Order of current variance in the simple exclusion process

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Joint work with

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(University of Wisconsin - Madison)

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1. ASEP: Interacting particles
2. ASEP: Surface growth
3. Growth fluctuations
4. The second class particle
5. The upper bound
6. The lower bound
7. Open questions

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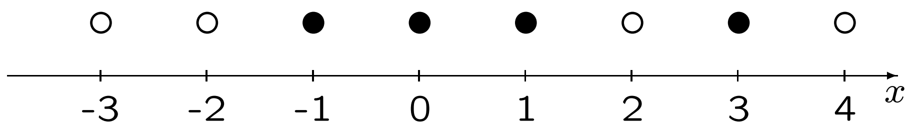

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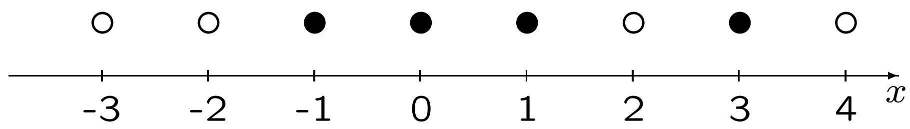

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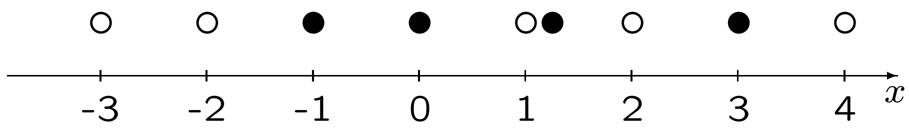

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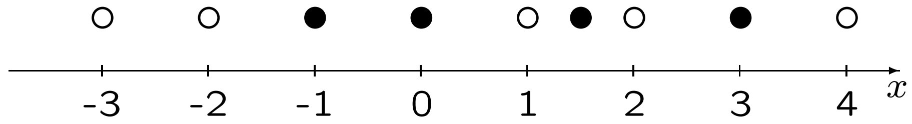

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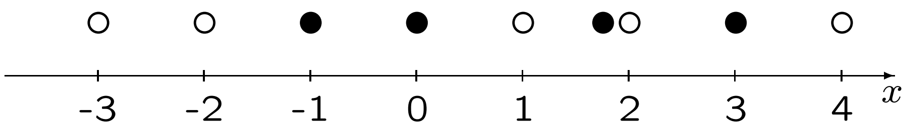

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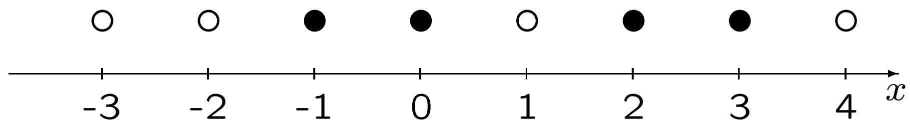

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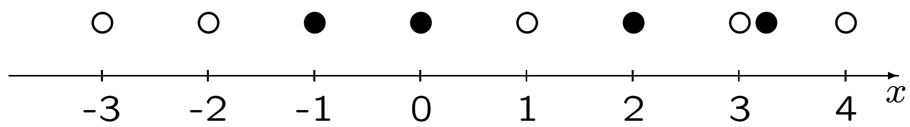

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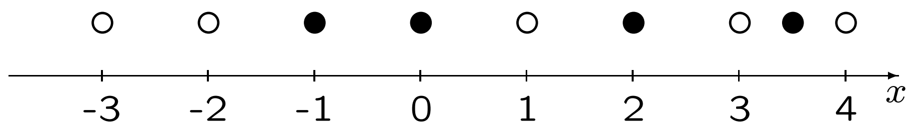

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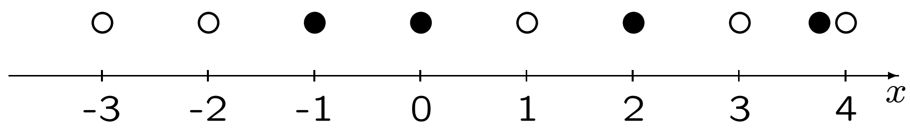

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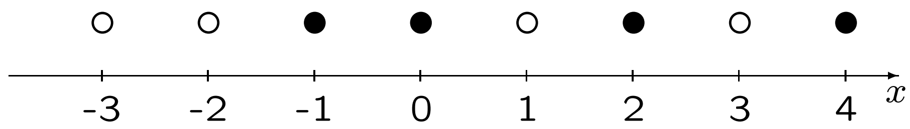

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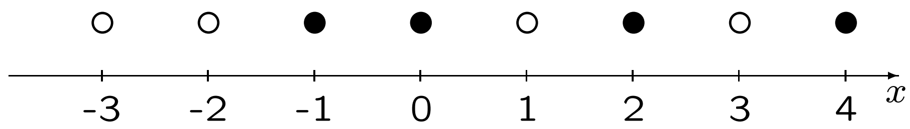

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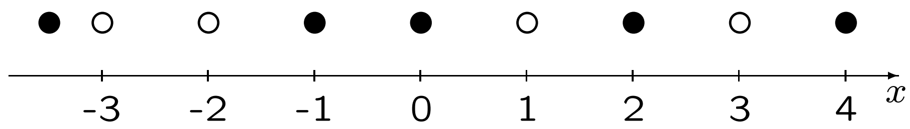

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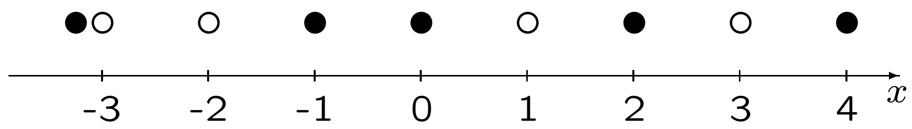

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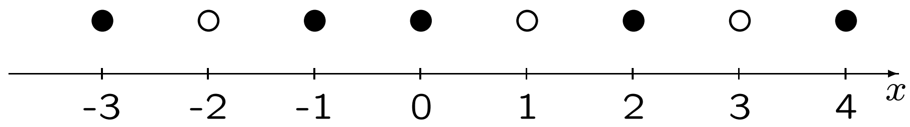

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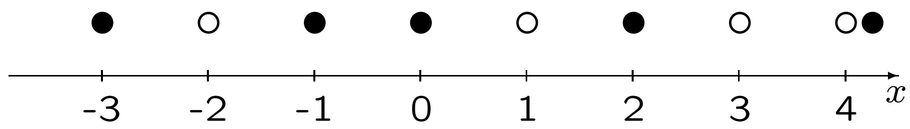

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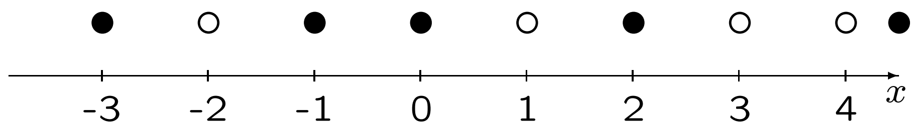

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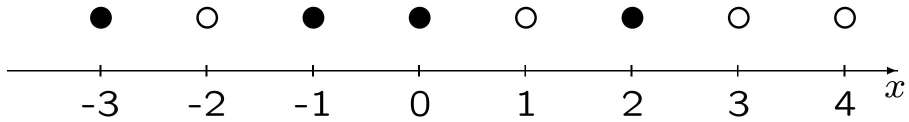

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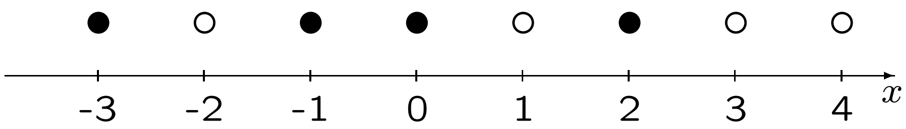

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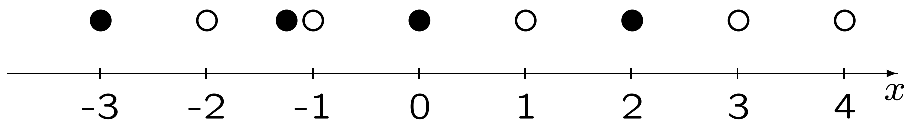

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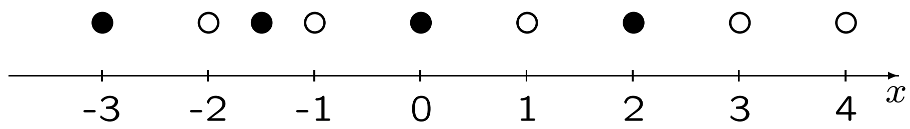

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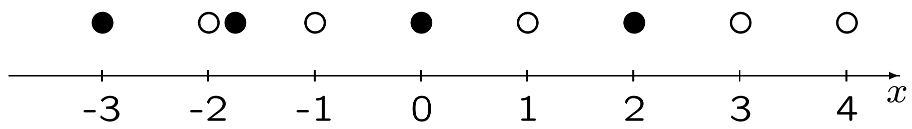

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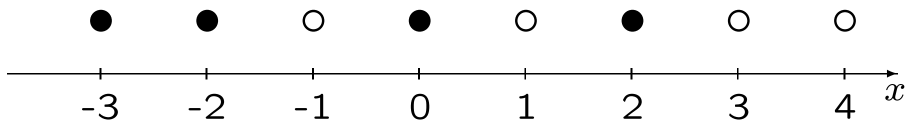

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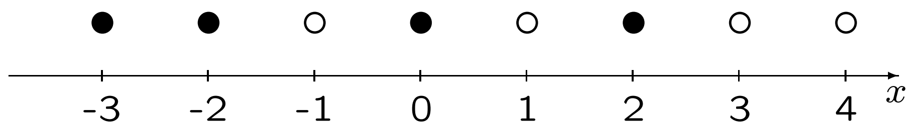

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The Bernoulli(ϱ) distribution is time-stationary for any ($0 \leq \varrho \leq 1$).
Any translation-invariant stationary distribution is a mixture of Bernoullis.

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$\rightsquigarrow$ The characteristic speed $C(\varrho) := a[1-2\varrho]$.
(ϱ is constant along $\dot{X}(T) = C(\varrho)$.)

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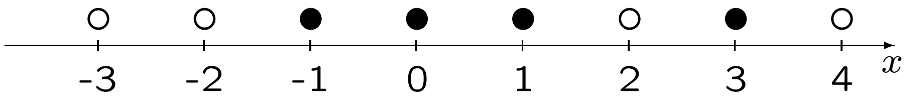

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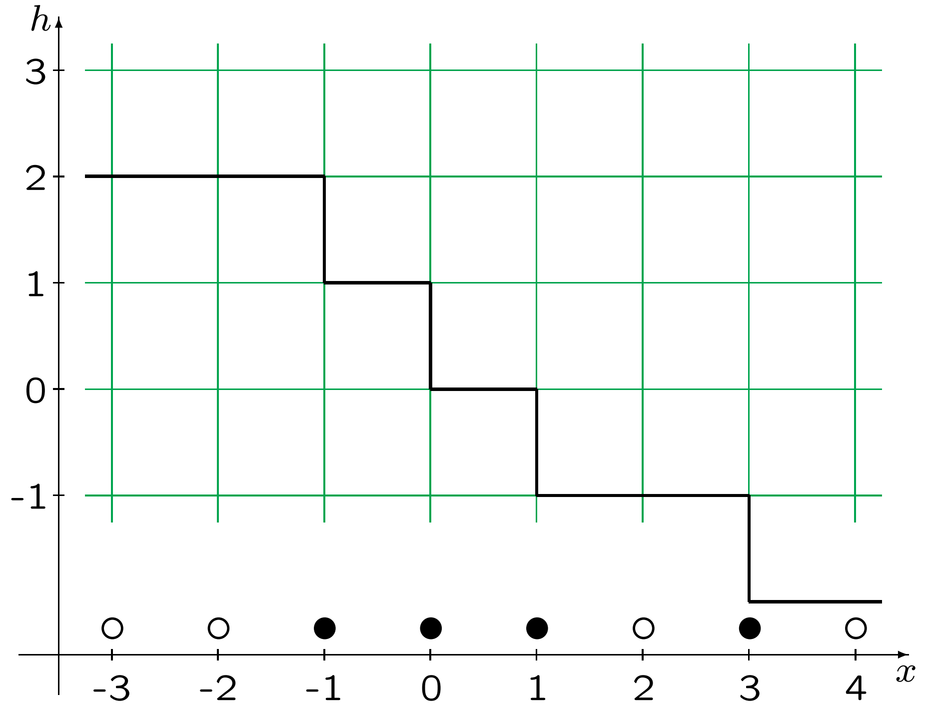

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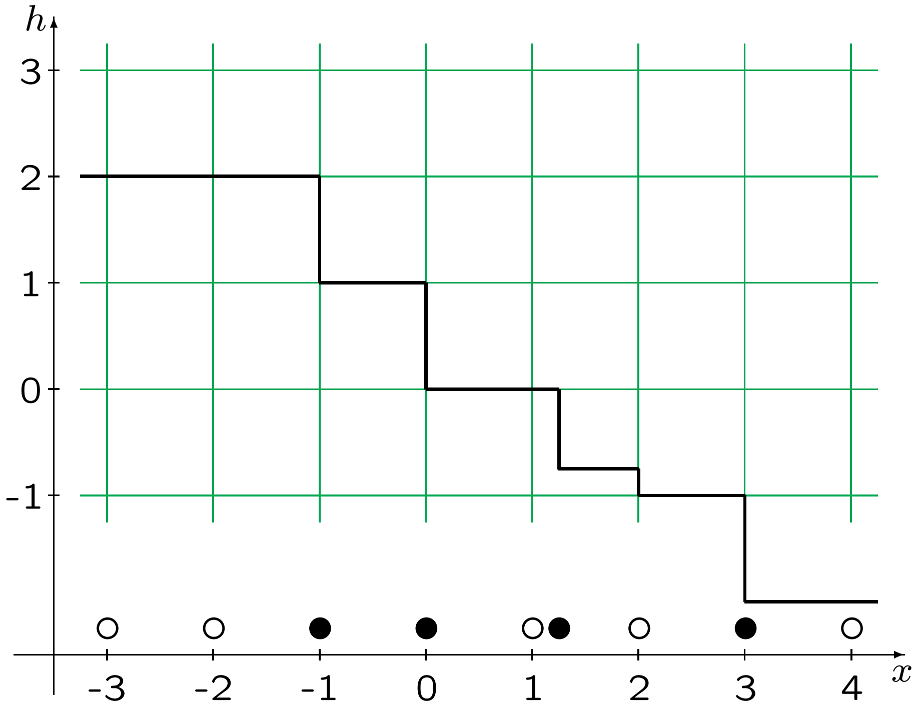

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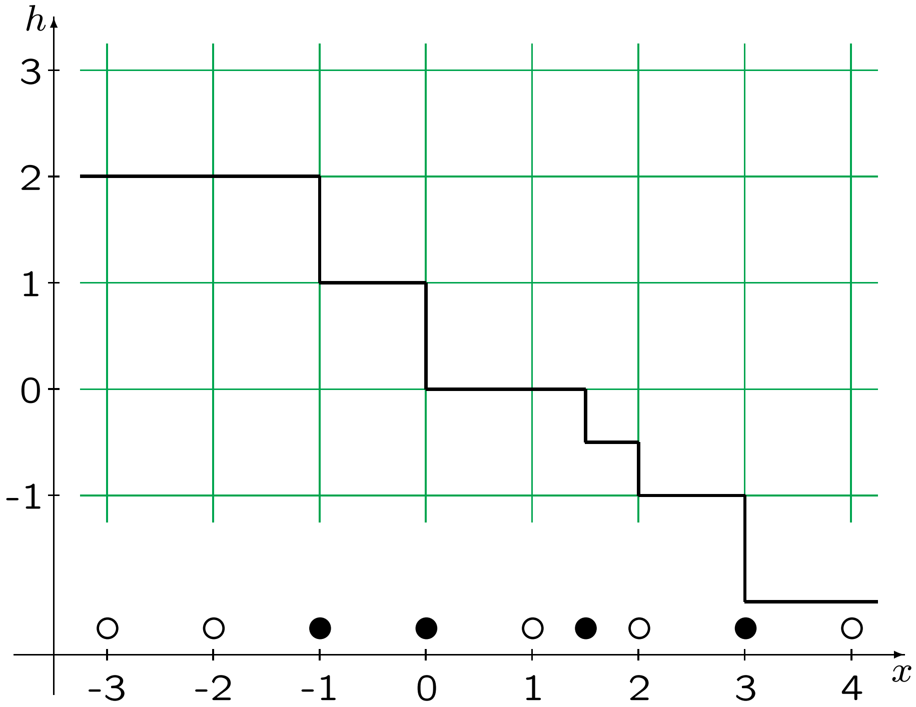

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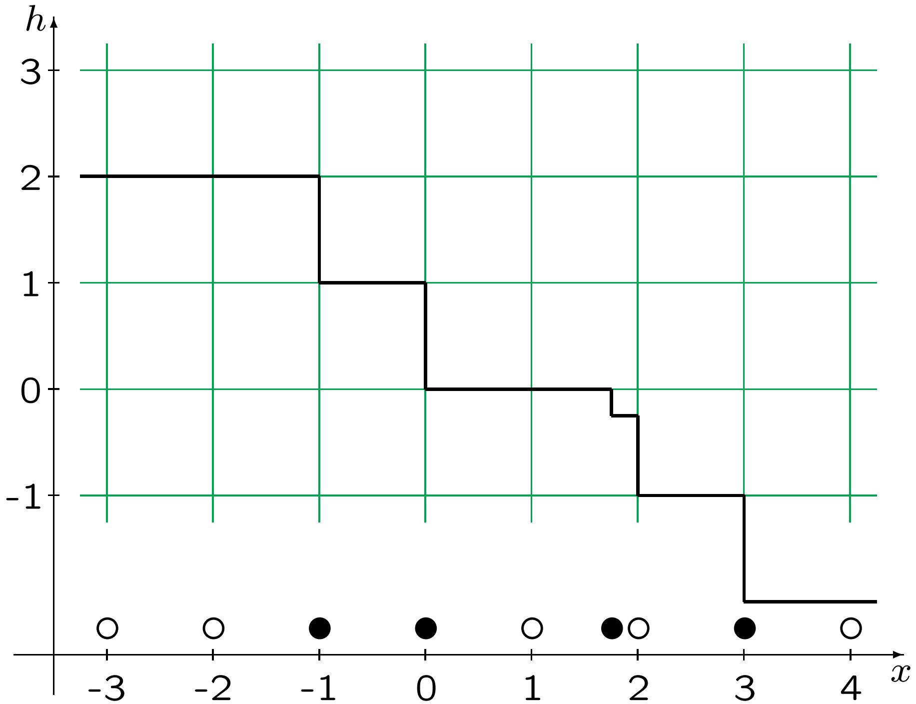

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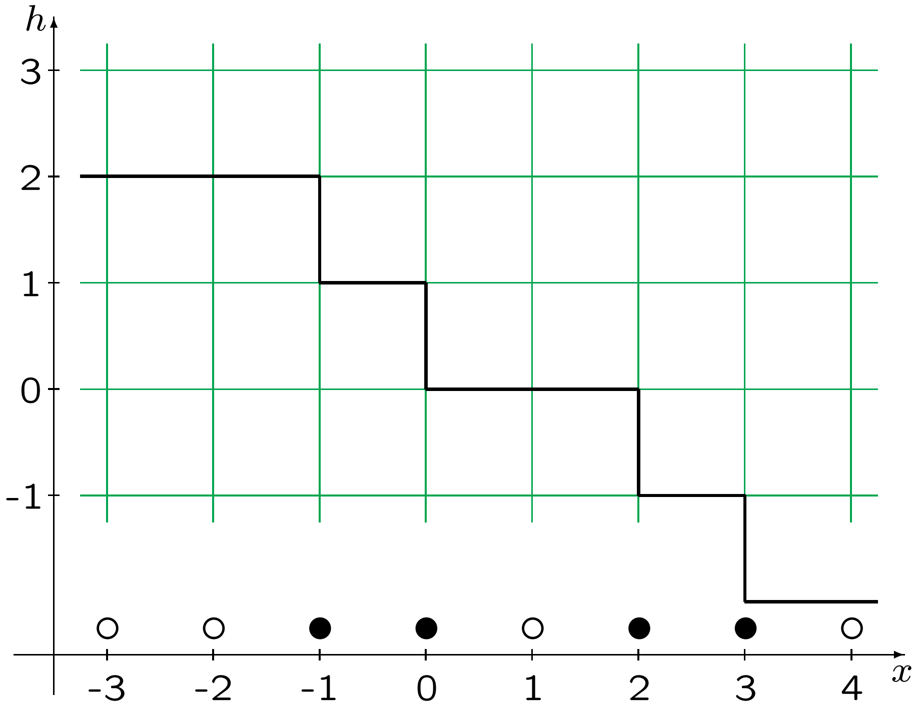

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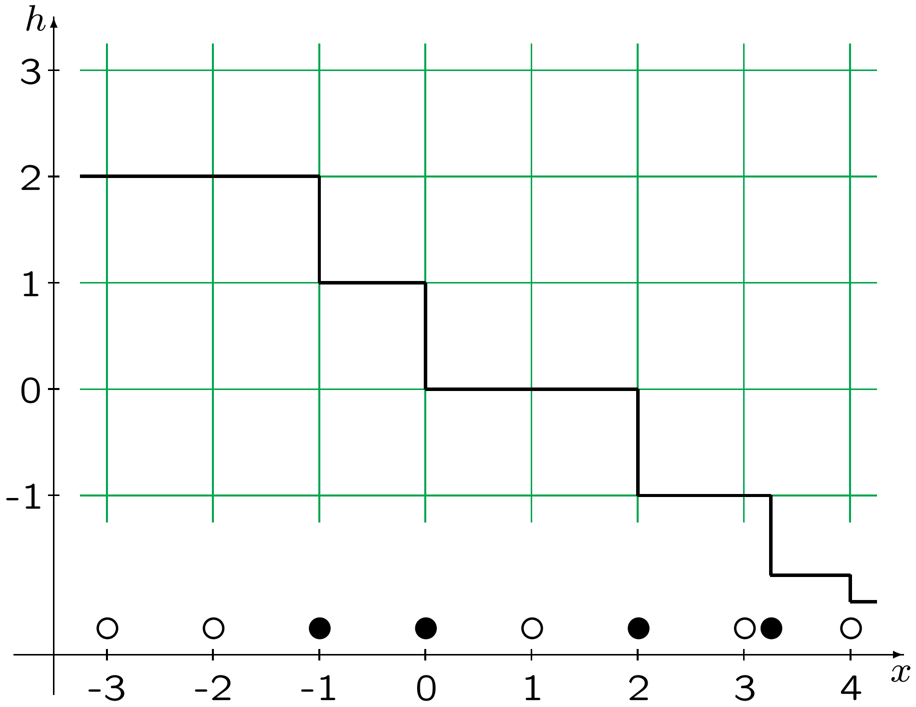

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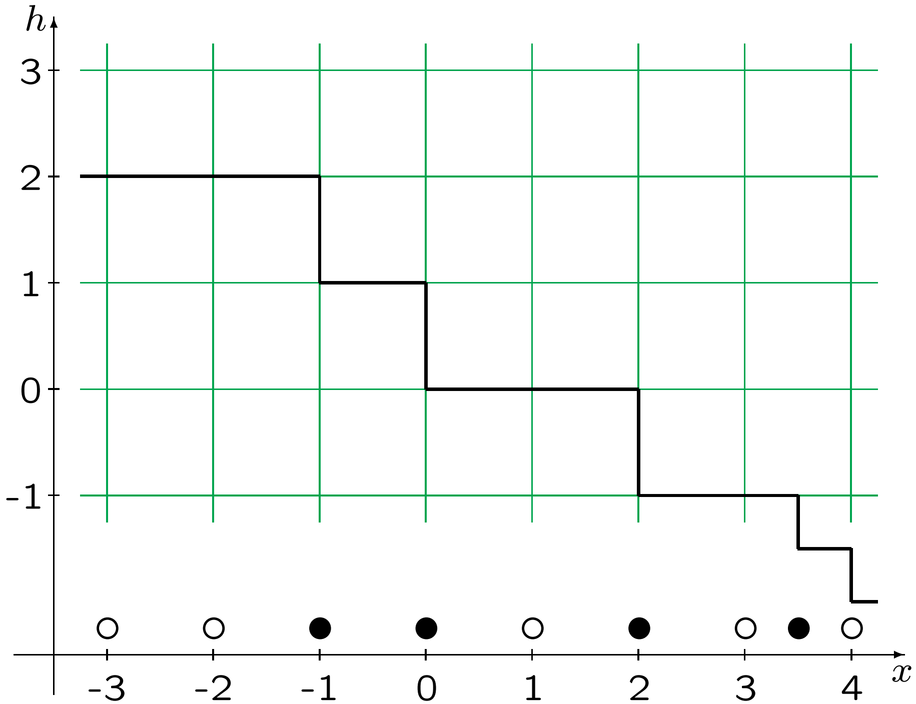

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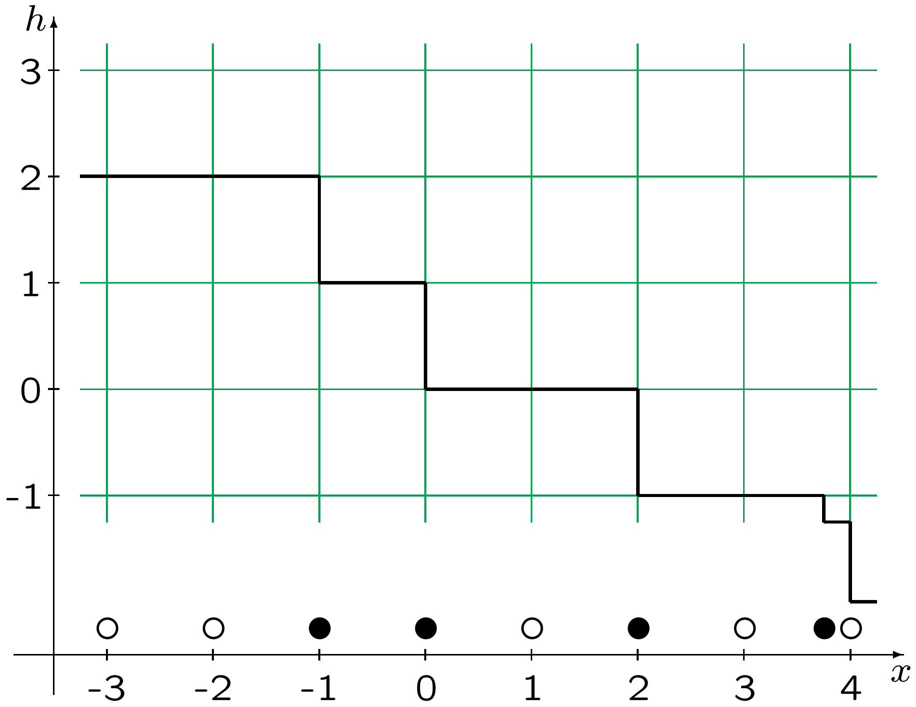

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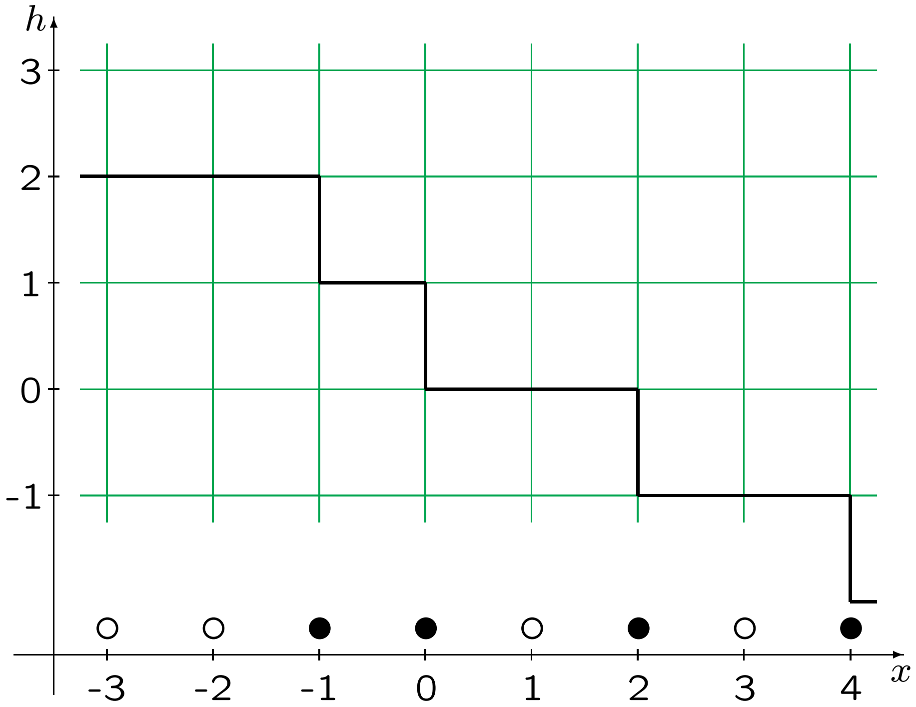

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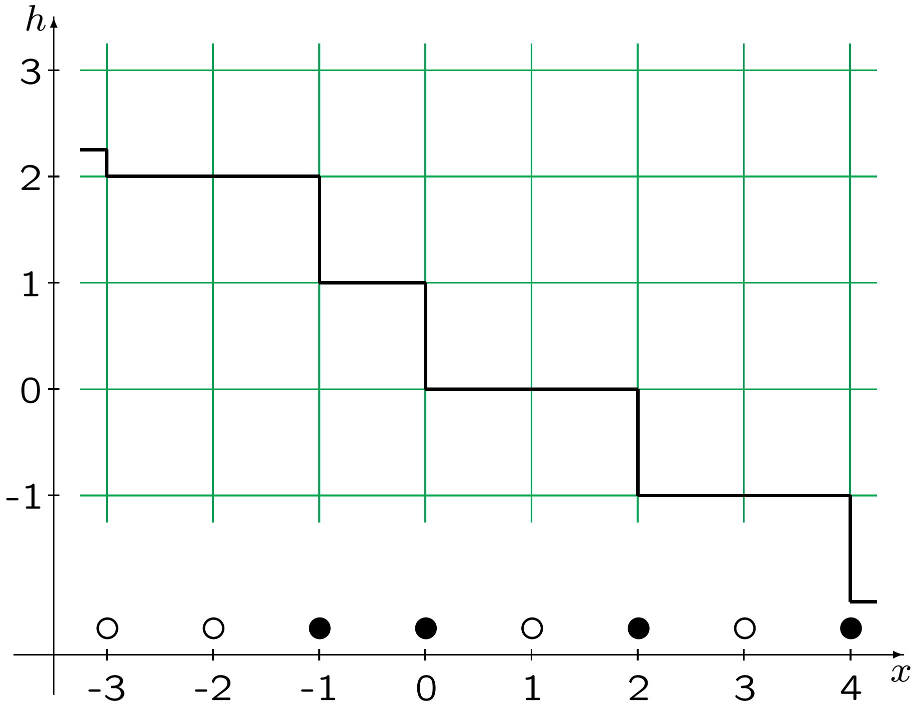

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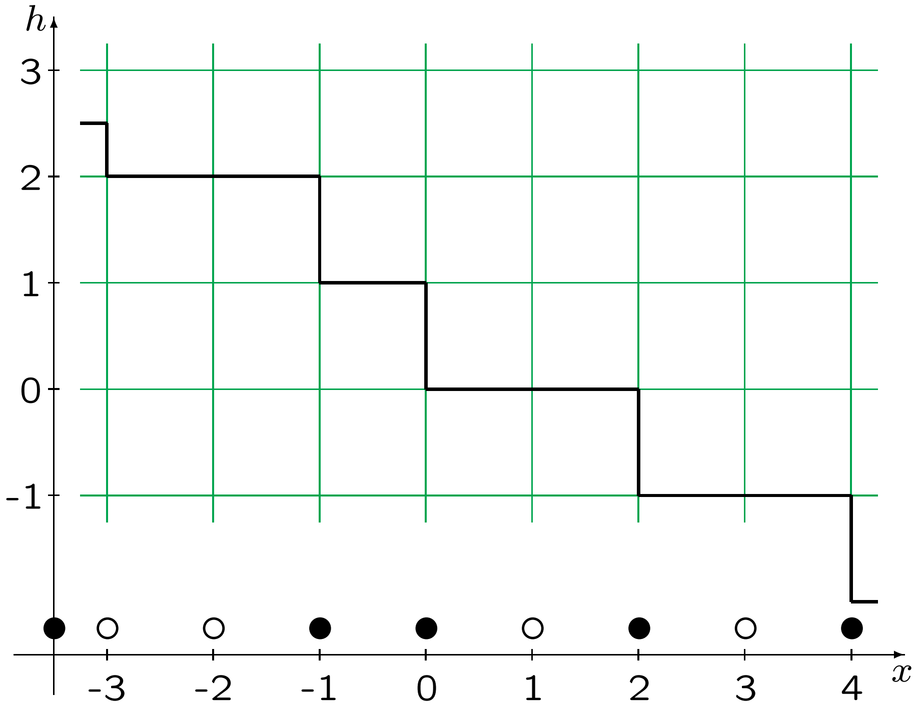

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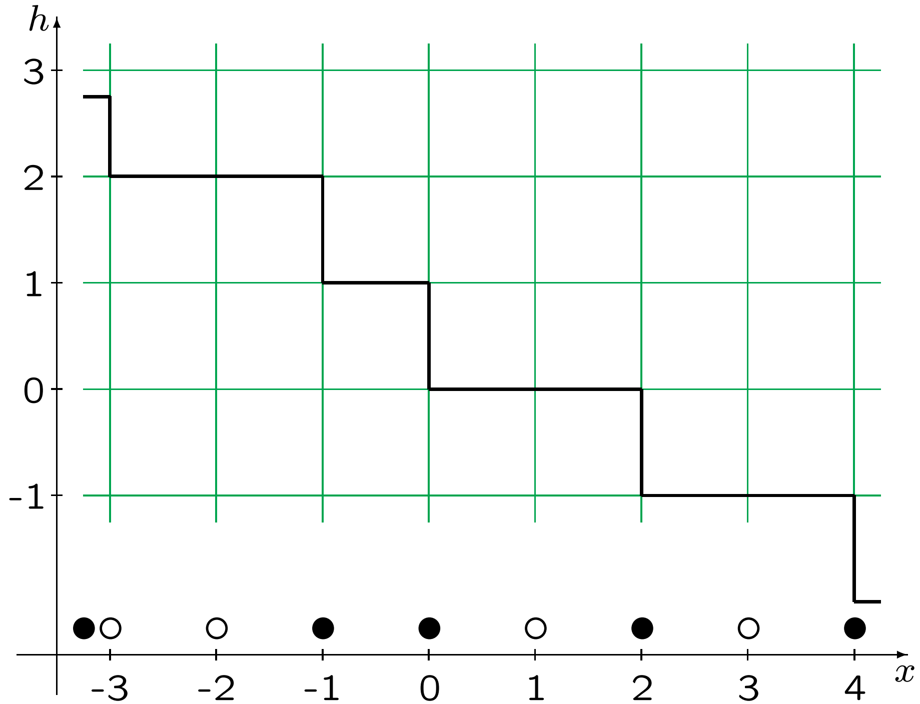

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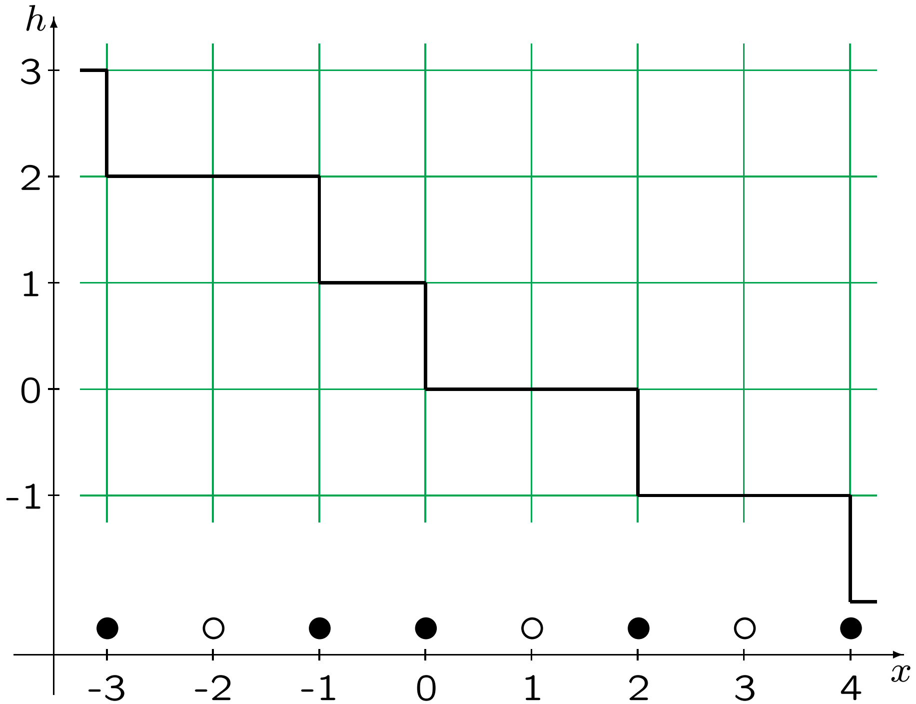

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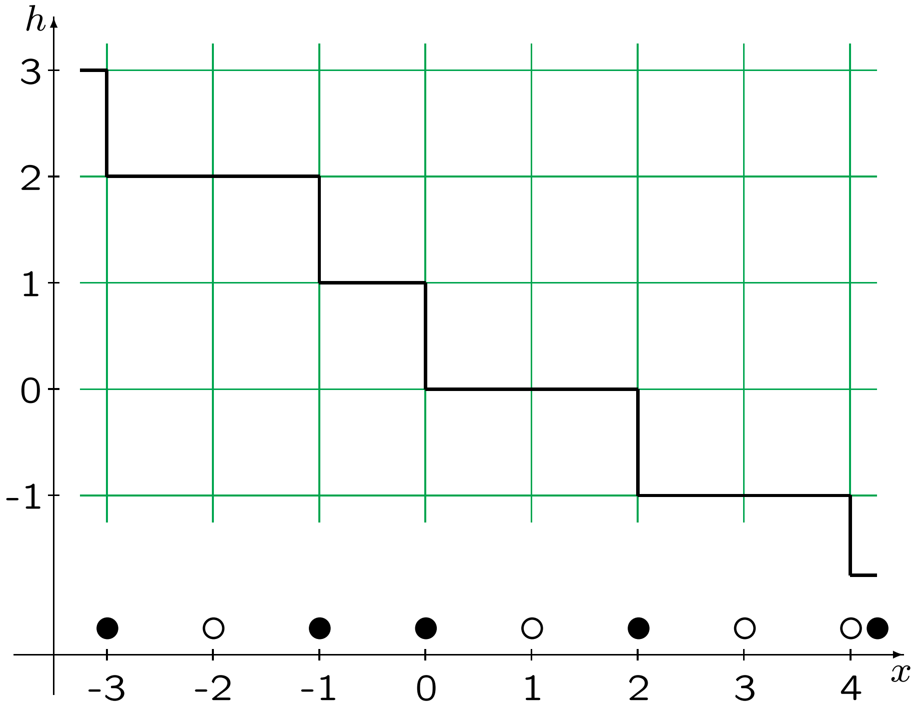

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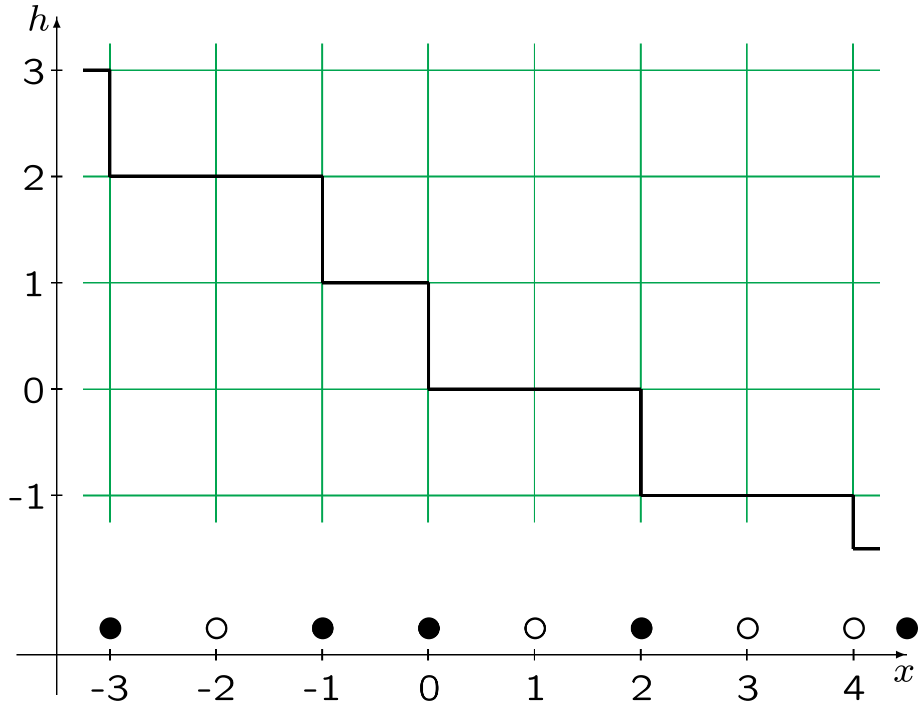

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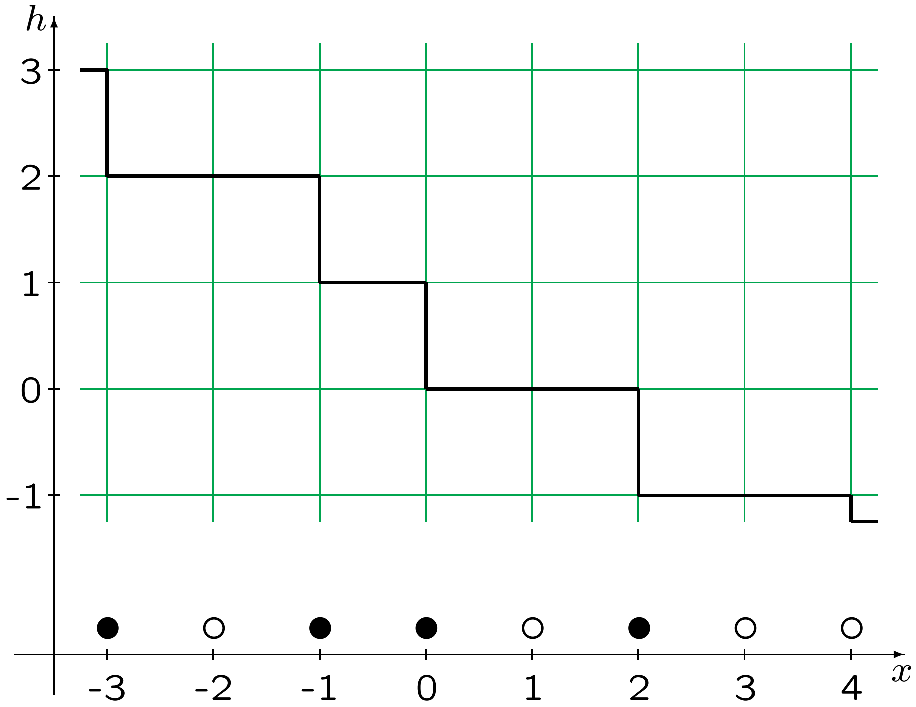

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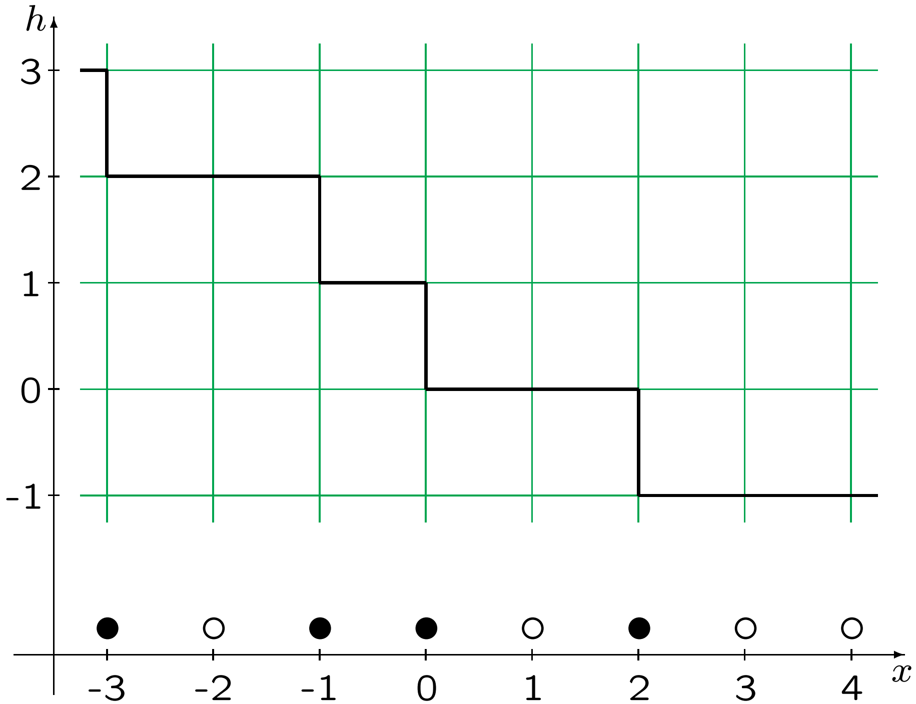

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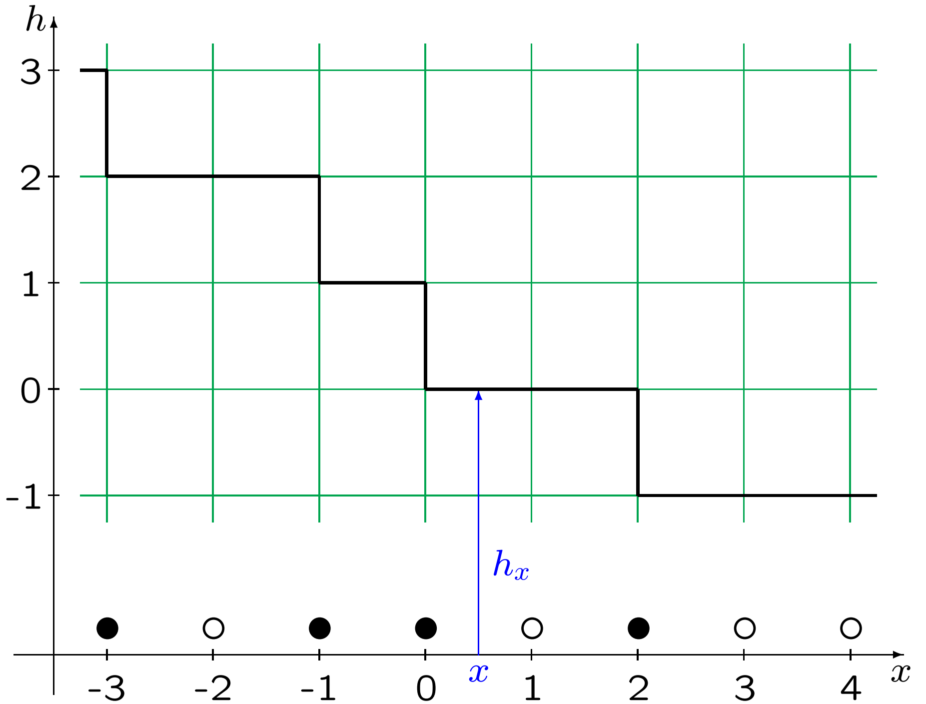

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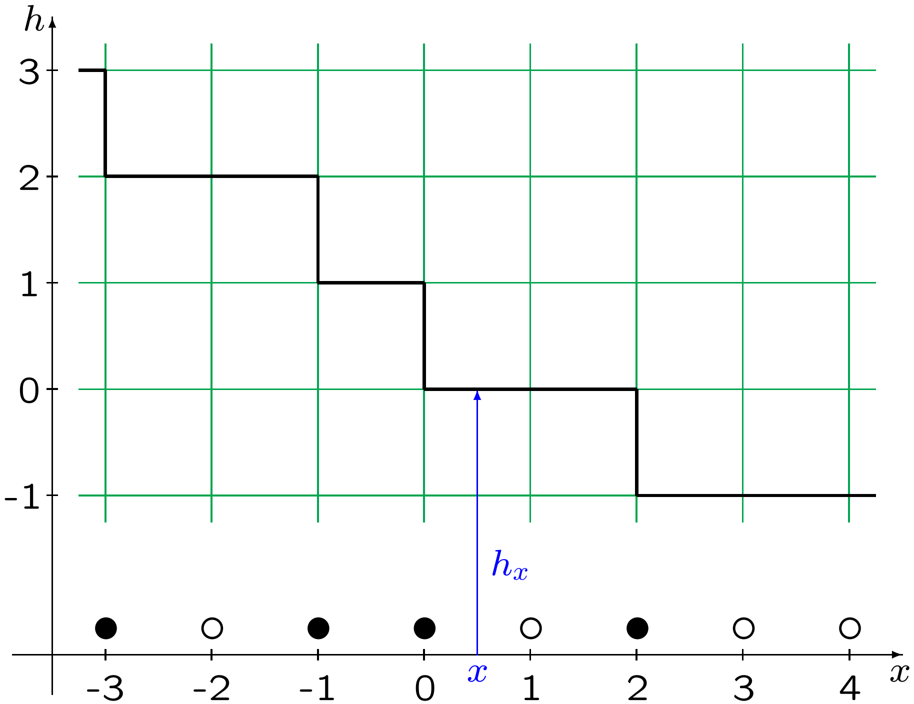

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$h_x(t) - h_x(0)$ = net number of particles passed above x.

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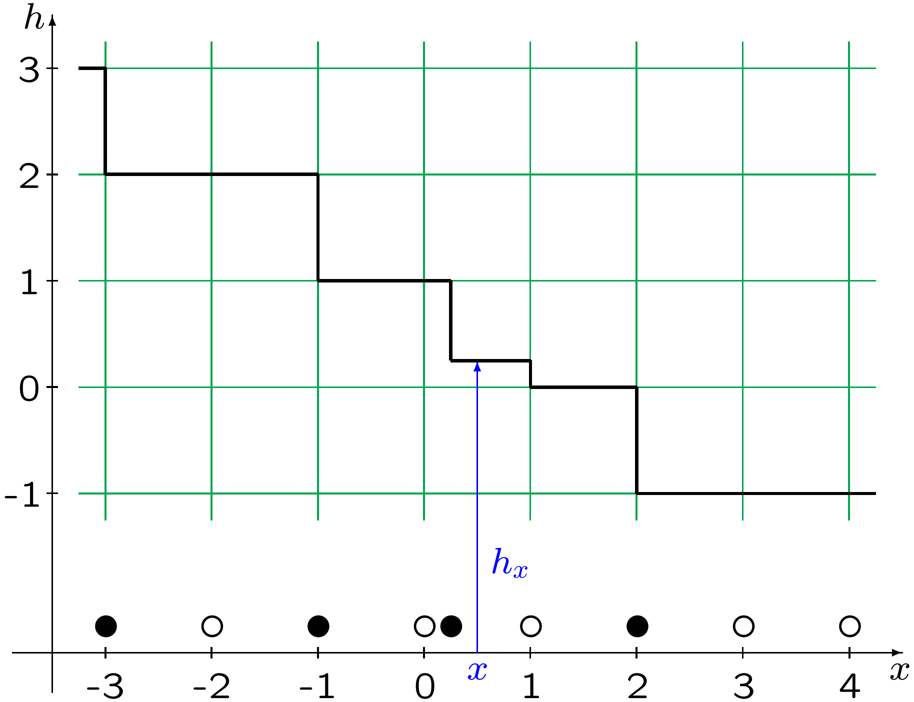

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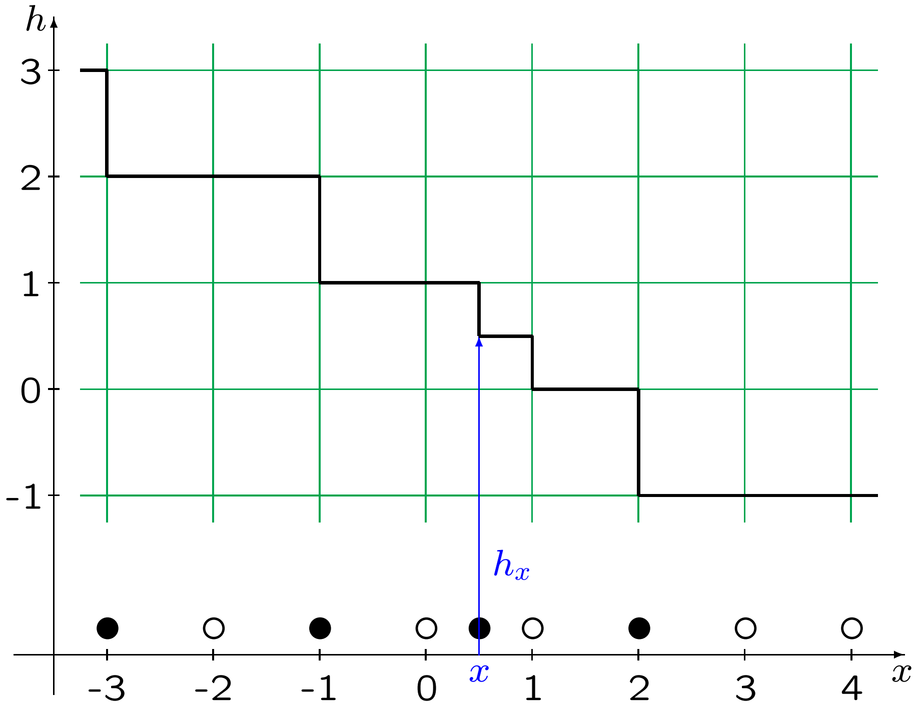

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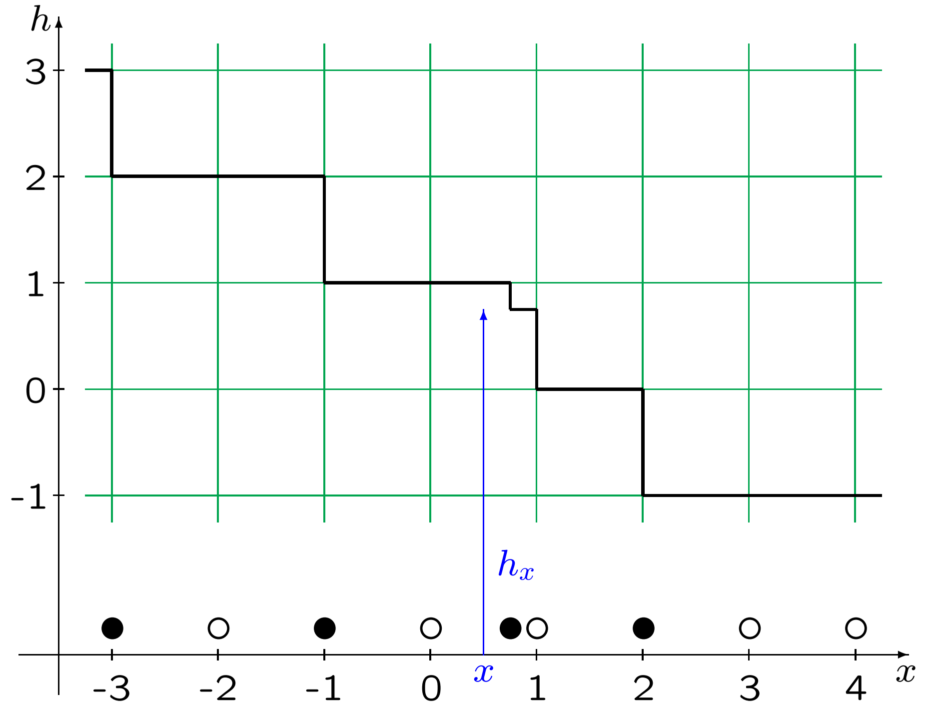

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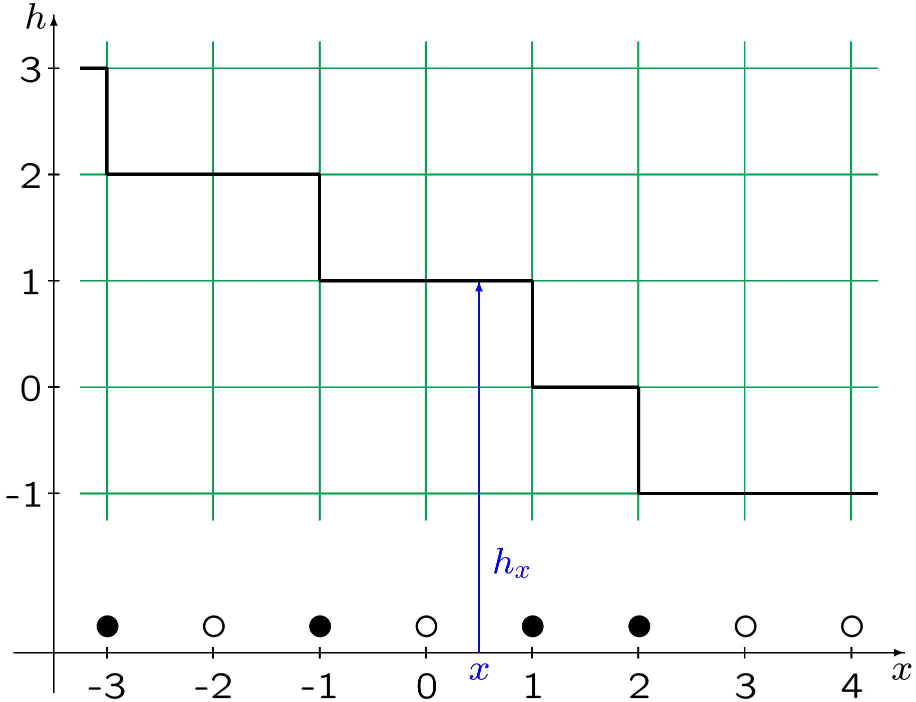

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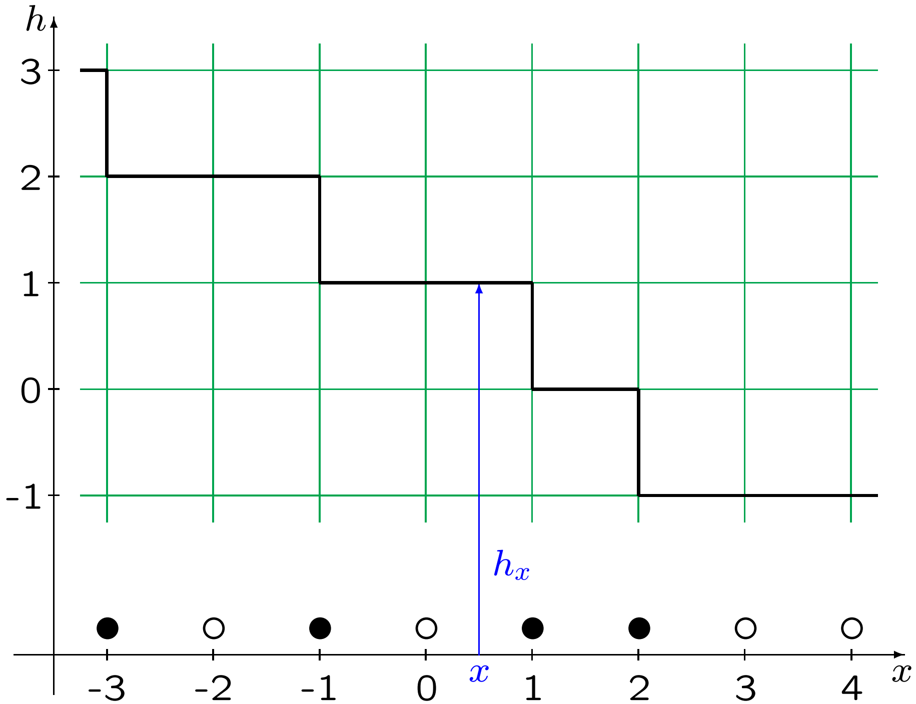

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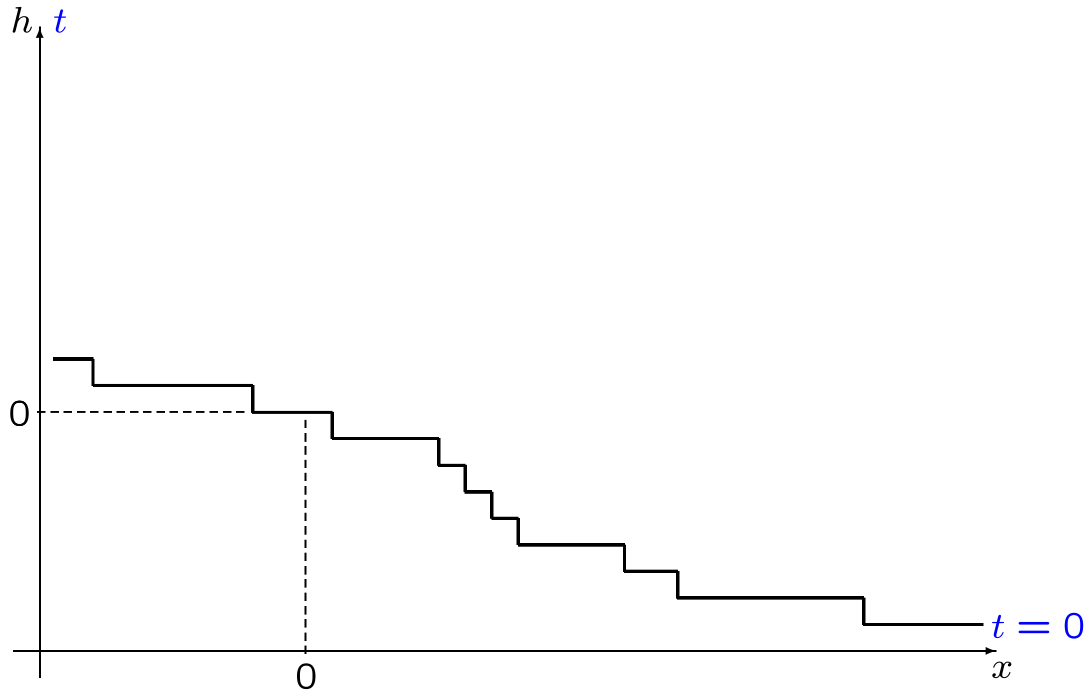

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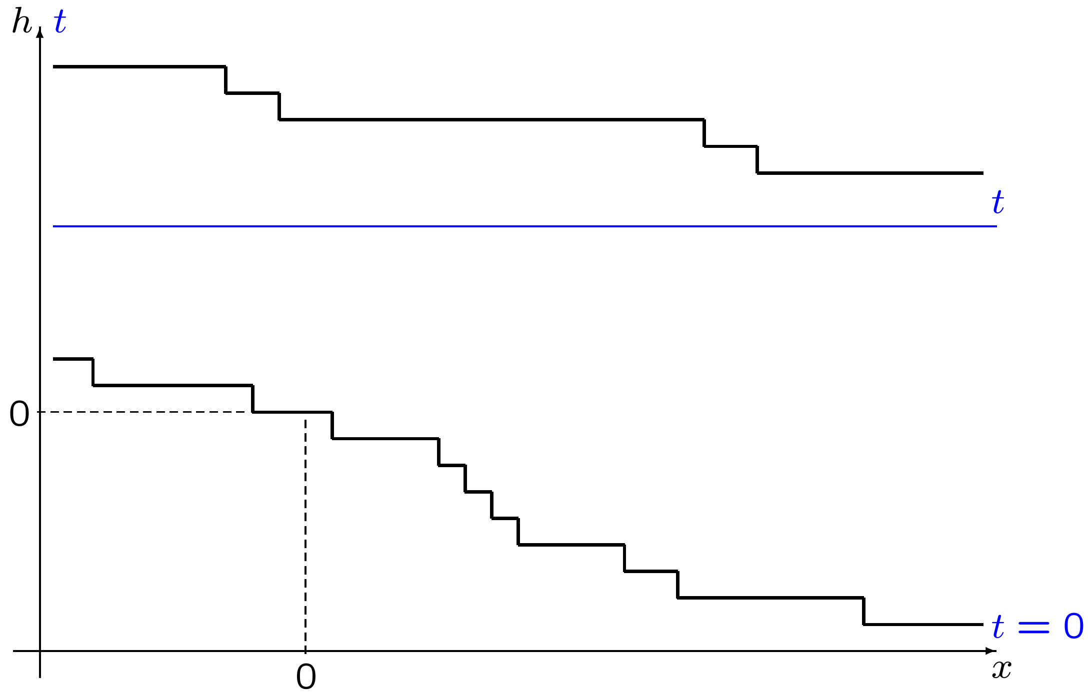

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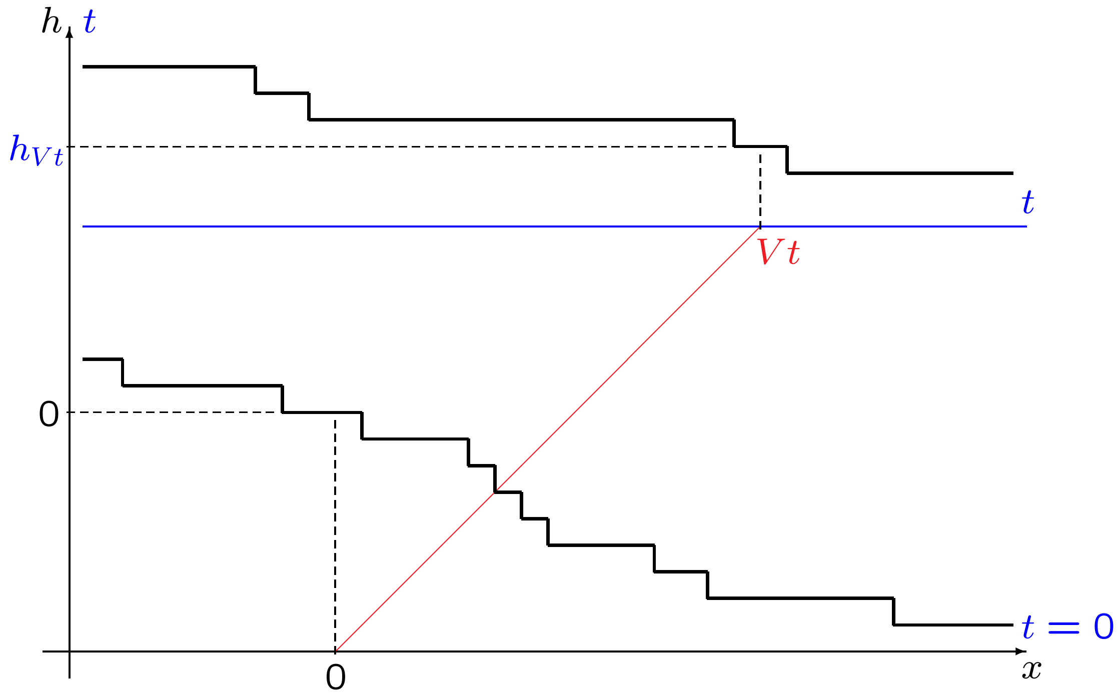

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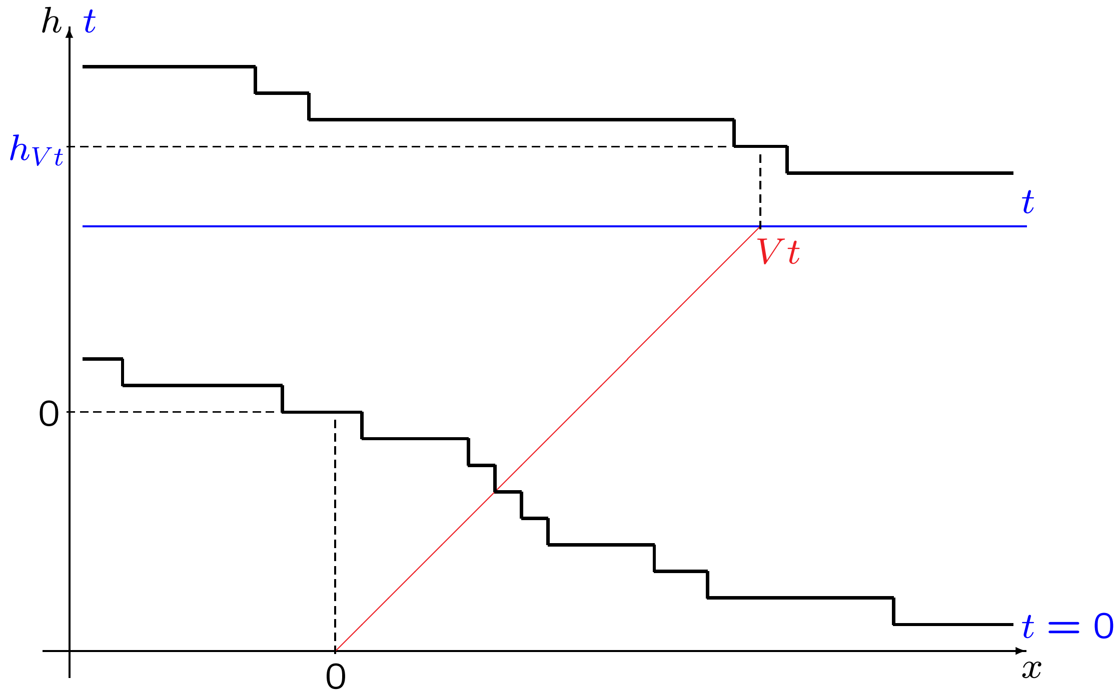

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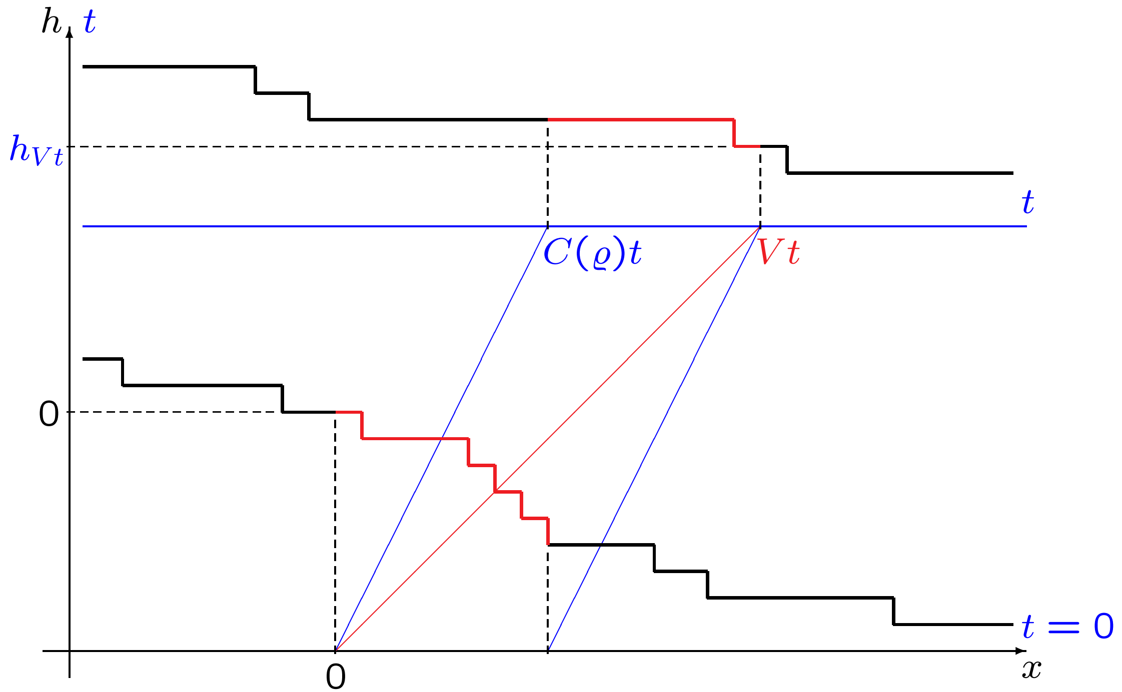

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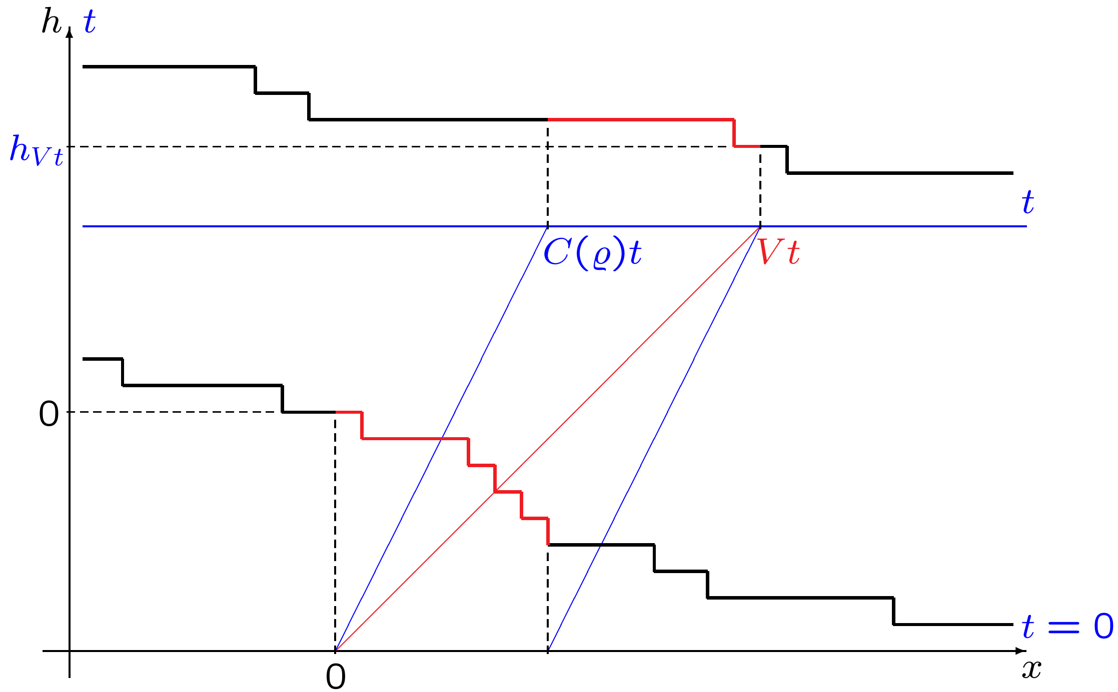

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$\rightsquigarrow$ Initial fluctuations are transported along the characteristics.

$\rightsquigarrow$ How about $V = C(\varrho)$?

3. Growth fluctuations

Conjecture:

$$\lim_{t\to\infty} \frac{\mathbf{Var}(h_{C(\varrho)t}(t))}{t^{2/3}} = [\text{sg. non trivial}].$$

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Theorem (B., Seppäläinen): For any $0 < \varrho < 1$, and any $q < p$,

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Corollary: The corresponding scaling of the diffusivity is also proved.

3. Growth fluctuations

Limit distributions (not yet controlling the second moment) in terms of the Tracy-Widom distribution (GUE random matrices) were found by Baik, Deift and Johansson 1999, Johansson 2000, and Ferrari and Spohn 2006 for the *totally* asymmetric exclusion (TASEP: $p = 1$, $q = 0$).

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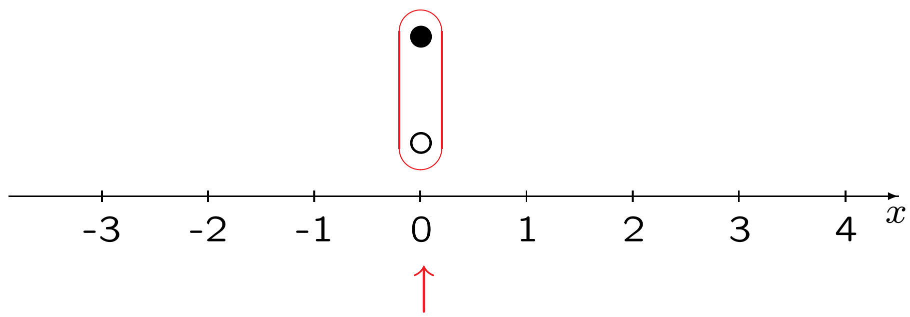
$\rightsquigarrow$ We needed to get rid of these tools. Premises: Cator and Groeneboom 2006 (Hammersley's process), B., Cator and Seppäläinen 2006 (TASEP, last passage).

4. The second class particle



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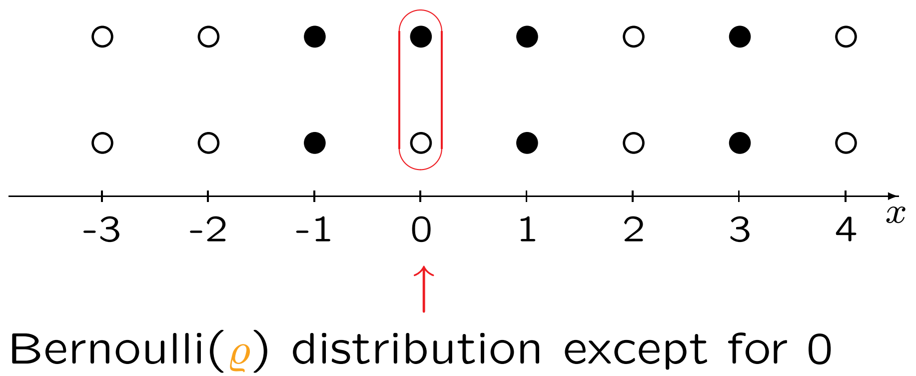

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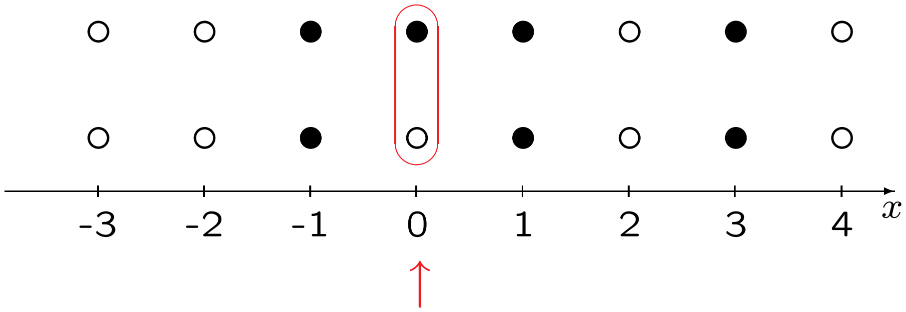

Bernoulli(ϱ) distribution except for 0

Coupling: A single discrepancy is always conserved

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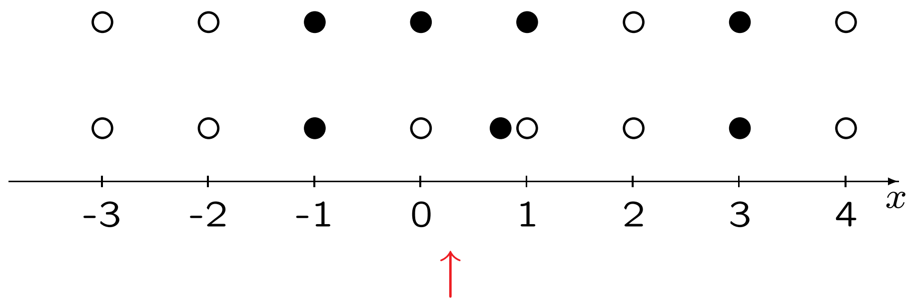

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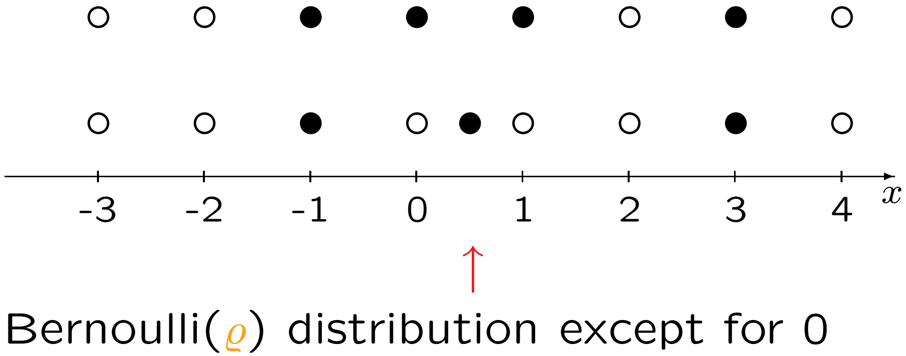

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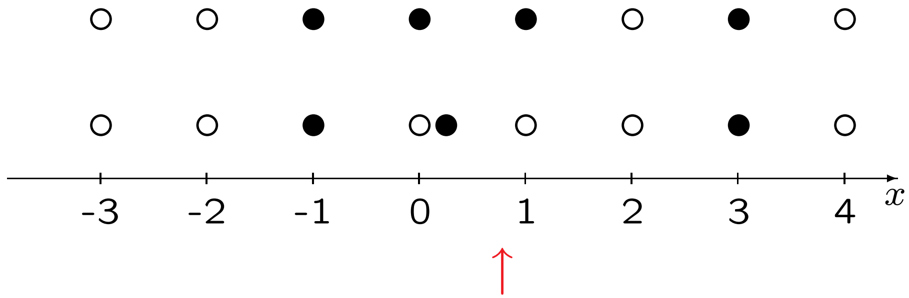

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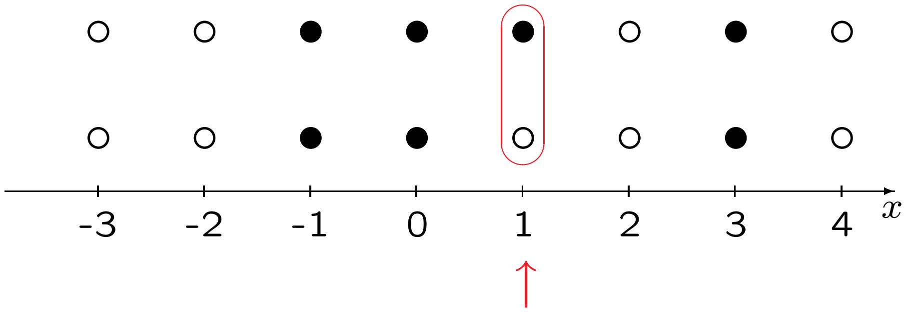

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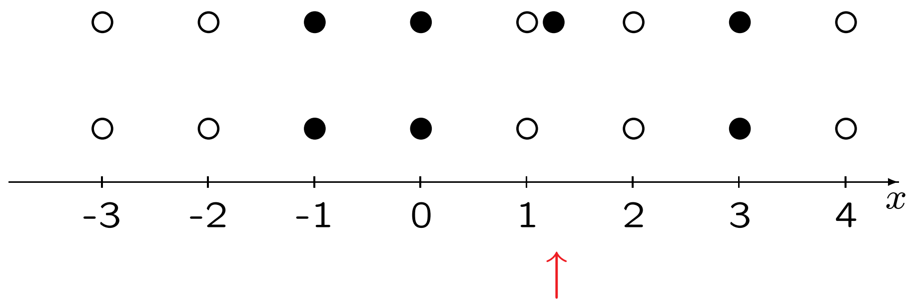

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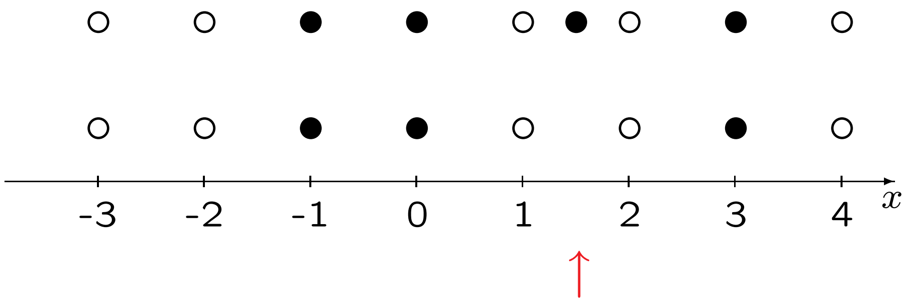

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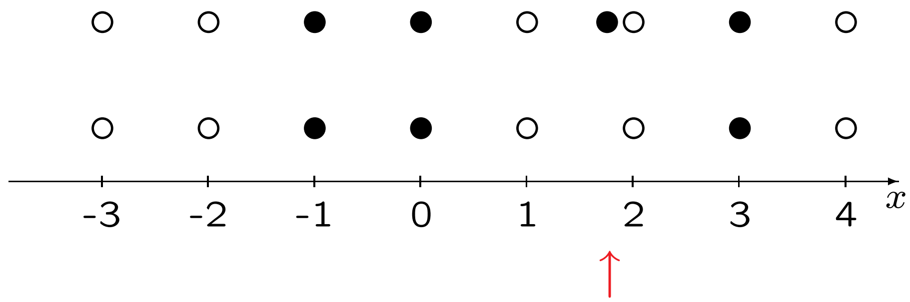

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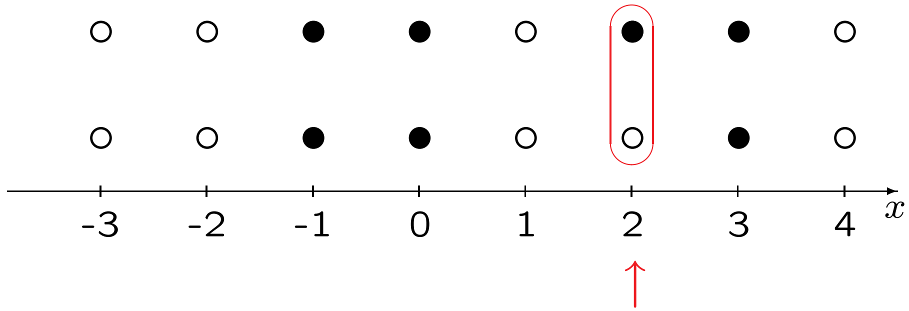

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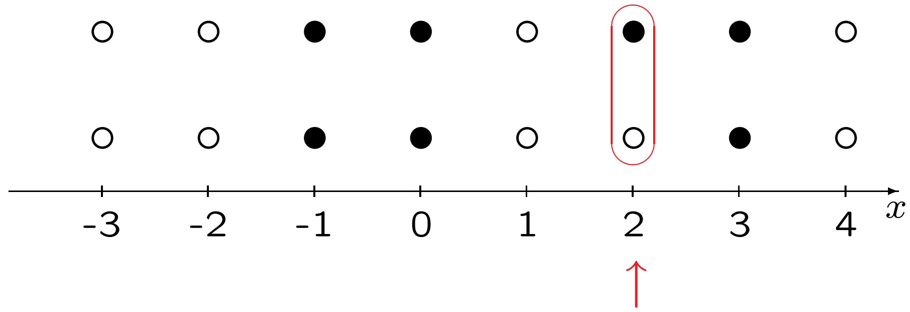

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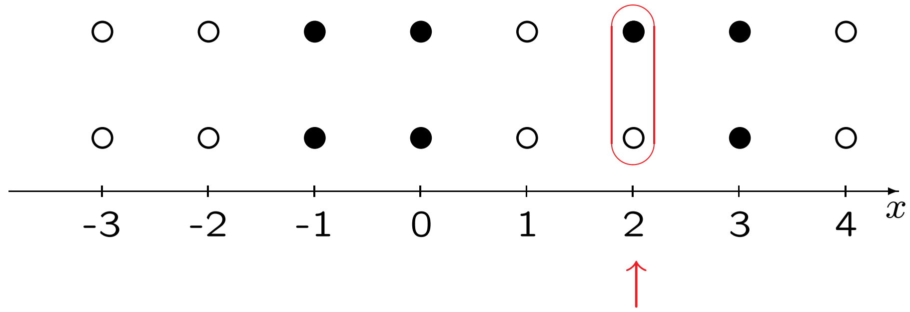

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The second class particle is a highly nontrivial object. For example, the Bernoulli(ϱ) distribution is *not* stationary as seen by the second class particle.

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$$\mathbf{E}(Q(t)) = C(\varrho)t$$

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The proof is based on ideas of Bálint Tóth, he said these ideas were standard.

Main idea for prooving $t^{1/3}$ scaling:

$$
\begin{tikzcd}
\boxed{\text{Deviations of } Q(t)} \arrow[r, bend left, "\text{Couplings}"] & \boxed{\text{Deviations of } h_{C(\varrho)}(t)} \arrow[l, bend left, "\text{The above thm.}"]
\end{tikzcd}
$$

The coupling measure

Let $\lambda < \varrho$, and

$$\mu\binom{\circ}{\circ} = 1 - \varrho, \quad \mu\binom{\bullet}{\circ} = \varrho - \lambda, \quad \mu\binom{\bullet}{\bullet} = \lambda.$$

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Then the “upper” marginal is Bernoulli(ϱ), and the “lower” marginal is Bernoulli(λ).

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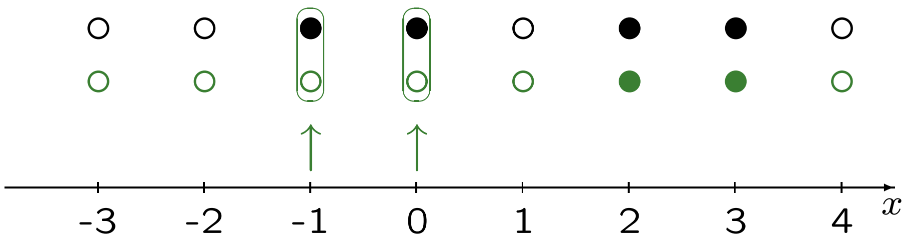

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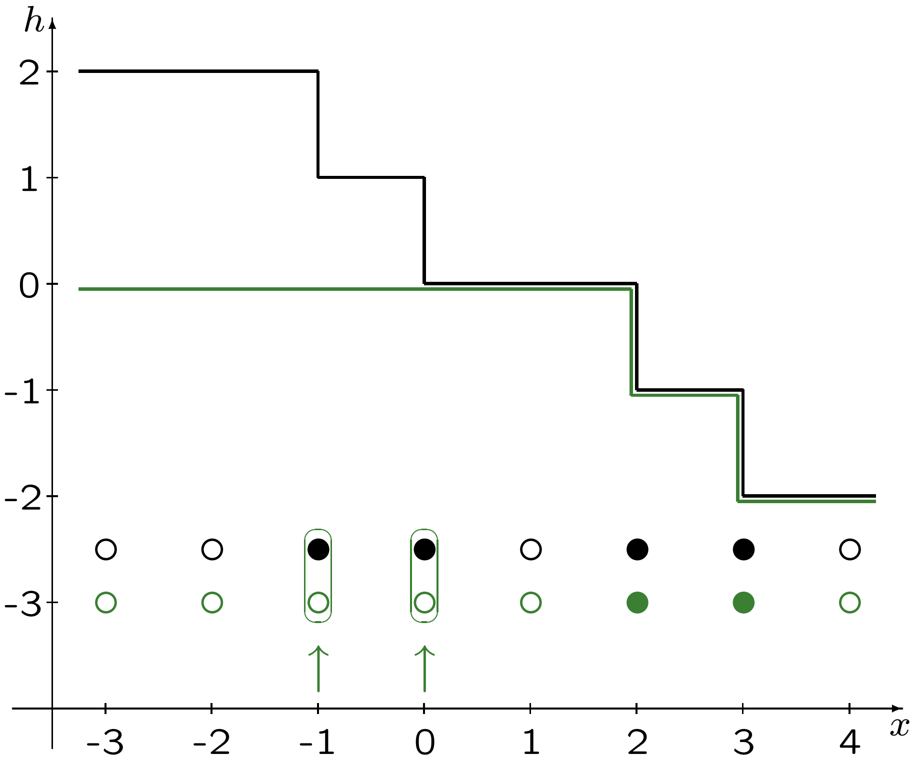

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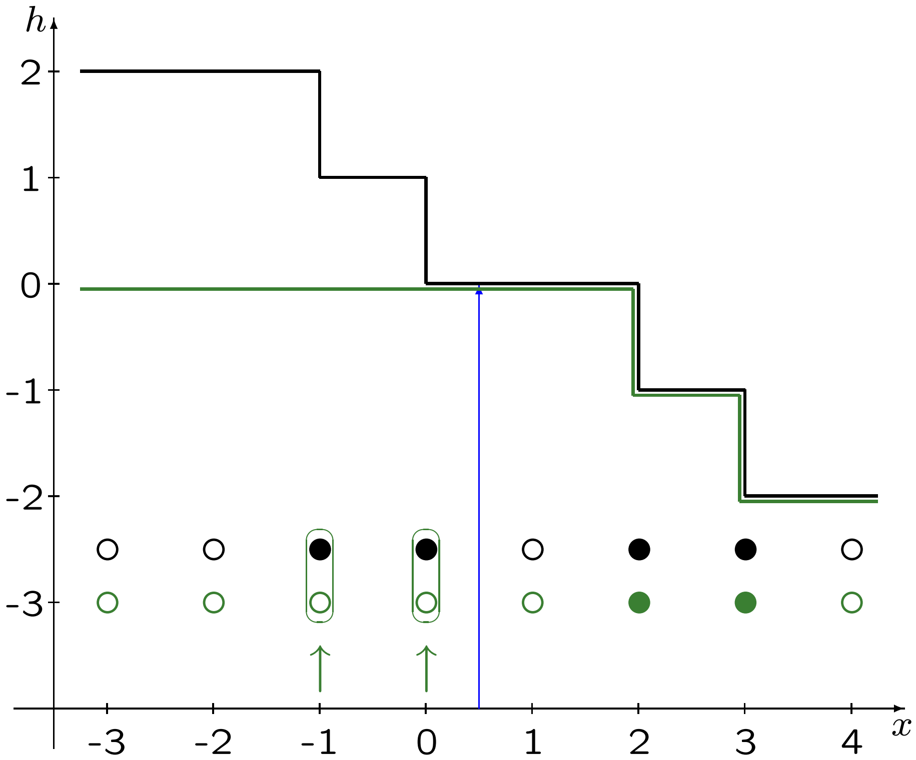

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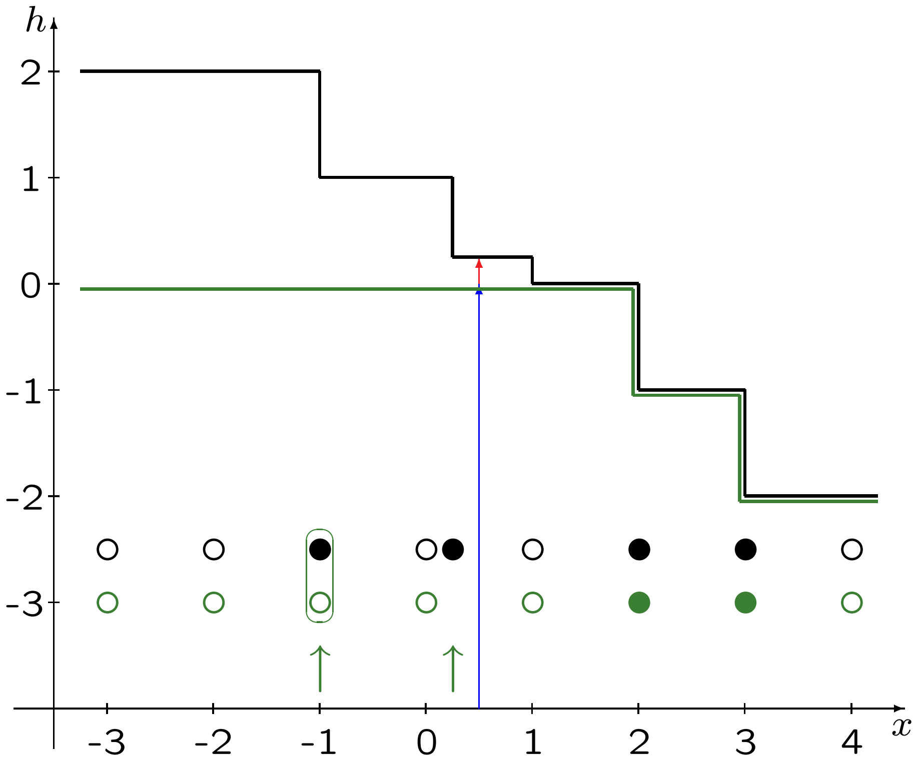

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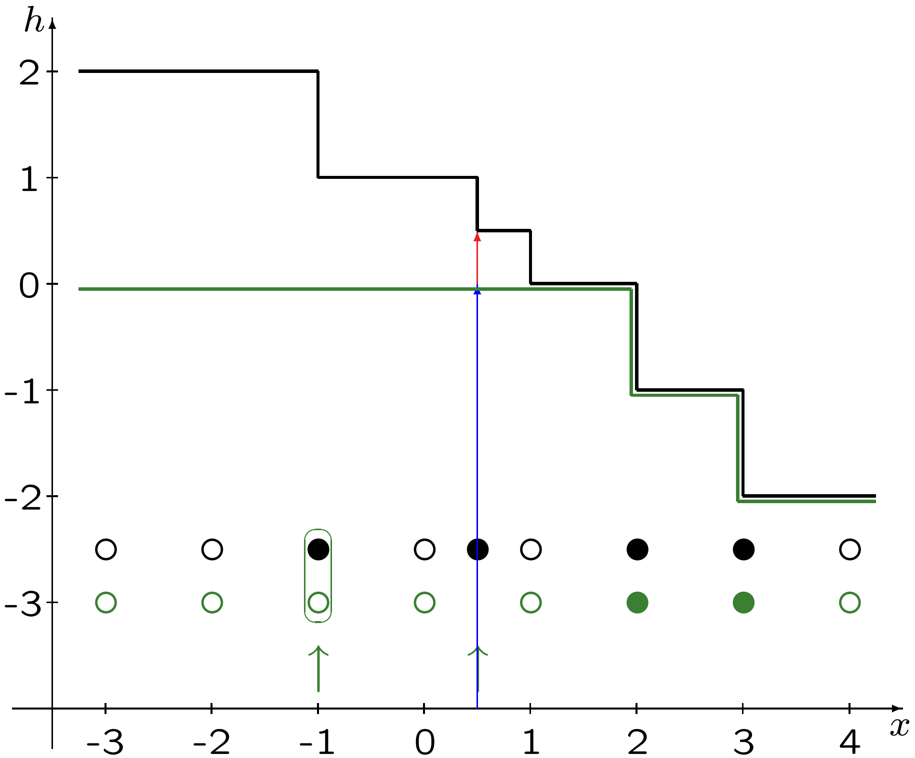

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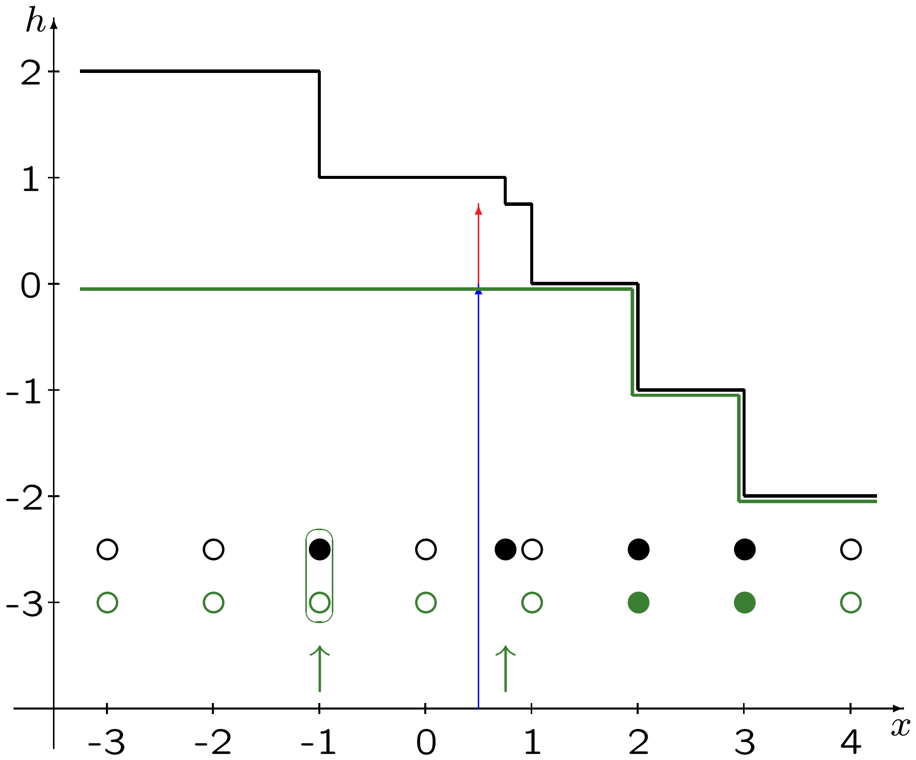

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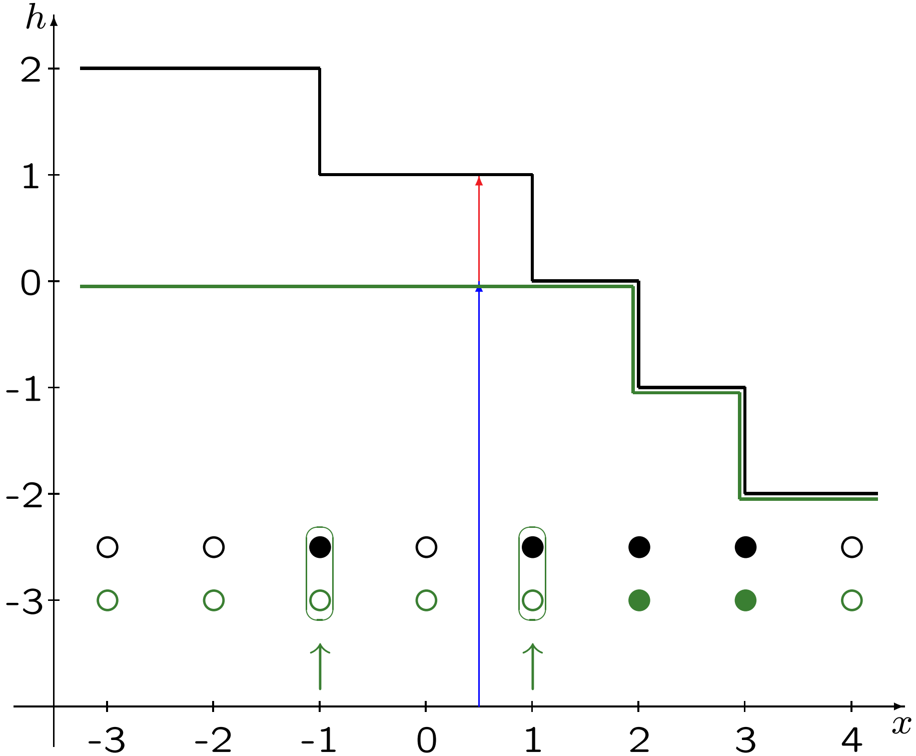

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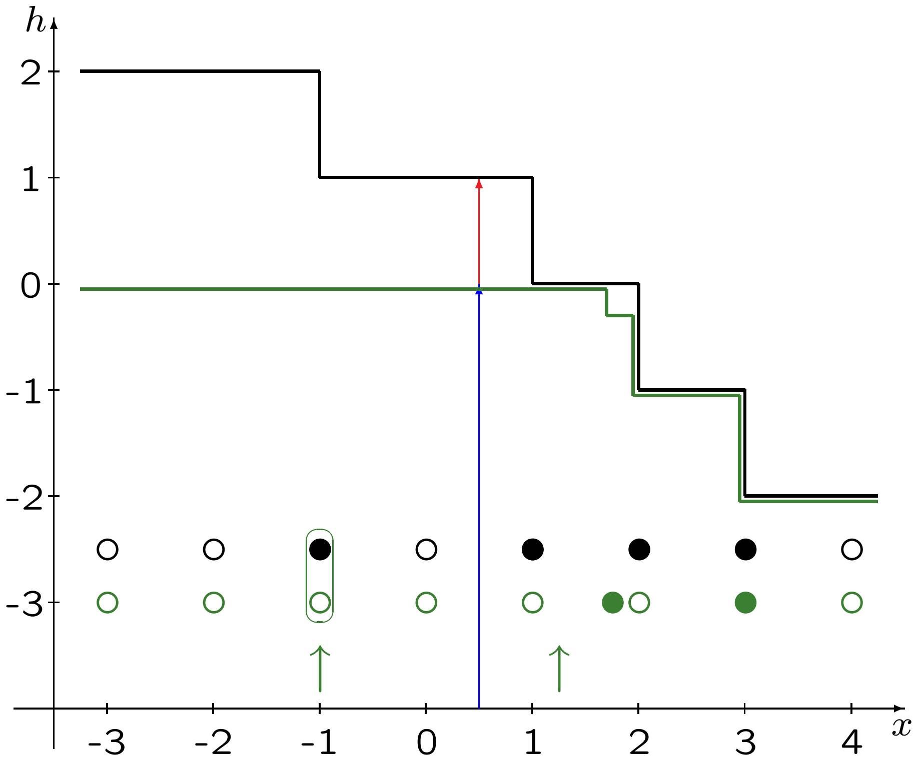

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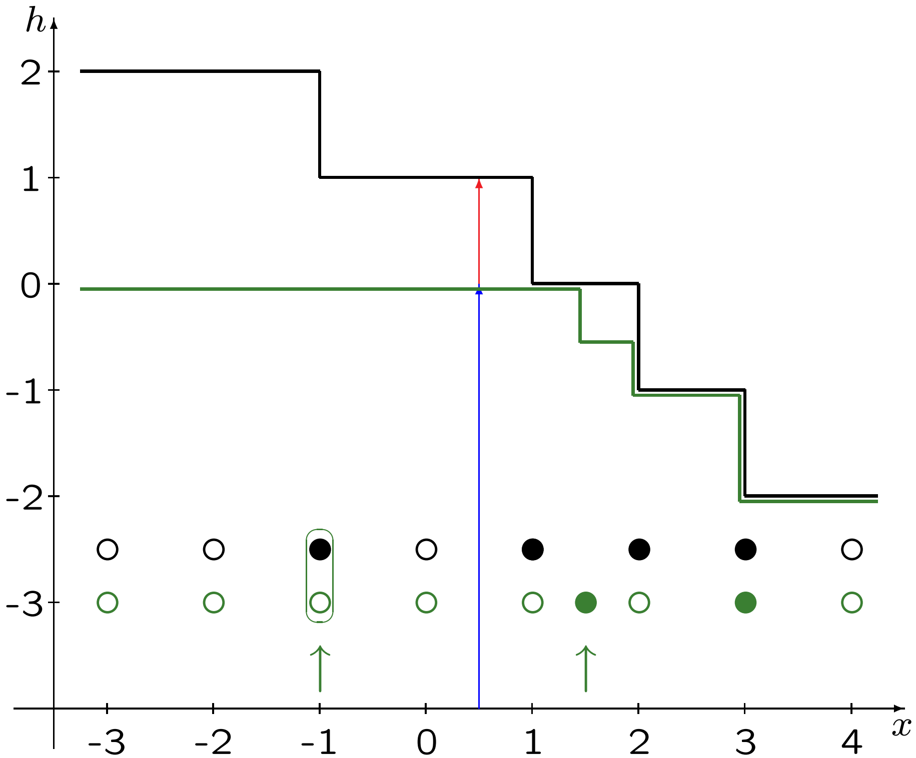

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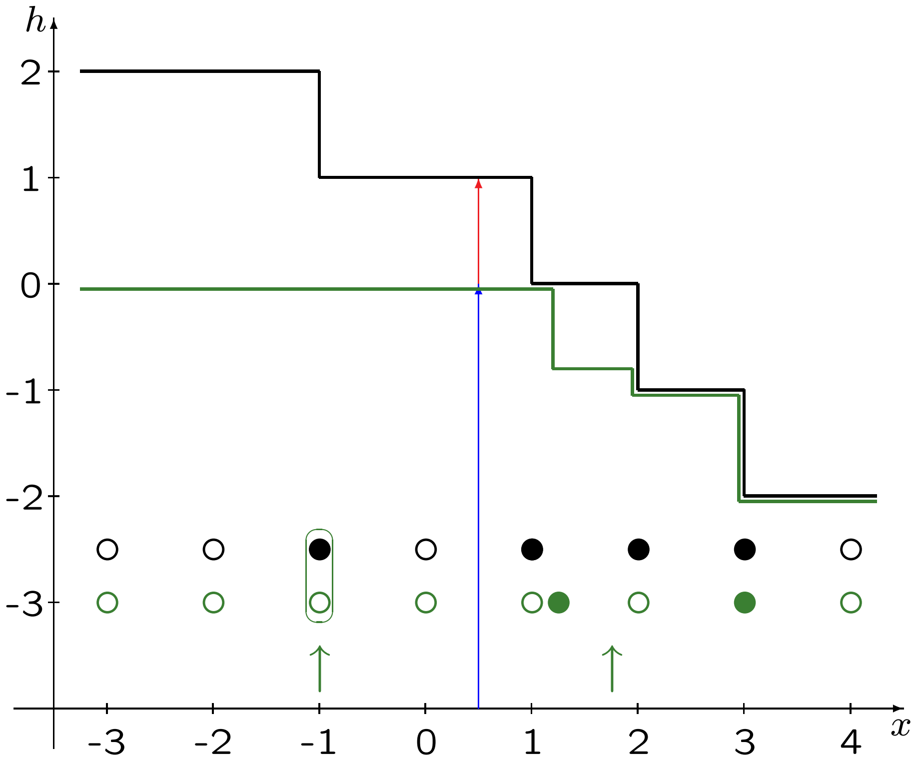

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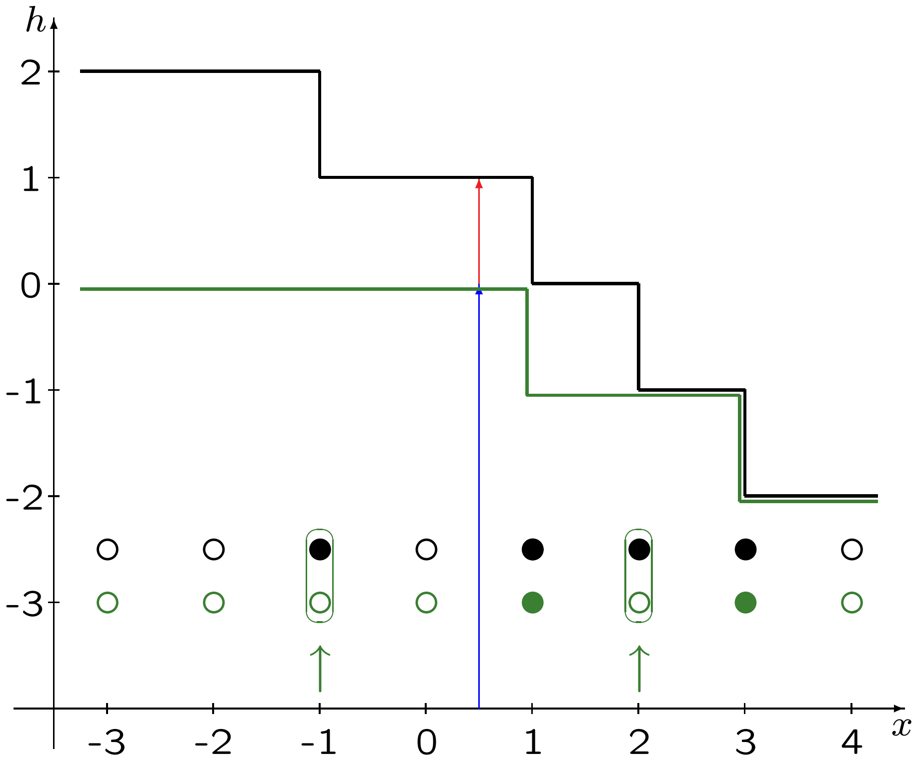

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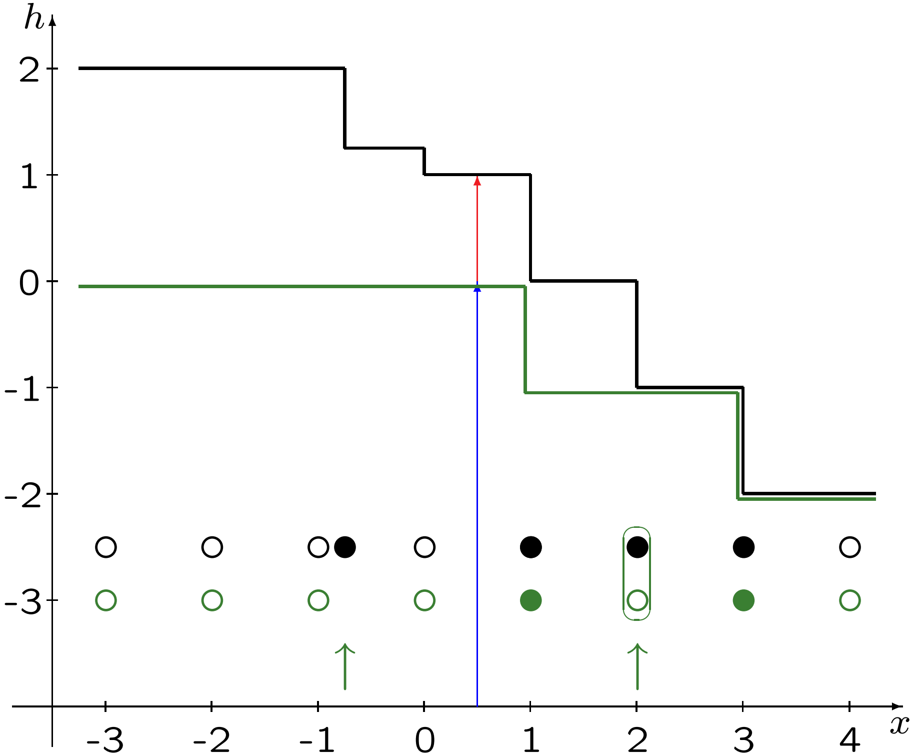

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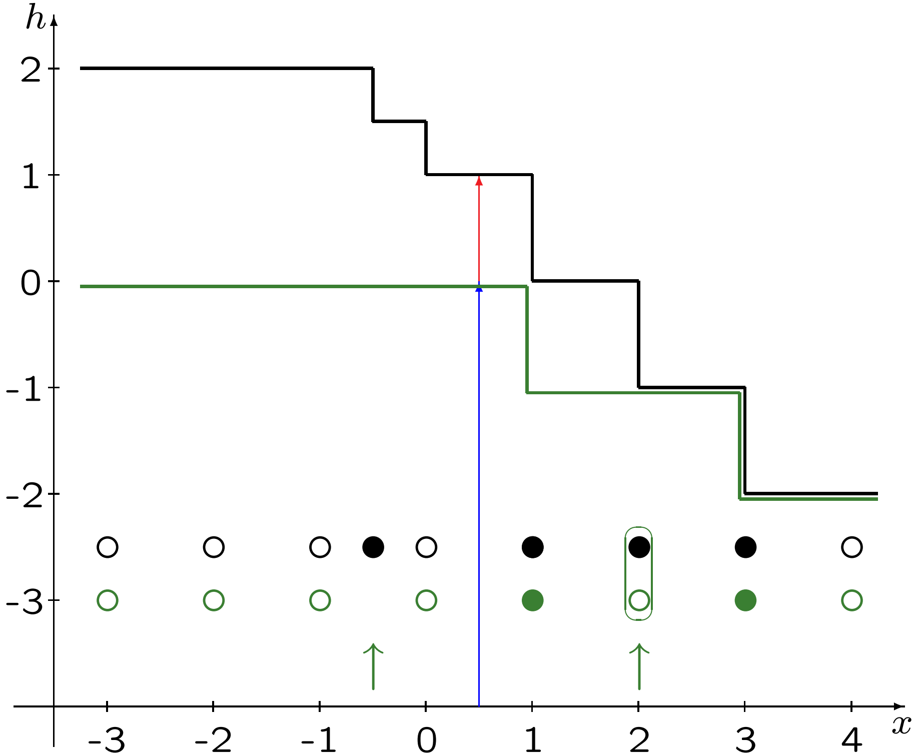

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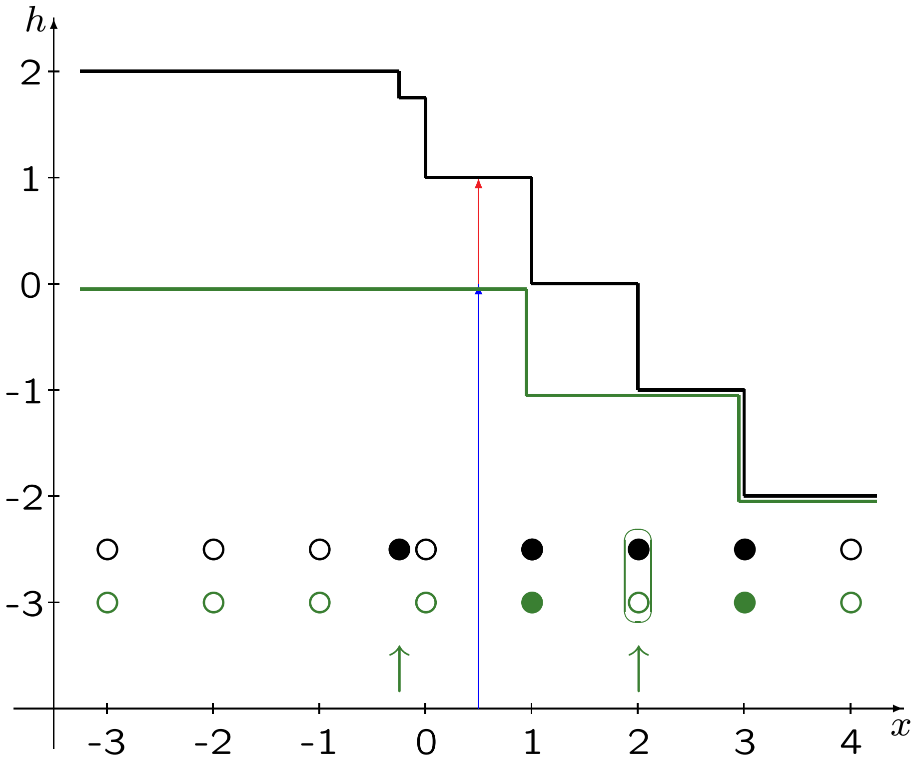

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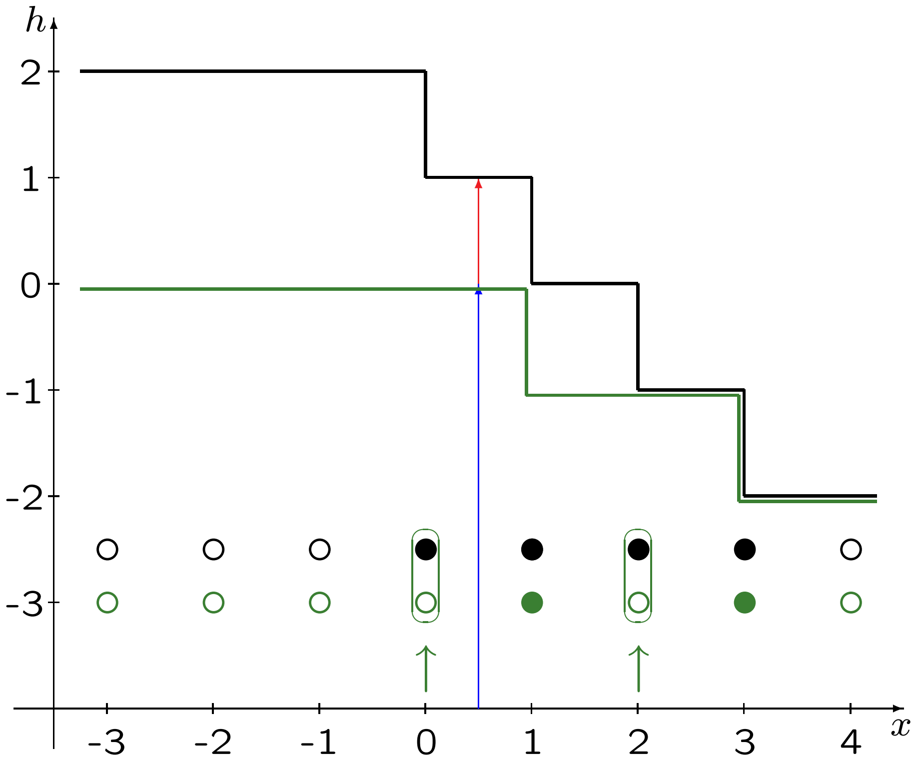

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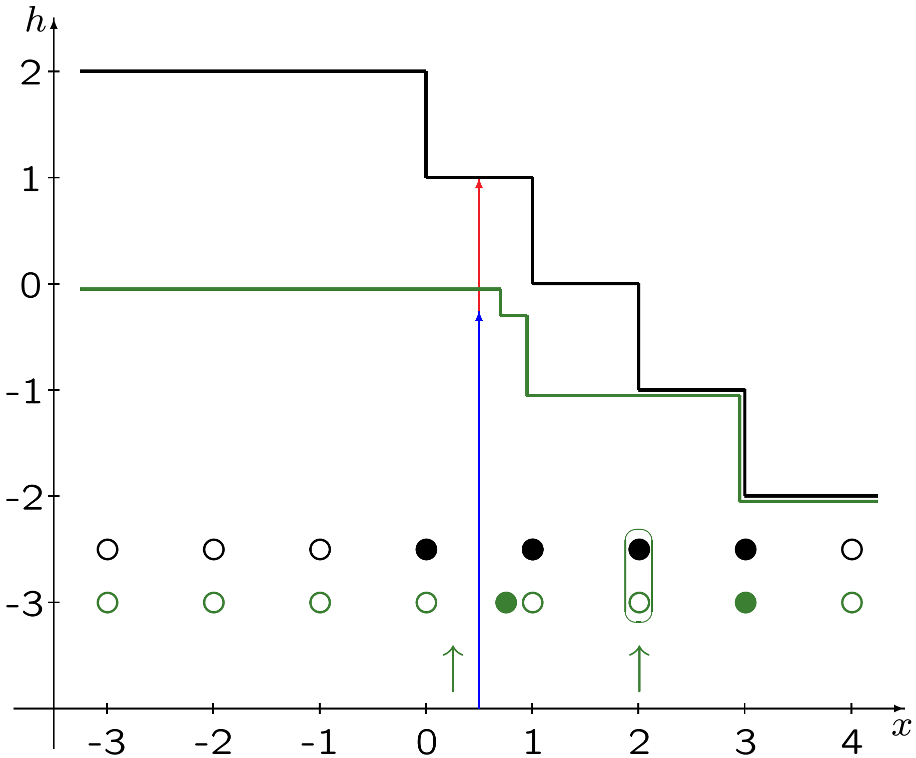

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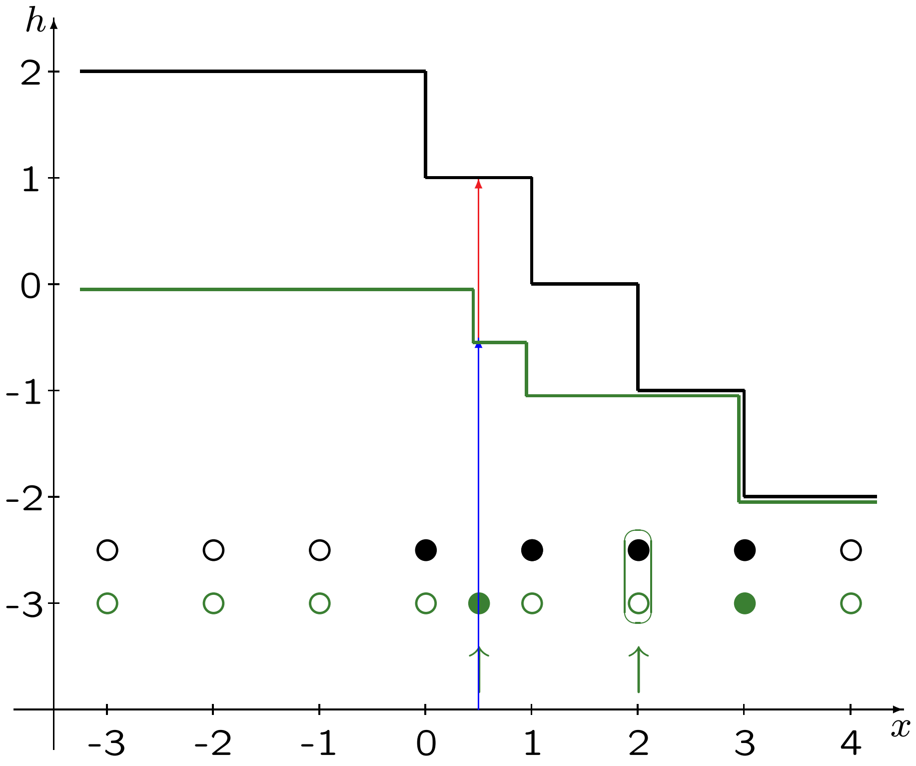

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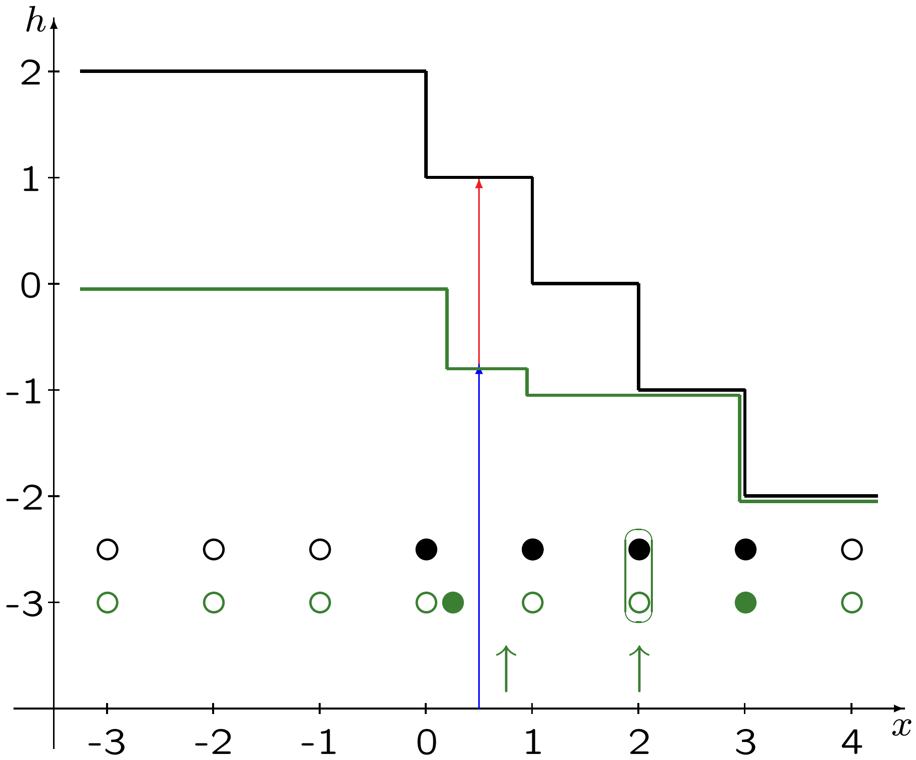

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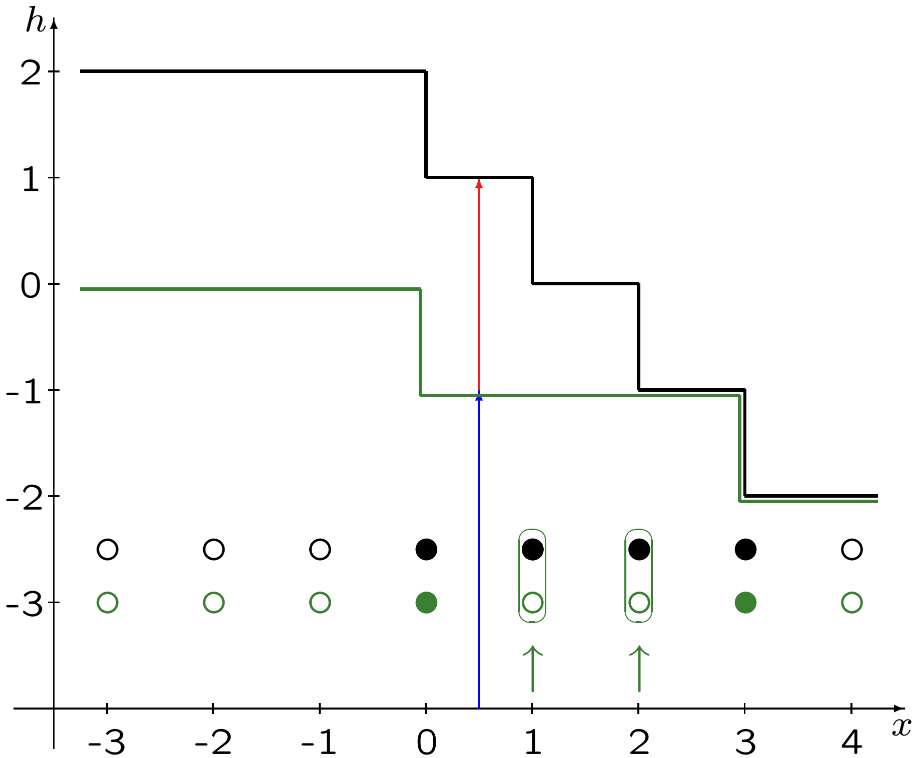

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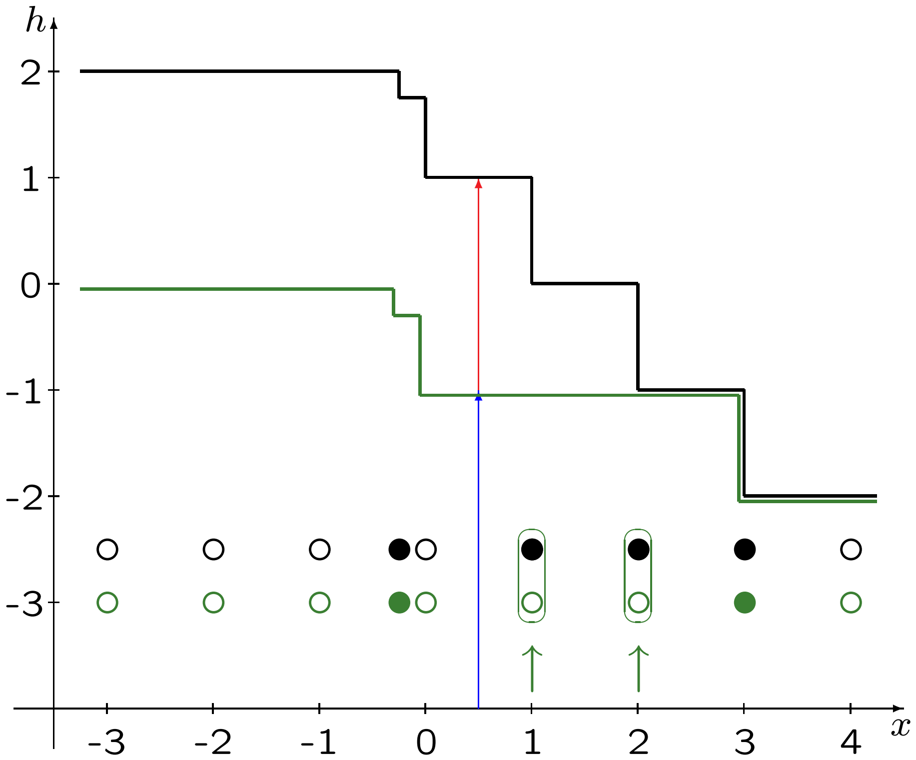

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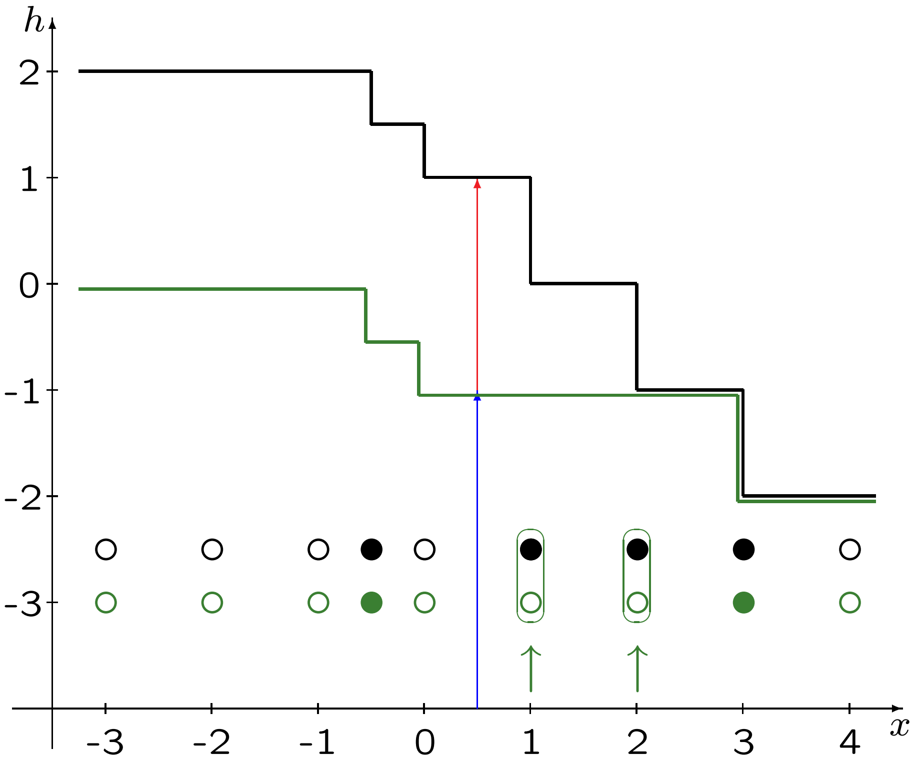

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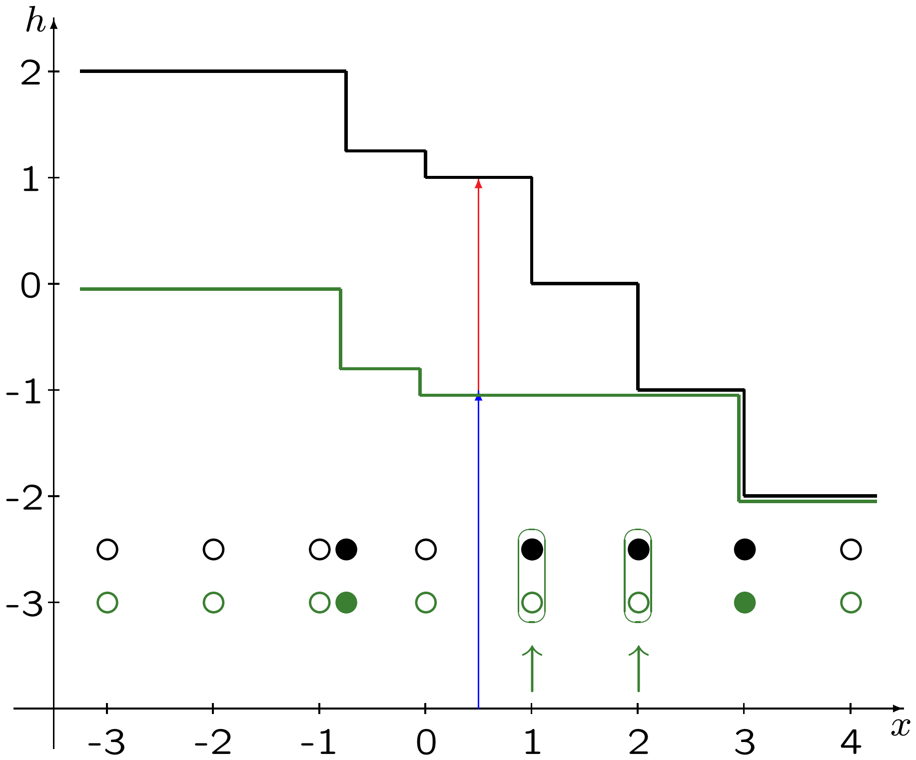

$h_{Vt}(t) - h_{Vt}(t)$ = the net number of $\uparrow$'s passed through the moving window at Vt $(V \in \mathbb{R})$.

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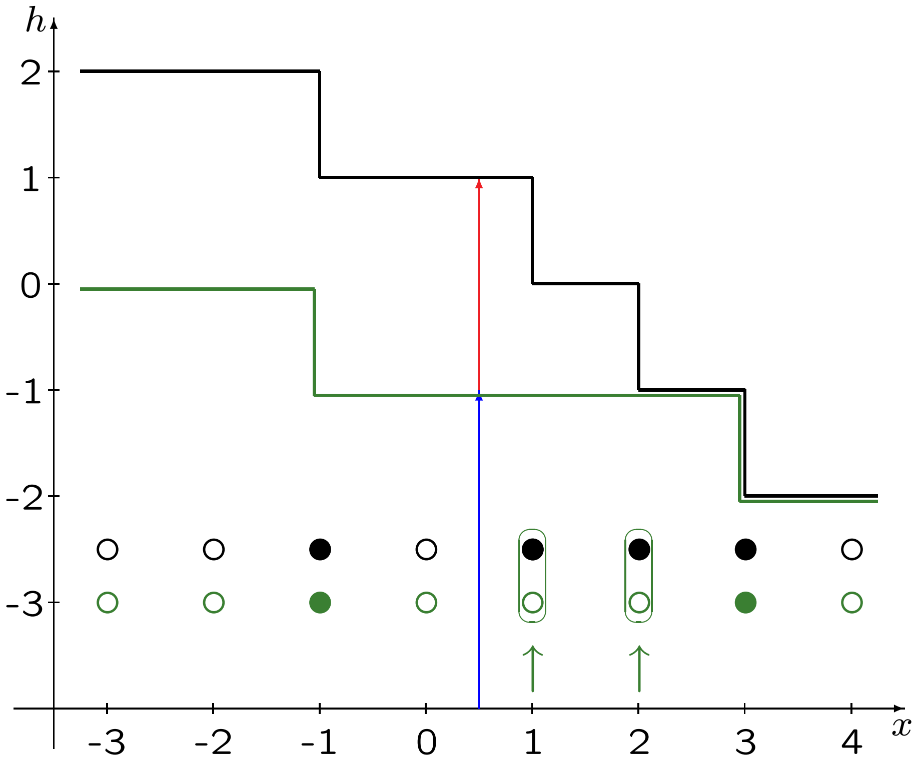

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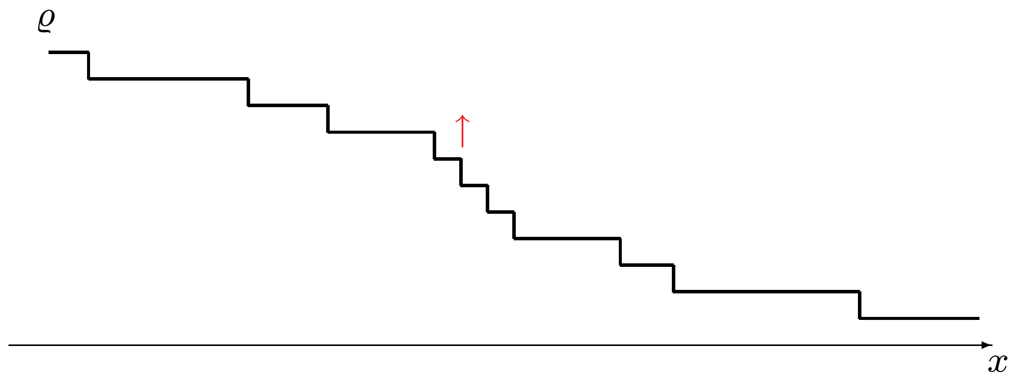
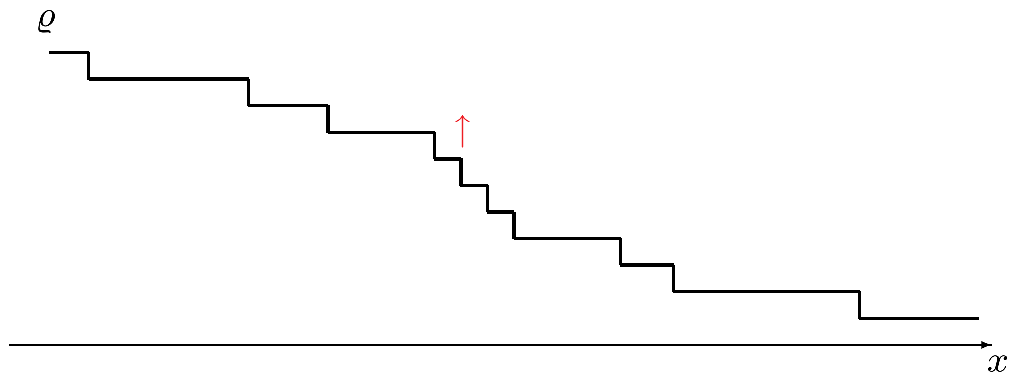
5. The upper bound

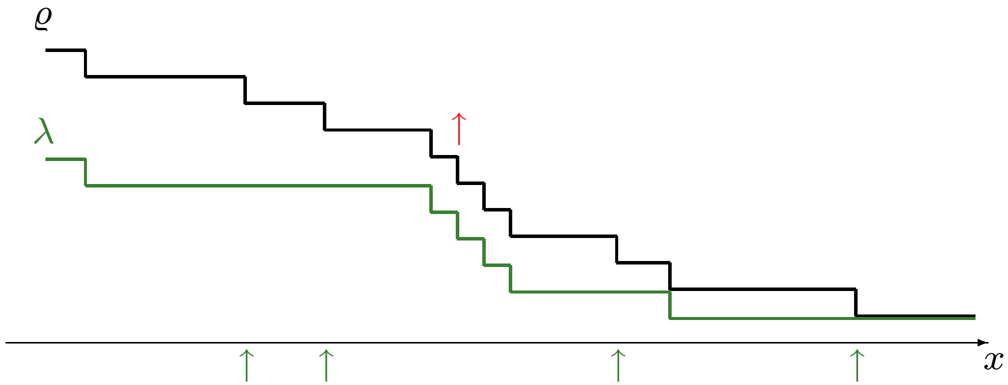


5. The upper bound



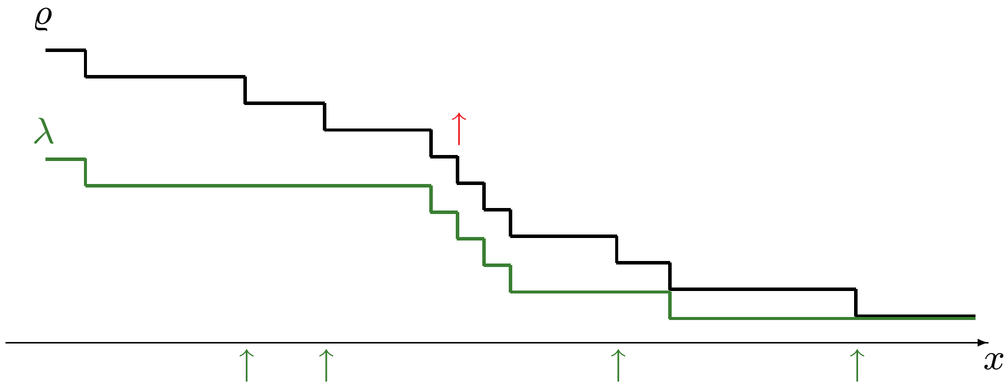
Connect $Q(t)$

5. The upper bound



Connect $Q(t)$ with the $\uparrow$'s

5. The upper bound



Connect $Q(t)$ with the $\uparrow$'s (this needs nontrivial couplings):

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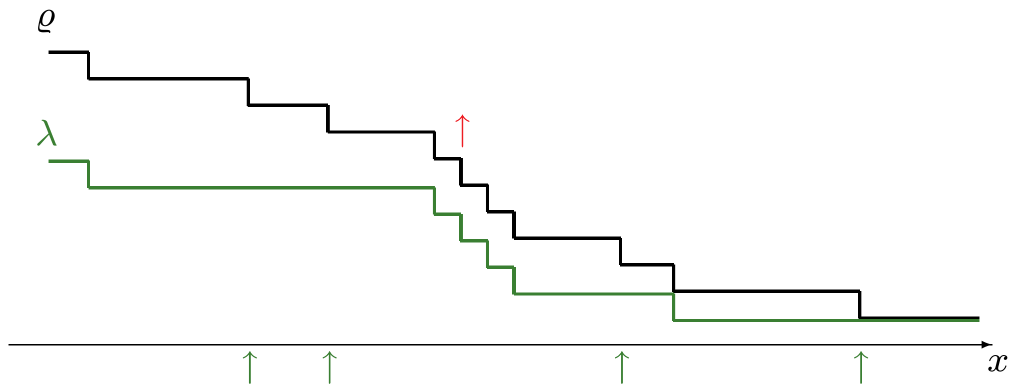

Connect $Q(t)$ with the $\uparrow$'s (this needs nontrivial couplings):

$$\mathbf{P}\{Q(t) \text{ is too large}\}$$

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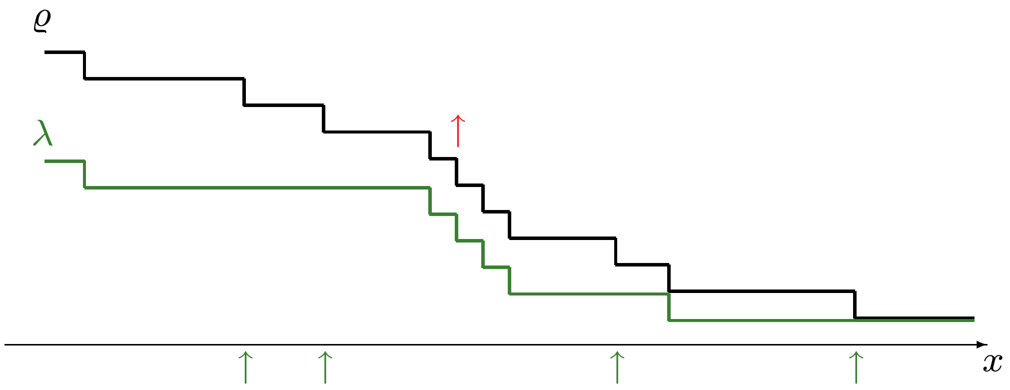

Connect $Q(t)$ with the $\uparrow$'s (this needs nontrivial couplings):

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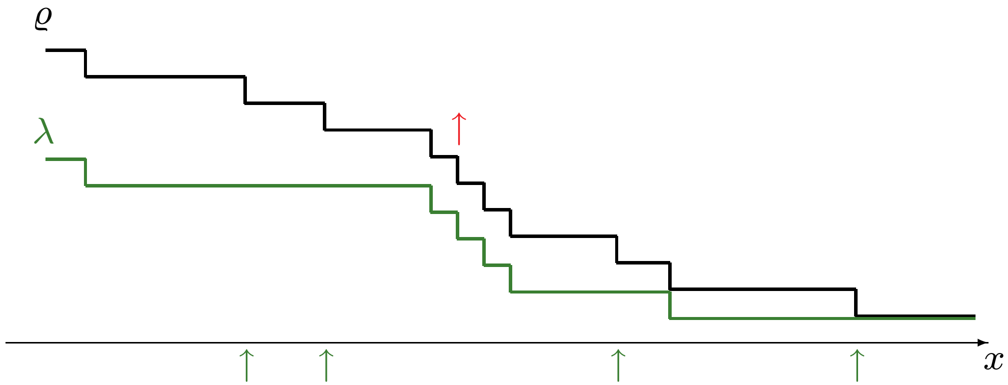

Connect $Q(t)$ with the $\uparrow$'s (this needs nontrivial couplings):

$$\begin{aligned}\mathbf{P}\{Q(t) \text{ is too large}\} &\le \mathbf{P}\{\text{too many } \uparrow\text{'s have crossed } C(\varrho)t\} \\ &\le \mathbf{P}\{h_{C(\varrho)t}(t) - h_{C(\varrho)t}(t) \text{ is too large}\}.\end{aligned}$$

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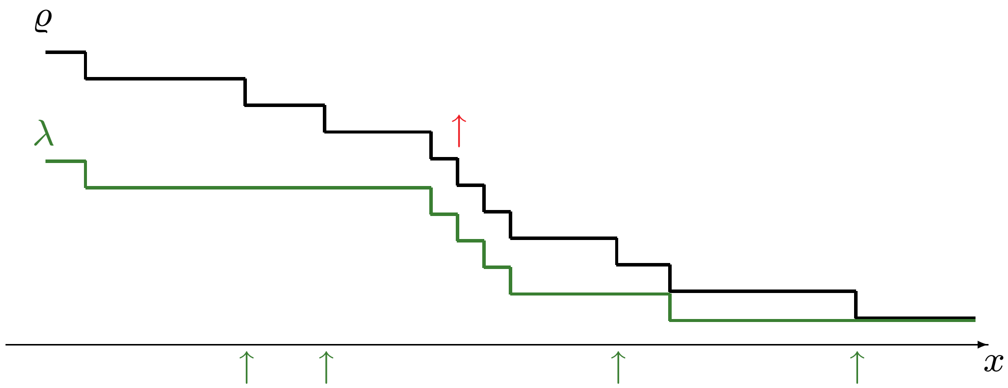

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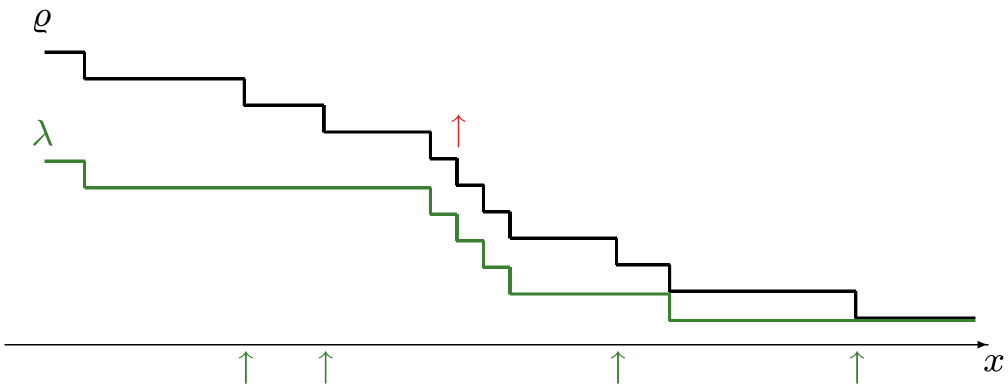

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Optimize "too large(λ)" in λ,

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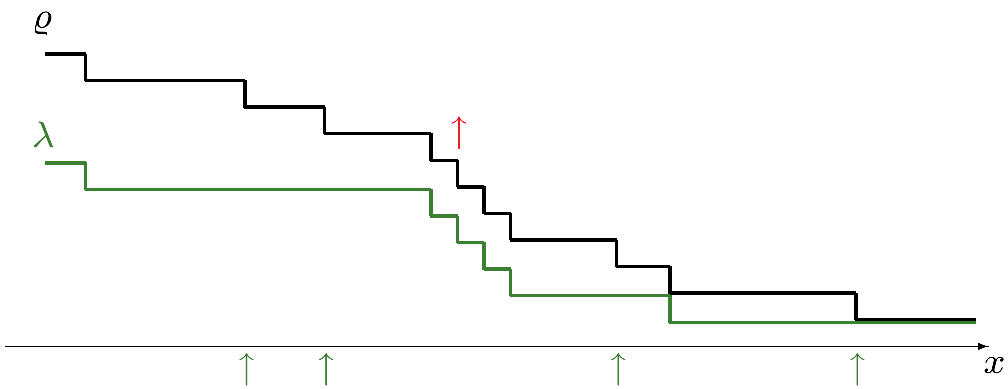

Connect $Q(t)$ with the $\uparrow$'s (this needs nontrivial couplings):

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$$\le \mathbf{P}\{h_{C(\varrho)t}(t) - h_{C(\varrho)t}(t) \text{ is too large}(\lambda)\}.$$

Optimize "too large(λ)" in λ, use Chebyshev's inequality and relate **Var**$(h_{C(\varrho)t}(t))$ to **Var**$(h_{C(\varrho)t}(t))$.

The computations result in

$$\mathbf{P}\{Q(t) - C(\varrho)t \geq u\} \quad \leq \quad c \cdot \frac{t^2}{u^4} \cdot \mathbf{Var}(h_{C(\varrho)t}(t))$$

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With

$$\widetilde{Q}(t) := Q(t) - C(\varrho)t \qquad \text{and} \qquad E := \mathbf{E}|\widetilde{Q}(t)|,$$

we have (with a similar lower deviation bound)

$$\mathbf{P}\{|\widetilde{Q}(t)| > u\} \leq c \cdot \frac{t^2}{u^4} \cdot E.$$

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Claim: this already implies the $t^{2/3}$ upper bound:

We had $\mathbf{P}\{|\widetilde{Q}(t)| > u\} \leq c \cdot \frac{t^2}{u^4} \cdot E.$

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&\leq E \int_{1/2}^\infty \mathbf{P}\{|\widetilde{Q}(t)| > vE\} \, \mathrm{d}v + \frac{1}{2}E
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that is, $E^3 \le c \cdot t^2$.

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$$\mathbf{Var}(h_{C(\varrho)t}(t)) \stackrel{\text{Thm}}{=} \text{const.} \cdot \mathbf{E}|Q(t) - C(\varrho)t|$$

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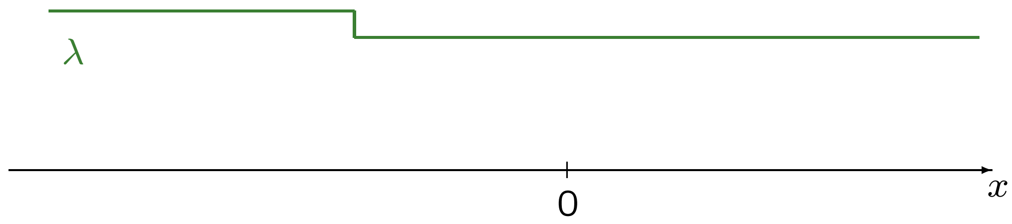
that is, $E^3 \leq c \cdot t^2$.

$$\begin{aligned} \mathbf{Var}(h_{C(\varrho)t}(t)) &\overset{\text{Thm}}{=} \text{const.} \cdot \mathbf{E}|Q(t) - C(\varrho)t| \\ &= \text{const.} \cdot E \leq c \cdot t^{2/3}. \qquad \blacksquare \end{aligned}$$

6. The lower bound



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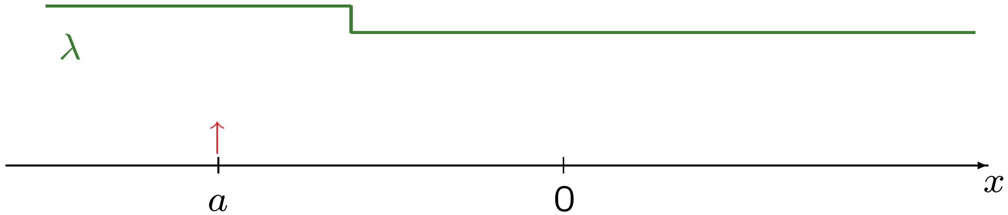

Let $Q^a(0) = a < 0$.

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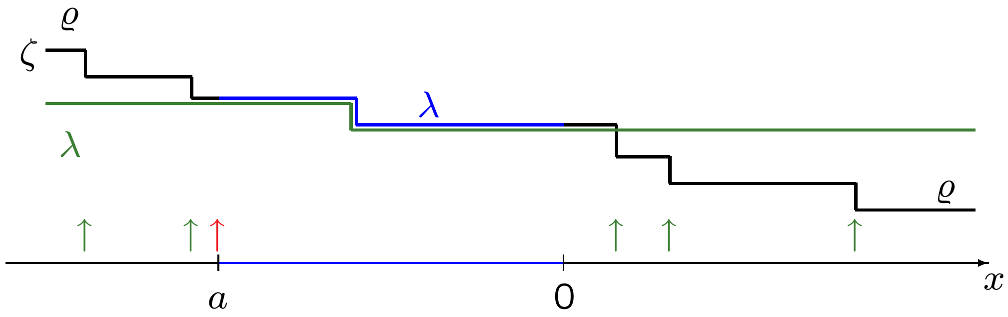

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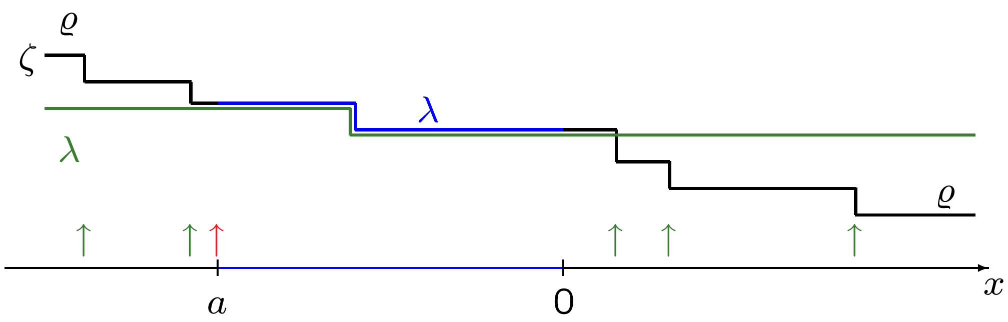

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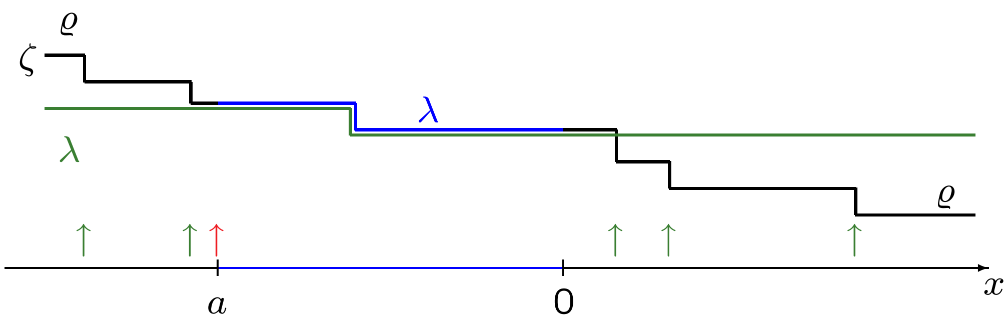

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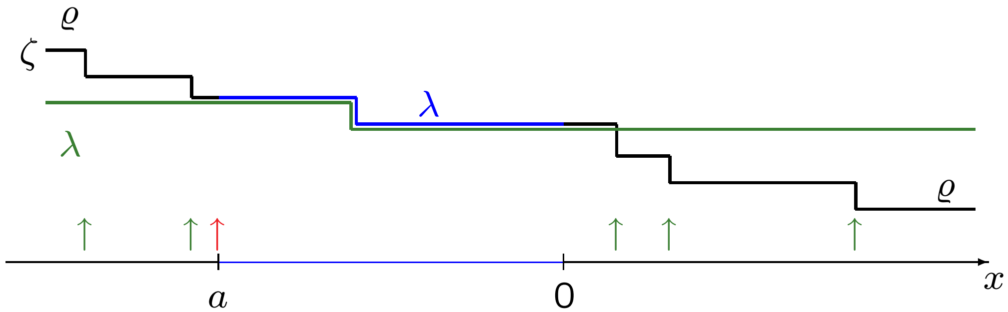
Therefore:

$$1 \le \mathbf{P}\{Q^a(t) > C(\varrho)t\} + \mathbf{P}\{h_{C(\varrho)t}(t) < h_{C(\varrho)t}(t)\}.$$



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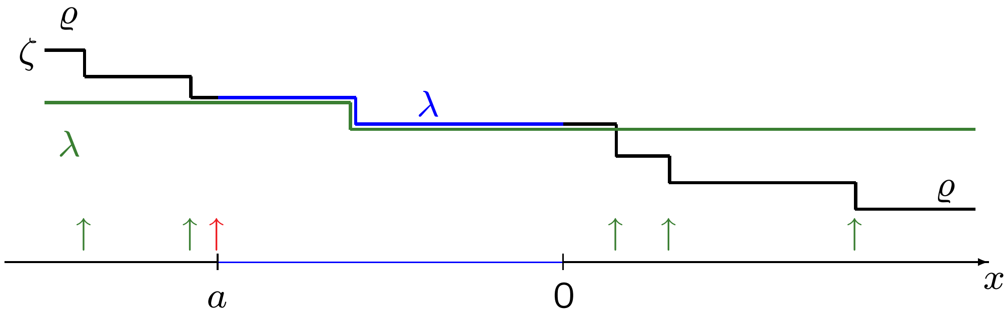

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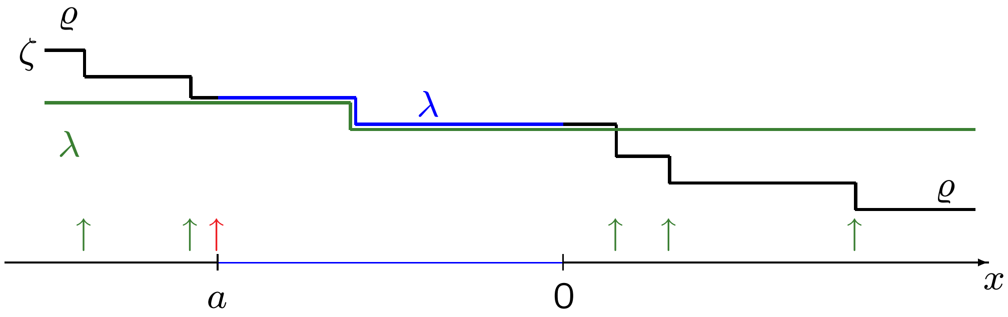

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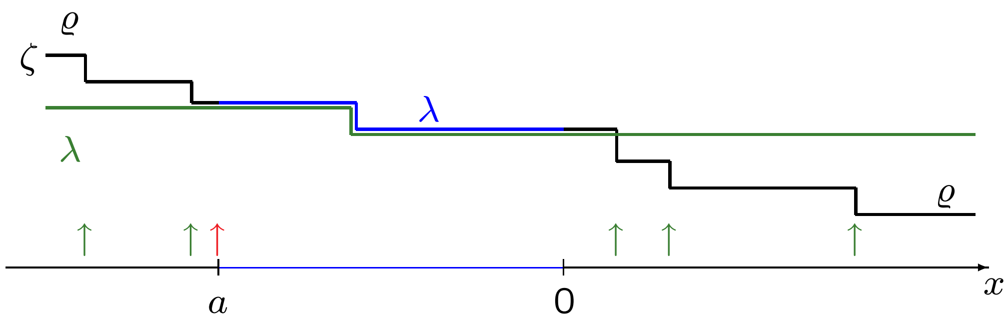

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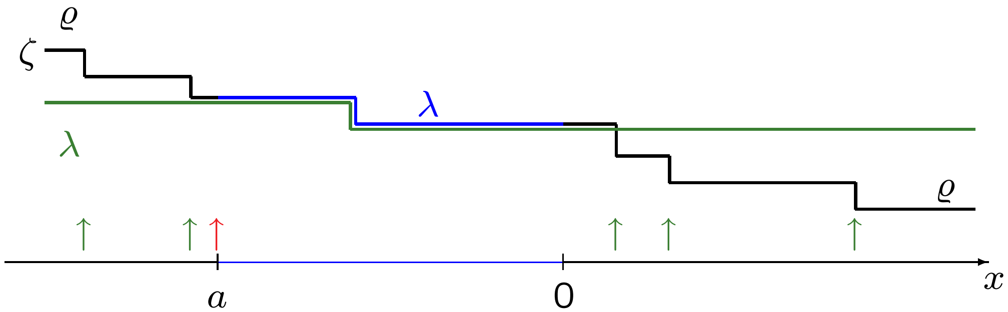

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$\Rightarrow$ Both probabilities are deviation probabilities.

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Apply Markov's inequality on the first, Chebyshev's on the second probability (use again the connection between **Var**$(h_{C(\varrho)t}(t))$ and **Var**$(h_{C(\varrho)t}(t))$).

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The correct scaling of the parameters is: $\varrho - \lambda \sim t^{-1/3}, \quad a \sim -t^{2/3}$.

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In this case

$$1 \;\; \le \;\; c_1 \cdot \frac{\mathbf{E}(|\widetilde{Q^a(t)}|)}{t^{2/3}} + c_2 \cdot \frac{\mathbf{Var}(h_{C(\varrho)t}(t))}{t^{2/3}}$$

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■

7. Open questions

$$\boxed{\mathbf{E}|\widetilde{Q}(t)|^1} \longleftrightarrow \boxed{\mathbf{E}|\widetilde{h}_{C(\varrho)t}(t)|^2}$$

7. Open questions

$$
\begin{tikzcd}
\boxed{\mathbf{E}|\widetilde{Q}(t)|^{2}} \arrow[d] & \\
\boxed{\mathbf{E}|\widetilde{Q}(t)|^{1}} \arrow[r, leftrightarrow] & \boxed{\mathbf{E}|\widetilde{h}_{C(\varrho)_t}(t)|^{2}}
\end{tikzcd}
$$

7. Open questions

$$
\begin{tikzcd}
\boxed{\mathbf{E}|\widetilde{Q}(t)|^{3-\varepsilon}} \arrow[d] & \\
\boxed{\mathbf{E}|\widetilde{Q}(t)|^{2}} \arrow[d] & \\
\boxed{\mathbf{E}|\widetilde{Q}(t)|^{1}} \arrow[r, leftrightarrow] & \boxed{\mathbf{E}|\widetilde{h}_{C(\varrho)t}(t)|^{2}}
\end{tikzcd}
$$

7. Open questions

$$\begin{tikzcd}
\mathbf{E}|\widetilde{Q}(t)|^4? \arrow[d, dotted] & \\
\boxed{\mathbf{E}|\widetilde{Q}(t)|^{3-\varepsilon}} \arrow[d] & \\
\boxed{\mathbf{E}|\widetilde{Q}(t)|^{2}} \arrow[d] & \\
\boxed{\mathbf{E}|\widetilde{Q}(t)|^{1}} \arrow[r, leftrightarrow] & \boxed{\mathbf{E}|\widetilde{h}_{C(\varrho)t}(t)|^2}
\end{tikzcd}$$

7. Open questions

$$
\begin{tikzcd}
\mathbf{E}|\widetilde{Q}(t)|^{4}? \arrow[r, dotted] \arrow[d, dotted] & \mathbf{E}|\widetilde{h}_{C(\varrho)t}(t)|^{5}? \arrow[d, dotted] \\
\boxed{\mathbf{E}|\widetilde{Q}(t)|^{3-\varepsilon}} \arrow[r, dotted] \arrow[d] & \mathbf{E}|\widetilde{h}_{C(\varrho)t}(t)|^{4}? \arrow[d, dotted] \\
\boxed{\mathbf{E}|\widetilde{Q}(t)|^{2}} \arrow[r, dotted] \arrow[d] & \mathbf{E}|\widetilde{h}_{C(\varrho)t}(t)|^{3}? \arrow[d, dotted] \\
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$\longrightarrow$ What is the limit $\lim_{t \to \infty} \frac{\textbf{Var}(h_{C(\varrho)t}(t))}{t^{2/3}} = ?$ What does it have to do with Gaussian random matrices? (Difficult.)

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$\longrightarrow$ Other processes (zero range, *Bricklayers'*, ...)?

$\longrightarrow$ Some processes (e.g. symmetric simple exclusion, linear rate zero range) show $t^{1/4}$ scaling (with Gaussian limits), rather than $t^{1/3}$. Where is the borderline? Are there other scalings as well?

Thank you.