Critical percolation on the Hamming graph

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Joint work with **Lorenzo Federico, Remco van der Hofstad & Frank den Hollander**

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Percolation

A simple model for geometric random graphs

Definition

Fix a graph $G = (\mathcal{V}, \mathcal{E})$ and $p \in [0, 1]$. Remove each edge $e \in \mathcal{E}$ independently with probability p: i.e., percolation is a product measure on $\{0, 1\}^{\mathcal{E}}$.

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Focus of this talk

Percolation on sequences of finite graphs.

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Example

The *Erdős-Rényi random graph:* Take $G = K_n$ (the complete graph on n vertices). Write $G(n, p)$ for the percolated graph. Study $G(n, p)$ as $n \to \infty$ (with $p = p(n) \to 0$).

The ERRG phase transition

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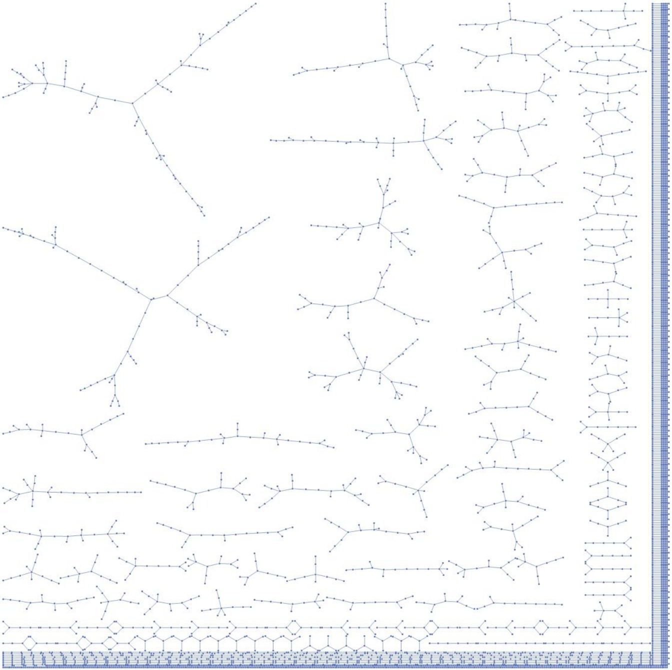
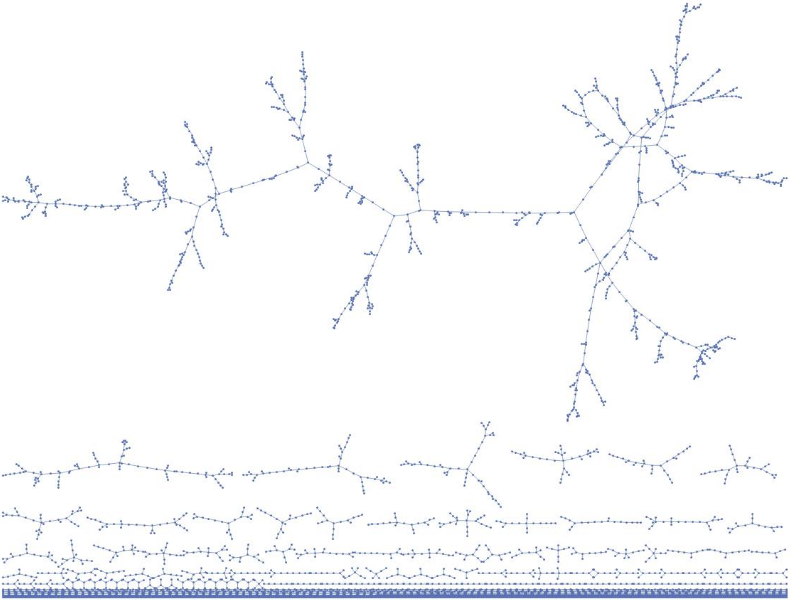
The critical window

We can zoom in on the phase transition by choosing $p = \frac{1+\varepsilon_n}{n}$ with $\varepsilon_n \to 0$. This shows a much richer structure around criticality. [Too much to discuss in detail here]





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$$(\gamma_i(t))_{i \ge 1} = \text{ the excursions of } R^\theta \text{ ordered s.t. } \gamma_1(\theta) > \gamma_2(\theta) > \dots$$

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$$\left(\frac{|\mathcal{C}_i|}{n^{2/3}} \right)_{i \ge 1} \xrightarrow{\text{d}} (\gamma_i(\theta))_{i \ge 1}$$

A graph exploration algorithm

Sketch of the proof (1/3)

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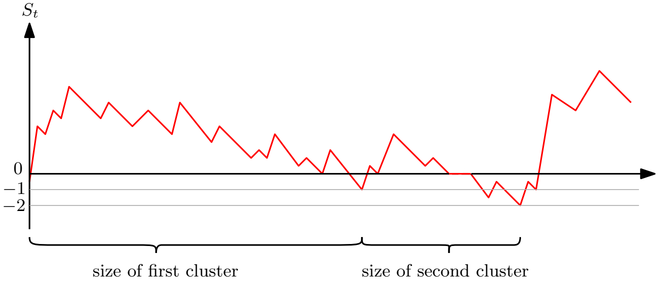
The exploration process and cluster sizes

Sketch of the proof (2/3)

Set $S_0 = 0$ and $S_i = S_{i-1} - 1 + X_i$. Observe that

- $\min\{j : S_j = -1\}$ = size of first explored cluster



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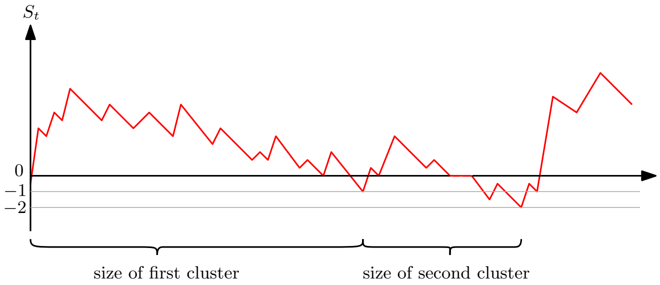

If $G(n, \frac{1+\theta n^{-1/3}}{n})$ has

$$\left(n^{-1/3} S_{tn^{2/3}}\right)_{t\geq 0} \xrightarrow{\text{d}} (B^\theta(t))_{t\geq 0},$$

then Aldous' Theorem follows (by relatively standard arguments)

A scaling limit for $(S_i)_{i\geq 1}$

Sketch of the proof (3/3)

Set $S_0 = 0$ and $S_i = S_{i-1} - 1 + X_i$. For $G = G(n,p)$,

$X_j \sim \mathrm{Bin}(\#\text{ neutral vertices}, p)$

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The main difficulty in going from the ERRG to geometric graphs is that K_n is highly *symmetric* and *self-similar,* which makes everything easier. For instance, if we remove a component of size k from $G(n, p)$, the (conditional) law of what remains is $G(n-k, p)$. This is obviously not true for percolation on any other graph.

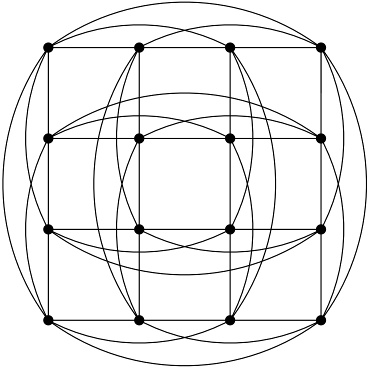
The Hamming graph

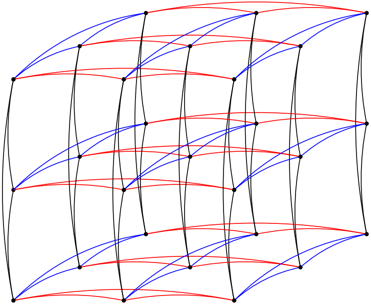
Definition of the Hamming graph

$H(d, n)$ is defined as the $(d-1)$-fold Cartesian product of K_n,

$$H(d, n) \simeq K_n \times K_n \times \cdots \times K_n$$

$H(d, n)$ has degree $m := d(n-1)$ and $V := n^d$ vertices.





The critical window

Theorem [FHHH]
For percolation on $H(d,n)$ with degree $m = d(n-1)$ and $d = 2, 3, \ldots, 6$,

$$p_c^{H(d,n)} = \frac{1}{m} + \frac{2d^2-1}{2(d-1)^2}\frac{1}{m^2}$$

is a point inside the critical window.

Critical percolation on the Hamming graph

An ERRG-type scaling limit

Theorem [FHHH]

For percolation on $H(d,n)$ with $d = 2, 3, 4$, fix $\theta \in \mathbb{R}$ and $p = p_c^{H(d,n)}(1 + \theta V^{-1/3})$. Then,

$$\left(\frac{|\mathcal{C}_i|}{V^{2/3}} \right)_{i \geq 1} \xrightarrow{\text{d}} (\gamma_i(\theta))_{i \geq 1}$$

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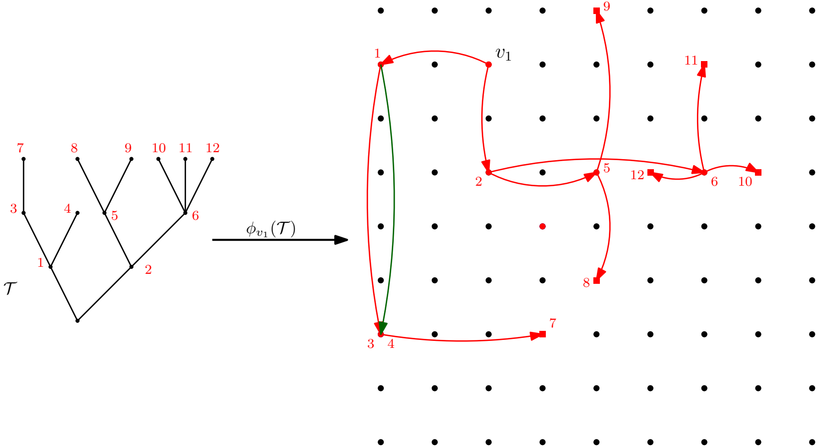
- Geometry $\Rightarrow$ current cluster is dependent on explored clusters

About the proof

Percolation = killed branching random walks

We describe percolation as a collection of randomly embedded $\mathsf{Bin}(m, p)$-Galton-Watson trees into $H(d, n)$, where particles are killed when they collide or visit a previously visited site. We call them *killed branching random walks.*



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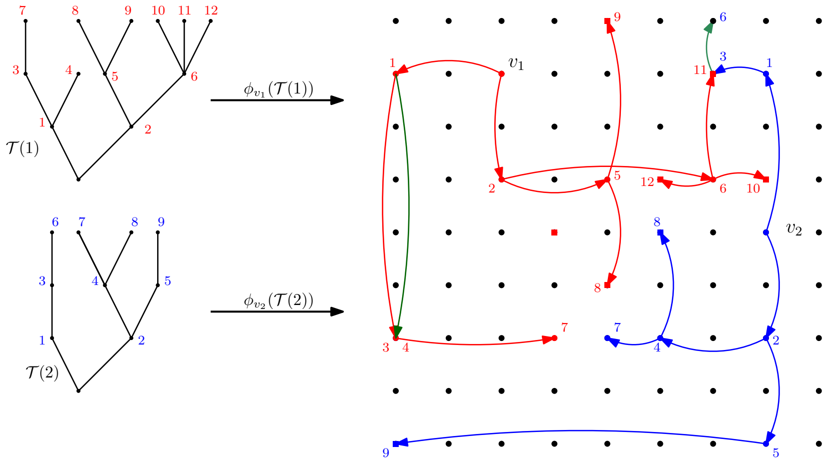

$\mathcal{T}(1)$
$\mathcal{T}(2)$
$\phi_{v_1}(\mathcal{T}(1))$
$\phi_{v_2}(\mathcal{T}(2))$
v_1
v_2

About the proof

Percolation = killed branching random walks

We describe percolation as a collection of randomly embedded $\mathsf{Bin}(m, p)$-Galton-Watson trees into $H(d, n)$, where particles are killed when they collide or visit a previously visited site. We call them *killed branching random walks*.

Advantages:

- The path between two particles in a (not killed) BRW has the same law as a simple random walk

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Disadvantage:

- The measure of killed BRW's on $H(d,n)$ is much more complicated than the percolation product measure

About the proof

Reducing dependence between exploration steps

A two-scale exploration

In Aldous' ERRG exploration process, we activate the direct neighbors. On the Hamming graph, this gives too much dependence. Instead, we explore a large chunk of the cluster at once, corresponding to the first $r_n \gg \log^2 n$ generations in the GW-tree. We only activate the boundary.

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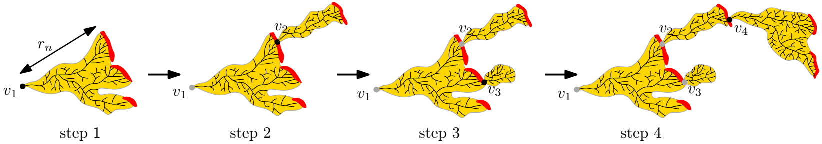

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- Random walk on $H(d, n)$ mixes fast $[t_{\text{mix}}(H(d, n)) = O(d \log d)]$, so the r_n-th generation of the BRW is very well mixed $\Rightarrow$ no dependence between large-scale exploration steps

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Disadvantage:

- The number of dead vertices is no longer deterministic. But for the right choice of r_n (not too large or small) the number concentrates.

About the proof

Reducing dependence between current cluster and explored clusters

A sticky coupling

In Aldous' ERRG exploration process, the geometry of the already explored clusters does not matter much (removing a cluster of size k from $G(n, p)$ gives $G(n-k, p)$). On the Hamming graph, this is not true.

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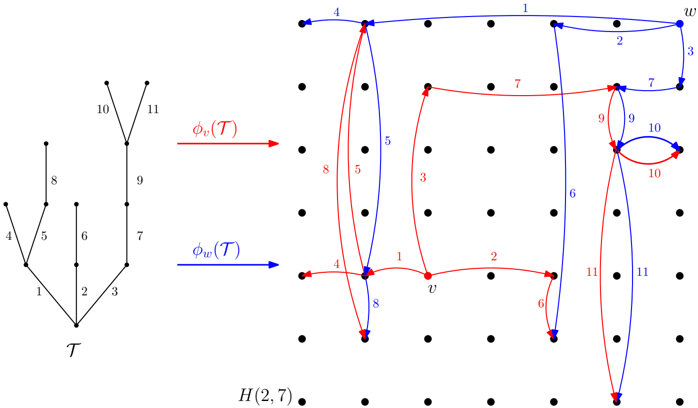

$\mathcal{T}$
$\phi_v(\mathcal{T})$
$\phi_w(\mathcal{T})$
$H(2,7)$
v
w

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Disadvantage:

- Many different processes and couplings going on at the same time

Thank you