta)-\psi(\gamma/\beta)}) > 2.$

Part II

Swarms with leader-followers dynamics versus swarms driven by a shill

- The leader behavior is not influenced by her followers.
- The shill interacts with her fellows via "usual" swarm rule - however the shill itself can be externally driven - (i.e basic idea behind the concept of *"soft-control"*).

Leader-follower dynamics - Stochastic feedback particles filters (FPF)



filtering task - "express overview for the continuous discrete case"

$$\begin{cases} dX(t) = F(X(t))dt + \sigma_B dW_t, & \text{system's signal}, dW_t \ \text{WGN} \\ Z_k = h(X_k) + \sigma_o dB_k, & \text{system's observation}, dB_t \ \text{WGN}. \end{cases}$$

Basic goal: merge information from the model evolution and successive observations

$$\begin{cases} Z_k := \{z_\tau : \tau \leq k\}, \\ P(x, t_k | Z_{k-1}) = \mathcal{F}\{P(x, t_{k-1} | Z_{k-1})\}, \quad \text{Fokker–Planck evolution for } t_{k-1} \leq t < t_k, \\ P(x, t_k | Z_k) := P(x, t_k | z_k, Z_{k-1}), \quad \text{updating after observation } z_k \text{ at time } t_k, \\ P(x, t_k | z_k, Z_{k-1}) P(z_k | Z_{k-1}) = P(x, z_k, t_k | Z_{k-1}) = P(z_k, t_k | x_k, Z_{k-1}) P(x, t_k | Z_{k-1}), \quad \text{Bayes}, \\ P(x, t_k | Z_k) = \dfrac{\overbrace{P(z_k, t_k | x_k, Z_{k-1})}^{\text{Gauss}} P(x, t_k | Z_{k-1})}{P(z_k | Z_{k-1})} = \dfrac{\left[\dfrac{\exp\left\{ -\frac{(z_k - h(x_k))^2}{2\sigma_o^2} \right\}}{\sqrt{2\pi\sigma_o^2}} \right] P(x, t_k | Z_{k-1})}{P(y_k | Z_{k-1})}. \end{cases}$$

Feedback Particles (nonlinear) Filters (FPF)

$$\begin{cases} dX(t) = F(X(t))dt + \sigma_B dW_t, & \text{system's signal,} \\ dZ(t) = h(X(t))dt + \sigma_o dB_t, & \text{system's observation.} \end{cases}$$

Filtering task: Get $P^*(X_t|\mathcal{Z}_t)$ posterior prob. given the history: $\mathcal{Z}_t := \sigma(Z_s; s \leq t)$.

****

$$\begin{cases} dX_m(t) = F(X_m(t))dt + \sigma_B dW_{m,t} + \overbrace{U_m(t)dt}^{\text{control}}, & N \text{ particles with feedback control } U_m(t), \\ p^{(N)}(x,t)dx = \frac{1}{N}\sum_{m=1}^{N} \mathbb{I}\left(x \leq X_m(t) \leq (x+dx)\right), & \text{empirical probability density} \\ \lim_{N\to\infty} p^{(N)}(x,t)dx = P^*(X_t|\mathcal{Z}_t). \end{cases}$$

FPF filtering algorithm: $\min\left(KL\left\{P^{(N)}(x,t)||P^*(\hat{x},t)\right\}\right)$, Kullback-Leibner distance.

T. Yang, P. G. Metha, S. P. Meyn. "*A mean-field control-oriented approach to particle filtering*". *IEEE Trans. Autom. Contr.* (2013).

Heterogeneous situation: swarm driven by a leader

$$\vec{X}(t) = \left(X_1(t), X_2(t), \cdots, X_{(N-1)}(t)\right),$$

$$\begin{cases} \left.\begin{aligned} dX_1(t) &= \Big[f(X_1(t)) + \overbrace{U(X_1(t), \vec{X}(t), Z(t))}^{\text{interaction kernel}} \Big] dt + \sigma_B dW_{1,t}, \\ dX_2(t) &= \Big[f(X_2(t)) + U(X_2(t), \vec{X}(t), Z(t)) \Big] dt + \sigma_B dW_{2,t}, \\ & \cdots \\ dX_N(t) &= \Big[f(X_N)(t) + U(X_N(t), \vec{X}(t), Z(t)) \Big] dt + \sigma_B dW_{N,t} \end{aligned} \right\} N \text{ followers} \\ \mathbf{dX(t) = f(X(t))dt + \sigma_B dW_t}, \qquad \textbf{leader's dynamics}, \\ dZ(t) = h[X(t)]dt + \sigma_o\, dB_t, \qquad \text{information available to followers.} \end{cases}$$

Stochastic optimal control problem

System $\leftrightarrow$ Leader

Feedback particles $\leftrightarrow$ Followers

Observation noise $\leftrightarrow$ Signal delivered to the followers

Followers cooperative task: minimize Kulback "distance" between P_{leader} and P_{swarm}

P_{leader} := true prob. density of the position of the leader

P_{swarm} := empirical prob. density of the positions of the $(N-1)$ followers

$\Downarrow$

Stochastic optimal control problem $\rightarrow$ explicitly soluble variational problem

Optimal control algorithm - explicit control $U(X_k(t), \vec{X}(t), Z(t))$

$$\begin{cases} U(X_k(t), \vec{X}(t), Z(t)) = \nu(X_k(t), \vec{X}(t), Z(t)) \overbrace{\circ}^{\text{Straton.}} \left\{ dZ(t) - \left[\frac{1}{2} h(X_k(t)) + \hat{h}(t) \right] \right\} \\ \nu(x(t), \vec{X}(t), Z(t)) = \frac{1}{\sigma_o^2 P(y,t|\mathcal{Z}(t))} \int_{-\infty}^{x} dy \left\{ \hat{h}(t) - h(y) \right\} P(y, t \mid \mathcal{Z}(t)), \\ \hat{h}(t) = \int_{\mathbb{R}} h(x,t) P(x, t \mid \mathcal{Z}(t)) dx, \\ P(x, t \mid \mathcal{Z}(t)) dx := P^*(X_t | \mathcal{Z}_t) \qquad \text{filtererd conditional probability density.} \end{cases}$$

- $P(x, t \mid \mathcal{Z}(t))$ known explicitly, (finite dimensional filters) $\Rightarrow$ solvable leader-follower dynamics

- The swarm density (tightness) is controlled by σ_o, (i.e. strength of the observation noise)

****

Fully solvable class of dynamics

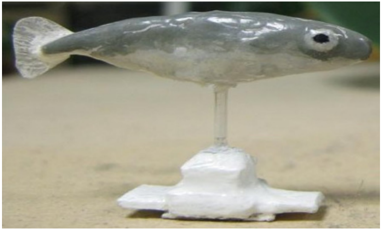
a) Linear drift; (Ornstein - Uhlenbeck (OU)+ linear observation $\Rightarrow$ Kalman filters).

b) Nonlinear drift: $\left(\left[f(x)^2 + \frac{d}{dx} f(x) \right] = Ax^2 + Bx + C \right)$ (V. Benes, Stochastics (1981) and MOH, Physica D (1981)).

Part II - (continued)

Inhomogeneous swarms infiltrated by a (controllable) complice, (shill).

Heterogeneous swarms infiltrated by a complice



Scalar dynamics - Ranked Brownian Motions - "Hybrid-Atlas" model, (T. Ichiba et al. (2011))

$$\mathrm{d}X_i(t) = \left(\underbrace{\sum_{k=1}^{N} g_k 1_{Q_k(i)} \{X(t)\}}_{ranked-based\ \ interaction} + \gamma_i + \gamma \right) \mathrm{d}t + \sigma_i \, \mathrm{d}W_i(t), \qquad \begin{cases} X_i(0) = x_i, \\ 1 \leq i \leq N. \end{cases}$$

$X_i(t)$	position of agent i
g_k	rank-dependent constant drift ($X_i(t)$) occupies rank k at time t)
γ_i	constant agent-dependent drift
γ	constant drift
$\mathrm{d}W_i(t)$	WGN (mutually indep. for $i = 1, 2, .., N$)

(set $\sum_{k=1}^{N} [g_k + \gamma_k] = 0$)

$\Downarrow$

γ: average barycentric speed of the swarm.

T. Ichiba, V. Papathanakos, A. Banner, I. Karatzas and R. Fernholz. *" Hybrid Atals models"*. Ann. Appl. Probab. (2011).

Theorem: (T. Ichiba et al., 2011):

1. Swarm tightness condition.

$$\sum_{k=1}^{m}\left[g_k + \gamma_{p(l)}\right] < 0, \qquad \begin{pmatrix} p = (p(1), ..., p(N)) \in \Sigma_N \\ 1 \leq m \leq N-1 \end{pmatrix}$$

Σ_N set of permutations of the N agents

2. Stationary measure for the $(N-1)$ gaps between agents $\Psi(z), z \in \mathbb{R}_+^{N-1}$.

$$\psi(z) = \underbrace{\left(\sum_{p\in\Sigma_N}\prod_{k=1}^{N-1}\lambda_{p,k}^{-1}\right)^{-1}}_{\text{normalization factor}} \sum_{p\in\Sigma_N}\exp\left(-\langle\lambda_p, z\rangle\right)$$

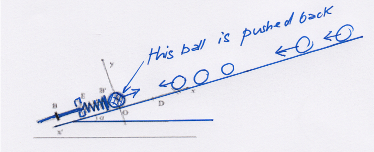
$$\lambda_p = (\lambda_{p,k})_{k=1}^{N-1} \qquad \lambda_{p,k} = \frac{-4\sum_{l=1}^{k}\left(g_l + \gamma_{p(l)}\right)}{\sigma_k^2 + \sigma_{k+1}^2}.$$

Swarm of $(N-1)$ identical agents infiltrated by a single super-diffusive fellow

- $(N-1)$ Brownian agents driven constant drift $(-g)$ and by WGN's .
- One "Shill" agent $(i=1)$ driven by drift $(N-1)g$ and by super-diffusive noise.

Specific choice of of the control parameters in the "Hybrid-Atlas" model

- $\gamma_i \equiv 0,$.
- $$g_k = \begin{cases} -g, & 1 \leq k < N, \\ (N-1)g, & k = N. \end{cases}$$



$$\Downarrow$$

$$\left(\sum_{k=1}^{l} g_k < 0 \quad \text{for} \quad l = 1, 2, ..., (N-1) \quad \Rightarrow \quad \text{swarm tightness cond. for BM agents} \right)$$

Dynamic mean preserving spread (MPS) noise - super-diffusive, (ballistic) noise source

$$dZ(t) = \beta \tanh\left[\beta Z(t)\right] dt + dW_t \quad \Leftrightarrow \quad P(z,t|z_0) = \frac{1}{2\sqrt{2\pi t}} \left\{ e^{-\frac{[(z-z_0)+\beta t]^2}{2t}} + e^{-\frac{[(z-z_0)-\beta t]^2}{2t}} \right\}$$

alternative $\Updownarrow$ representation

$Z(t) = \mathcal{B}\, \beta dt + dW_t,$ $\mathcal{B}$ a Bernouli random variable

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- L. C. G. Roger and J. Pitman. *"Markov functions"*, Annals of Probab. (1981).
- MOH *"Exact solutions for a class of nonlinear Fokker-Planck equations"*, Phys. Letters A. (1979).
- J.-L. Arcand, D. Rinaldo and MOH *"Dynamic Mean Preserving Spreads"*, Preprint - Graduate institute - Geneva, (2016).

Heterogenous swarm driven by a shill with MPS noise

$$\begin{cases} dX_1(t) = \left(\sum_{k=1}^{N} g_k \mathbb{1}_{Q_k(1)}\{X(t)\} + \gamma_1 + \gamma\right) dt + dZ_1(t), & X_1(t) = x_{1,0}, \\ dX_i(t) = \left(\sum_{k=1}^{N} g_k \mathbb{1}_{Q_k(i)}\{X(t)\} + \gamma_i + \gamma\right) dt + dW_i(t), & \begin{cases} X_i(0) = x_{i,0}, \\ 2 \leq i < N. \end{cases} \end{cases}$$

$$\Downarrow$$

alternative a)
$$\begin{cases} dX_1(t) = \left(\sum_{k=1}^{N} g_k \mathbb{1}_{Q_k(1)}\{X(t)\} + \gamma_1 + \beta + \gamma\right) dt + dW_1(t), \\ dX_i(t) = \left(\sum_{k=1}^{N} g_k \mathbb{1}_{Q_k(i)}\{X(t)\} + \gamma_i + \gamma\right) dt + dW_i(t), \end{cases}$$

alternative b)
$$\begin{cases} dX_1(t) = \left(\sum_{k=1}^{N} g_k \mathbb{1}_{Q_k(1)}\{X(t)\} \gamma_1 - \beta + \gamma\right) dt + dW_1(t), \\ dX_i(t) = \left(\sum_{k=1}^{N} g_k \mathbb{1}_{Q_k(i)}\{X(t)\} + \gamma_i + \gamma\right) dt + dW_i.(t) \end{cases}$$

Tight, "semi-tight" and unstable regimes ($\beta_c^+ \leq \beta < \beta_c^-$)

$$\Downarrow$$

Stability character depends on the realisation of β defining the d$Z(t)$

Tight regime

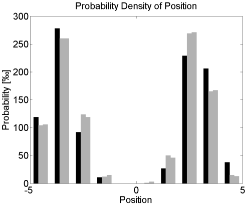

Figure: $\beta = 1.1 < \beta_c^+ = \frac{3}{2}$

"Semi-tight"regime

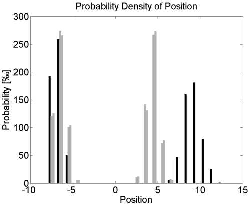

Figure: $\beta = 2 > \beta_c^+$

Unstable regime

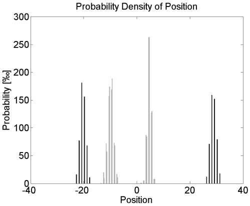

Figure: $\beta = 4 > \beta_c^-$

End position distributions ($t = 10$, $N = 3$ and $g = 1$):

R. Filliger, O. Gallay and MOH, *"Local versus nonlocal barycentric interaction in 1D agents' dynamics"*, Math, Biosciences and Engineering **11**(2), (2014).

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