Eigenvalue fluctuations for lattice Anderson Hamiltonians

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Anderson Hamiltonian

Anderson Hamiltonian is the random Schrödinger operator of the form

$$H_\omega = -\kappa\Delta + V_\omega$$

defined on $L^2(\mathbb{R}^d)$ or $\ell^2(\mathbb{Z}^d)$, where V_ω is random, stationary and ergodic.

Typical choices of V_ω include the alloy model

$$V_\omega(x) = \sum_{q\in\mathbb{Z}^d} \omega_q v(x-q)$$

and the random displacement model

$$V_\omega(x) = \sum_{q\in\mathbb{Z}^d} v(x-q-\omega_q).$$

Localizations

Due to the randomness, V_ω creates deep "traps" in well separated small regions. Consequently, various localization phenomenon emerges:

Spectral localization
: The spectrum of H_ω consists of eigenvalues around the bottom and the corresponding eigenfunctions decay exponentially.

Dynamical localization
: Starting from a low energy state ϕ, the bulk of wave function $e^{itH_\omega}\phi$ stays bounded.

Localization of diffusion
: The diffusion $e^{-tH_\omega}\delta_0$ concentrates in some small regions.

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Roughly speaking, the trapping effect of V_ω is stronger than the smoothing effect of Δ.

Setting of the problem

We are interested in the so-called "homogenization" problem.

- $D \subset \mathbb{R}^d$: a bounded domain with smooth boundary;
- $D_\epsilon = D \cap \epsilon\mathbb{Z}^d$: a natural discretization;
- $\Delta_\epsilon f(x) = \epsilon^{-2} \sum_{|y-x|=\epsilon} (f(y) - f(x))$;
- $\xi = \{\xi(x) \colon x \in D_\epsilon\}$: a random potential.

Let $\{\lambda^{(k)}_{D_\epsilon,\xi}\}_{k\geq 1}$ be the eigenvalues of the operator (matrix)

$$-\Delta_\epsilon + \xi$$

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Remark

$-\Delta_\epsilon + \xi \longleftrightarrow \epsilon^{-2}(-\Delta + \epsilon^2 \xi(\epsilon\cdot))$: potential weakened.

Homogenization of eigenvalues

- $\lambda_D^{(k)}$: k-th smallest eigenvalue of $-\Delta$ on D.

Theorem (homogenization, Biskup-F.-König)

If ξ is IID with $\mathbb{E}[|\xi|^K] < \infty$ for some $K > 1 \vee d/2$,

$$\lambda_{D_\epsilon,\xi}^{(k)} \to \lambda_D^{(k)} + \mathbb{E}[\xi] \quad \textit{as} \quad \epsilon \downarrow 0$$

in probability for each $k \geq 1$.

Remark

The moment condition is optimal in the sense that if $\mathbb{E}[\xi(x)_-^K] = \infty$ for some $K < d/2$, then $\underline{\lim}_{\epsilon \downarrow 0} \lambda_{D_\epsilon,\xi} = -\infty$.

Fluctuation around the mean

- $\lambda_D^{(k)}$: k-th smallest eigenvalue of $-\Delta$ on D.
- $\varphi_D^{(k)}$: corresponding eigenfunction, $\|\varphi_D^{(k)}\|_2 = 1$.

Theorem (fluctuation, BFK)

If ξ is IID with $\mathbb{E}[|\xi|^K] < \infty$ for some $K > 2 \vee d/2$ and $\lambda_D^{(k_1)}, \ldots, \lambda_D^{(k_n)}$ are distinct simple eigenvalues. Then,

$$\epsilon^{-d/2}\left(\lambda_{D_\epsilon,\xi}^{(k_1)} - \mathbb{E}\lambda_{D_\epsilon,\xi}^{(k_1)}, \ldots, \lambda_{D_\epsilon,\xi}^{(k_n)} - \mathbb{E}\lambda_{D_\epsilon,\xi}^{(k_n)}\right) \xrightarrow{\epsilon\downarrow 0} \mathcal{N}(0,\sigma)$$

in law, where

$$\sigma_{ij}^2 := \mathrm{var}(\xi) \int_D \varphi_D^{(k_i)}(x)^2 \varphi_D^{(k_j)}(x)^2 \,\mathrm{d}x.$$

Remark

When K is close to $2 \vee d/2$, ξ in the expectation need to be replaced by $\xi \vee (-\epsilon^{-d/K - o(1)})$ to make the expectation finite.

Where does the fluctuation come from?

Note that the weighted sum

$$\langle \xi, (\varphi_D^{(k)})^2 \rangle := \sum_{x \in D_\epsilon} \epsilon^d \xi(x) \varphi_D^{(k)}(x)^2$$

obeys the same CLT.

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obeys the same CLT. On the other hand, the eigenvalue can be expressed as

$$\lambda_{D_\epsilon,\xi}^{(k)} = \underbrace{\|\nabla_\epsilon g_{D_\epsilon,\xi}^{(k)}\|_2^2}_{\text{kinetic energy}} + \underbrace{\langle \xi^{(\epsilon)}, (g_{D_\epsilon,\xi}^{(k)})^2 \rangle}_{\text{potential energy}}$$

by using the random eigenfunction $g_{D_\epsilon,\xi}^{(k)}$. It seems as if the eigenvalue fluctuation comes only from the potential energy part. This is indeed the case and we can prove

$$\mathrm{Var}(\|\nabla_\epsilon g_{D_\epsilon,\xi}^{(k)}\|_2^2) = o(\epsilon^d).$$

Related works 1

- Bal (2008): Consider

$$-\Delta + \xi(\cdot/\epsilon) \text{ on } D \subset \mathbb{R}^d \ (d \leq 3),$$

where ξ is stationary, centered and assume either

1. boundedness and a certain mixing condition or
2. $\mathbb{E}[\xi^6(0)] < \infty$ and a stronger mixing condition.

Then for each $k \geq 1$,

$$\lambda^{(k)}_{D_\epsilon,\xi} \to \lambda^{(k)}_D \text{ as } \epsilon \downarrow 0 \text{ in probability.}$$

Moreover, for distinct simple eigenvalues $\lambda^{(k_1)}_D, \dots, \lambda^{(k_n)}_D$,

$$\epsilon^{-d/2}\left(\lambda^{(k_1)}_{D_\epsilon,\xi} - \lambda^{(k_1)}_D, \dots, \lambda^{(k_n)}_{D_\epsilon,\xi} - \lambda^{(k_n)}_D\right) \xrightarrow{\epsilon\downarrow 0} \mathcal{N}(0,\sigma)$$

in law, where $\sigma^2_{ij} := \mathrm{var}(\xi) \int_D \varphi^{(k_i)}_D(x)^2 \varphi^{(k_j)}_D(x)^2 \, dx$.

Remark

1. $\mathbb{E}[\xi^4(0)] < \infty$ *suffices for our discrete IID setting.*
2. *The Green function* $(-\Delta)^{-1}(x, \cdot) \in L^{2+}_{\mathrm{loc}}$ *is essential in his argument which is based on the asymptotic expansion of* $G_\xi = (-\Delta + \xi)^{-1}$*:*

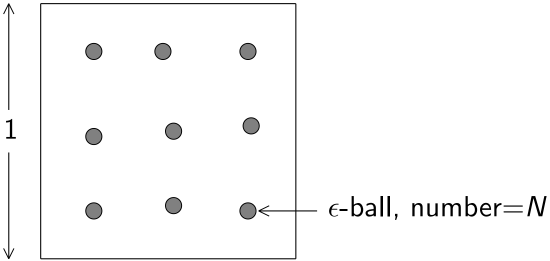
$$G_\xi = G_0 - G_0 \xi G_0 + G_0 \xi G_0 \xi G_0 - \cdots$$

   This causes the restriction $d \leq 3$.

Related works 2

Crushed ice problem

- Kac (1974) and Rauch-Taylor (1975): homogenization of eigenvalues of $-\Delta$ in a randomly perforated domain;



When $d = 3$,

$$\lambda^{(k)}_{D\setminus \mathrm{balls}} \to \infty \quad \text{as } N\epsilon^2 \to 1.$$

Surface area does not control the cooling efficiency.

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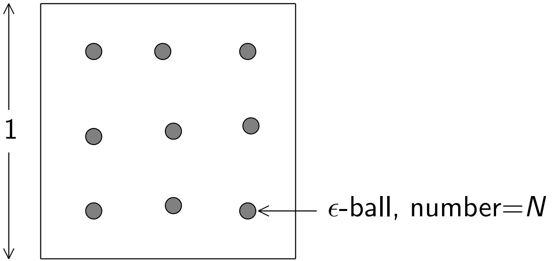

When $d = 3$,

$$\lambda^{(k)}_{D\setminus\mathrm{balls}} \to \lambda^{(k)}_{D} + \alpha \quad \text{as } N\epsilon \to 1$$

by using the so-called Wiener sausage.

Kac, in his 1974 paper:

"Here the probabilistic treatment is extremely useful, because from the analytic point of view the problem looks impossible, unless you do it by the perturbation method, which few of us are willing to buy."

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Ozawa (1992):

To probabilists: "Find a probabilistic proof of the CLT."

Proof of the homogenization

Let $\mathbb{E}[\xi] = 0$ for simplicity. We also focus on the first eigenvalue and drop the superscript $^{(1)}$.

Rayleigh-Ritz formula

$$\lambda_{D_\epsilon,\xi} = \inf_{g \in \ell_0^2(D_\epsilon), \|g\|_2 = 1} \left\{ \|\nabla_\epsilon g\|_2^2 + \langle \xi, g^2 \rangle \right\},$$

$$\lambda_D = \inf_{\psi \in H_0^1(D), \|\psi\|_2 = 1} \|\nabla \psi\|_2^2.$$

$\rightarrow$ $g_{D_\epsilon,\xi}$ and φ_D are minimizers.

- $\lambda_{D_\epsilon,\xi} \lesssim \lambda_D$ by substituting φ_D to the first formula;
- $\lambda_{D_\epsilon,\xi} \gtrsim \lambda_D$ by substituting $g_{D_\epsilon,\xi}$ to the second formula.

Proof of the homogenization 2

The first step

$$\lambda_{D_\epsilon,\xi} \le \|\nabla_\epsilon \varphi_D\|_2^2 + \langle \xi, \varphi_D^2 \rangle \xrightarrow{\epsilon \downarrow 0} \|\nabla \varphi_D\|_2^2 = \lambda_D$$

is nothing but the weak law of large numbers.

Proof of the homogenization 2

The first step

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is nothing but the weak law of large numbers.

The second step

$$\underbrace{\|\nabla g_{D_\epsilon,\xi}\|_2^2}_{\text{need an interpolation}} \sim \|\nabla_\epsilon g_{D_\epsilon,\xi}\|_2^2 + \underbrace{\langle \xi, g_{D_\epsilon,\xi}^2 \rangle}_{\text{randomly weighted sum}}$$

is more problematic.

Proof of the homogenization 3

We use the following two tools:

Finite element method

$\exists$ piecewise affine interpolation $\widetilde{g_{D_\epsilon,\xi}}$ such that $\|\nabla_\epsilon g_{D_\epsilon,\xi}\|_2 = \|\nabla \widetilde{g_{D_\epsilon,\xi}}\|_2$.

Elliptic regularity

$\|\nabla_\epsilon g_{D_\epsilon,\xi}\|_2^2$ is bounded (with high probability). This follows by a Moser's iteration combined with some probabilistic estimates.

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H^1-boundedness & Poincaré inequality

$$\Downarrow$$

$g_{D_\epsilon,\xi}$ can be well-aproximated by a step function with large plateaus.

For a step function, we can use weak LLN (with a tail bound) step-wise. $\square$

Proof of the fluctuation (martingale decomposition)

We use a martingale CLT. Assume $\mathbb{E}[\xi] = 0$ and $\mathrm{Var}(\xi) = 1$.
Let $D_\epsilon = \{x_1, \ldots, x_n\}$ and $\mathcal{F}_m = \sigma[\xi(x_1), \ldots, \xi(x_m)]$.

$$
\begin{aligned}
\lambda_{D_\epsilon,\xi} - \mathbb{E}[\lambda_{D_\epsilon,\xi}] &= \sum_{m=1}^{n} \mathbb{E}[\lambda_{D_\epsilon,\xi}|\mathcal{F}_m] - \mathbb{E}[\lambda_{D_\epsilon,\xi}|\mathcal{F}_{m-1}] \\
&=: \sum_{m=1}^{n} Z_m.
\end{aligned}
$$

Need to check:

(1) $\epsilon^{-d} \sum_m \mathbb{E}[Z_m^2|\mathcal{F}_{m-1}] \xrightarrow{\epsilon\downarrow 0} \int_D \varphi_D(x)^4 \mathrm{d}x$ in prob.;

(2) $\epsilon^{-d} \sum_m \mathbb{E}[Z_m^2 1_{\{|Z_m| > \delta\epsilon^{d/2}\}}|\mathcal{F}_{m-1}] \xrightarrow{\epsilon\downarrow 0} 0$ in prob. (easy)

Proof of the fluctuation (Hadamard's formula)

By independence,

$$\begin{aligned}
Z_m &= \mathbb{E}[\lambda_{D_\epsilon,\xi}|\mathcal{F}_m] - \mathbb{E}[\lambda_{D_\epsilon,\xi}|\mathcal{F}_{m-1}] \\
&= \hat{\mathbb{E}}\left[\lambda_{D_\epsilon,\,\xi_{\le m},\,\widehat{\xi}_{>m}} - \lambda_{D_\epsilon,\,\xi_{<m},\,\widehat{\xi}_{\ge m}}\right] \\
&= \hat{\mathbb{E}}\left[\int_{\widehat{\xi}_m}^{\xi_m} \partial_m \lambda_{D_\epsilon,\,\xi_{<m},\,\widetilde{\xi}_m,\,\widehat{\xi}_{>m}} \mathrm{d}\widetilde{\xi}_m\right] \\
&= \hat{\mathbb{E}}\left[\int_{\widehat{\xi}_m}^{\xi_m} \epsilon^d g^2_{D_\epsilon,\,\xi_{<m},\,\widetilde{\xi}_m,\,\widehat{\xi}_{>m}}(x_m) \mathrm{d}\widetilde{\xi}_m\right].
\end{aligned}$$

The last $=$ is a consequence of Hadamard's first variation formula.

$$\partial_m \lambda_{D_\epsilon,\xi} = \epsilon^d g_{D_\epsilon,\xi}(x_m)^2.$$

Proof of the fluctuation (heuristics)

We expect

$$\mathbb{E}[Z_m^2|\mathcal{F}_{m-1}] = \epsilon^{2d}\int \mathbb{P}(\mathrm{d}\xi_m)\hat{\mathbb{E}}\left[\int_{\widehat{\xi}_m}^{\xi_m} g^2_{D_\epsilon,\xi_{<m},\widetilde{\xi}_m,\widehat{\xi}_{>m}}(x_m)\mathrm{d}\widetilde{\xi}_m\right]^2$$

$$\overset{?}{\sim} \epsilon^{2d}\int \mathbb{P}(\mathrm{d}\xi_m)\hat{\mathbb{E}}\left[\int_{\widehat{\xi}_m}^{\xi_m} \varphi_D^2(x_m)\mathrm{d}\widetilde{\xi}_m\right]^2$$

$$= \epsilon^{2d}\varphi_D(x_m)^4$$

$$\Rightarrow \epsilon^{-d}\sum_m \mathbb{E}[Z_m^2|\mathcal{F}_{m-1}] \sim \sum_m \epsilon^d \varphi_D(x_m)^4 \sim \int_D \varphi_D(x)^4 \mathrm{d}x.$$

But the dummy variable $\widetilde{\xi}_m$ prevent us from using ANY probability estimates to establish $\overset{?}{\sim}$.

Proof of the replacement

Essential part of the proof is

$$\int \mathbb{P}(\mathrm{d}\xi_m)\hat{\mathbb{E}}\left[\int_{\widehat{\xi}_m}^{\xi_m} g^2_{D_\epsilon,\,\xi_{<m},\,\widetilde{\xi}_m,\,\widehat{\xi}_{>m}}(x_m)\mathrm{d}\widetilde{\xi}_m\right]^2$$

$$\overset{?}{\sim} \int \mathbb{P}(\mathrm{d}\xi_m)\hat{\mathbb{E}}\left[\int_{\widehat{\xi}_m}^{\xi_m} g^2_{D_\epsilon,\,\xi_{<m},\,\xi_m,\,\widehat{\xi}_{>m}}(x_m)\mathrm{d}\widetilde{\xi}_m\right]^2 .$$

Lemma

$$\partial_m \log g_{D_\epsilon,\xi}(x_m) = P_1^\perp (H_{D_\epsilon,\xi} - \lambda_{D_\epsilon,\xi})^{-1} P_1^\perp (x_m, x_m)$$

with $P_1^\perp$ *the orthogonal projection onto* $\langle g_{D_\epsilon,\xi}\rangle^\perp$.

Proof of the replacement (comparison)

For some large $\lambda > 0$,

$$
\begin{aligned}
&(H_{D_\epsilon,\xi} - \lambda_{D_\epsilon,\xi})^{-1} P_1^{\perp}(x_m, x_m) \\
&\quad = \sum_{k \geq 2} \frac{1}{\lambda^{(k)}_{D_\epsilon,\xi} - \lambda_{D_\epsilon,\xi}} g^{(k)}_{D_\epsilon,\xi}(x_m)^2 \\
&\quad \lesssim \sum_{k \geq 1} \frac{1}{\lambda^{(k)}_{D_\epsilon,\xi} + \lambda} g^{(k)}_{D_\epsilon,\xi}(x_m)^2 \\
&\quad = (H_{D_\epsilon,\xi} + \lambda)^{-1}(x_m, x_m).
\end{aligned}
$$

If we can replace $H_{D_\epsilon,\xi}$ by $H_{D_\epsilon,0}$, we are done:

$$
(H_{D_\epsilon,0} + \lambda)^{-1}(x_m, x_m) \lesssim \begin{cases} 1, & d = 1, \\ \log \frac{1}{\epsilon}, & d = 2, \\ \epsilon^{2-d}, & d \geq 3. \end{cases}
$$

Proof of the replacement (Khas'minskii's lemma)

We write

$$(H_{D_\epsilon,\xi} + \lambda)^{-1}(x_m, x_m) = \int_0^\infty e^{-t(H_{D_\epsilon,\xi}+\lambda)}(x_m, x_m)\mathrm{d}t.$$

Khas'minskii's lemma

$$\exists \tau > 0, \sup_{z \in D_\epsilon} I_{\tau,z}(\xi) := \sup_{z \in D_\epsilon} \int_0^\tau e^{-sH_{D_\epsilon,0}}\xi_-(z)\mathrm{d}s < 1/2$$
$$\Rightarrow e^{-tH_{D_\epsilon,\xi}}(x_m, x_m) \le e^{t\zeta(\tau)} e^{-tH_{D_\epsilon,0}}(x_m, x_m).$$

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$$\Rightarrow e^{-tH_{D_\epsilon,\xi}}(x_m,x_m)\le e^{t\zeta(\tau)}e^{-tH_{D_\epsilon,0}}(x_m,x_m).$$

Remark

This is "incredible" at the first sight since it deduces a bound on $E^z[e^{-\int_0^\tau \xi(X_s)\mathrm{d}s}]$ from that of $E^z[\int_0^\tau \xi_-(X_s)\mathrm{d}s]$.

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$$\Rightarrow e^{-tH_{D_\epsilon,\xi}}(x_m, x_m) \leq e^{t\zeta(\tau)} e^{-tH_{D_\epsilon,0}}(x_m, x_m).$$

If we can find the above τ,

$$\begin{aligned}(H_{D_\epsilon,\xi} + \lambda)^{-1}(x_m, x_m) &\leq \int_0^\infty e^{-t(H_{D_\epsilon,0}+\lambda-\zeta(\tau))}(x_m, x_m)\mathrm{d}t \\ &= (H_{D_\epsilon,0} + \lambda - \zeta(\tau))^{-1}(x_m, x_m).\end{aligned}$$

Proof of the replacement (finding τ)

Note that $\mathbb{E}[I_{\tau,z}] = \mathbb{E}[\int_0^\tau e^{-sH_{D_\epsilon,0}}\xi_-(z)\mathrm{d}s] \le \tau \max_y \mathbb{E}[\xi_-(y)]$. Moreover, since

$$
\begin{aligned}
|I_{\tau,z}(\xi) - I_{\tau,z}(\eta)| &\le \int_0^\tau \|e^{-s\Delta_\epsilon}(z,\cdot)\|_2 \|\xi - \eta\|_2 \mathrm{d}s \\
&= \|\xi - \eta\|_2 \int_0^\tau e^{-2s\Delta_\epsilon}(z,z)^{1/2}\mathrm{d}s \\
&\lesssim \|\xi - \eta\|_2 \begin{cases} \tau^{1-d/4}\epsilon^{d/2}, & d \le 3, \\ \epsilon^2 \log(\tau\epsilon^{-2}), & d = 4, \\ \epsilon^2, & d \ge 5, \end{cases}
\end{aligned}
$$

Talagrand's inequality implies concentration around the mean.

Random vs. non-random error

- Perturbation methods $\longrightarrow$ CLT around the homogenized eigenvalues for $d \leq 3$, (under mixing condition)
- Probabilistic method $\longrightarrow$ CLT around the mean for any dimensions. (under independence)

Random vs. non-random error

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We can always write

$$\lambda_{D_\epsilon,\xi} - \lambda_D = \underbrace{\lambda_{D_\epsilon,\xi} - \mathbb{E}[\lambda_{D_\epsilon,\xi}]}_{\text{random shift}} + \underbrace{\mathbb{E}[\lambda_{D_\epsilon,\xi}] - \lambda_D}_{\text{non-random shift}}.$$

Question: Can we prove that the non-random part is

$$\begin{cases} = o(\epsilon^{d/2}), & \text{when } d \leq 3, \\ \gg \epsilon^{d/2}, & \text{when } d \geq 4? \end{cases}$$

Partial Answer: It is $\gtrsim \epsilon^2$ for continuous problem on $(\mathbb{R}/\mathbb{Z})^d$.

Random vs. non-random: local time heuristics

Let ξ be IID standard Gaussian and $D = (\mathbb{R}/\mathbb{Z})^d$. ($\lambda_D = 0$.)

$$\mathbb{E}\left[\exp\left\{-\epsilon^{-d/2}\lambda_{D_\epsilon,\xi}\right\}\right] \sim \mathbb{E}\left[e^{-\epsilon^{-d/2}H_\xi}1(0)\right]$$
$$= \mathbb{E}\left[E_0\left[\exp\left\{-\int_0^{\epsilon^{-d/2}}\xi(X_{\epsilon^{-2}s})\mathrm{d}s\right\}\right]\right]$$
$$= \mathbb{E}\left[E_0\left[\exp\left\{-\epsilon^2\sum_x \xi(x)\ell_{\epsilon^{-2-d/2}}(x)\right\}\right]\right]$$
$$= E_0\left[\exp\left\{\frac{\epsilon^4}{2}\sum_x \ell_{\epsilon^{-2-d/2}}(x)^2\right\}\right],$$

where X is SRW on $(\mathbb{R}/\epsilon^{-1}\mathbb{Z})^d$ and $\ell_t(x) = \int_0^t 1_{\{X_s=x\}}\mathrm{d}s$.

Random vs. non-random: local time heuristics

$$\mathbb{E}\left[\exp\left\{-\epsilon^{-d/2}\lambda_{D_\epsilon,\xi}\right\}\right] \sim E_0\left[\exp\left\{\frac{\epsilon^4}{2}\sum_x \ell_{\epsilon^{-2-d/2}}(x)^2\right\}\right].$$

Easy to check:

$$E_0\left[\|\ell_{\epsilon^{-2-d/2}}\|_2^2\right] \approx \begin{cases} \epsilon^{-4}, & d \le 3, \\ \epsilon^{-2-d/2} \gg \epsilon^{-4}, & d \ge 5. \end{cases}$$

This suggests (but does not prove) that we need a different scaling in higher dimensions.

Thank you!