$$\mathbb{E}X = \int_{-\infty}^{\infty} x f(x)\,dx$$

$$\mathbb{E}g(X) = \int_{-\infty}^{\infty} g(x) f(x)\,dx$$

Binomial R.V.

n indep. trials, success prob is p

X = no. of successes $\sim Binom(n,p)$

$$p(k) = \binom{n}{k} p^k (1-p)^{n-k}$$

$$\mathbb{E}X = np \qquad Var X = np(1-p)$$

$$\left.\begin{array}{l} n \to \infty \\ p \to 0 \end{array}\right\} n \cdot p \to \lambda > 0$$

$$Binom(n,p) \longrightarrow Poi(\lambda)$$

$$Y \sim Poi(\lambda)$$

$$p(i) = \mathbb{P}(Y = i) = \frac{\lambda^i}{i!} e^{-\lambda} \qquad i \geq 0$$

$\mathbb{E}X = \lambda \qquad \mathrm{Var}\,X = \lambda$

Geom(p)

$X \sim \mathrm{Geom}(p)$

= waiting time for the first success.

$\mathbb{P}(X = k) = p(k) = (1-p)^{k-1} \cdot p$

$\mathbb{P}(X > k) = (1-p)^{k}$

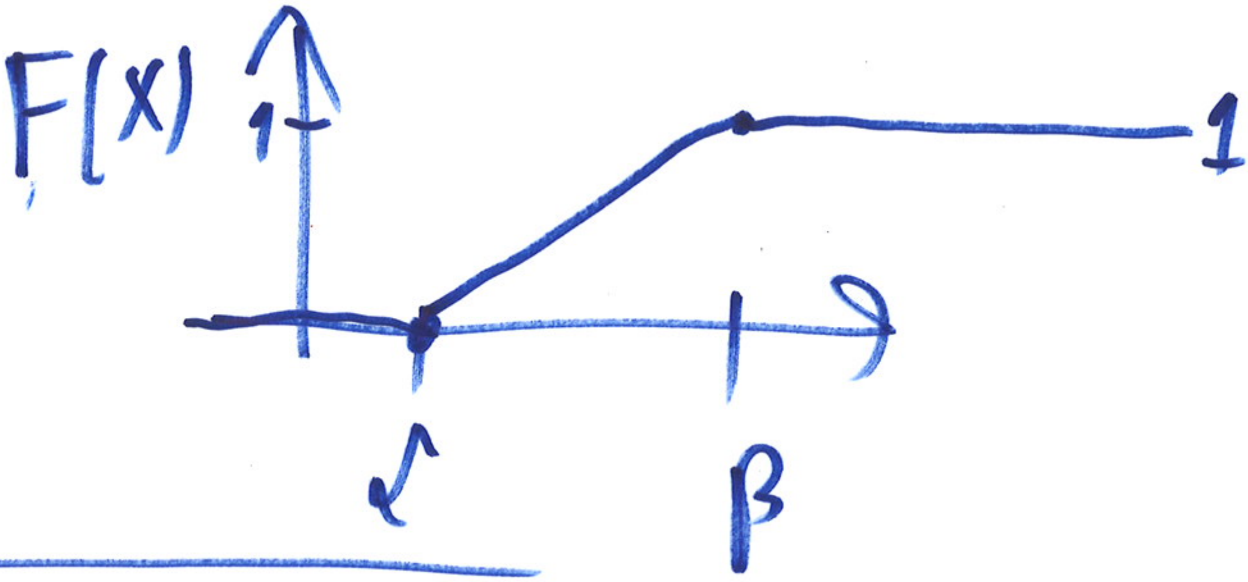
$\mathbb{E}X = \frac{1}{p} \qquad \mathrm{Var}\,X = \frac{1-p}{p^2}$

$X \sim \mathrm{Uniform}(\alpha, \beta)$

$$f(x) = \begin{cases} \frac{1}{\beta - \alpha} & \text{if } \alpha < x < \beta \\ 0 & \text{else} \end{cases}$$

$\mathbb{E}X = \frac{\alpha + \beta}{2} \qquad \mathrm{Var}\,X = \frac{(\beta - \alpha)^2}{12}$

$$\mathbb{P}(X \in B) = \frac{|B|}{\beta - \alpha} \qquad B \subseteq (\alpha, \beta)$$



$$X \sim \mathcal{N}(\mu, \sigma^2)$$

$$f(x) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \qquad \forall x$$

$$\mathbb{E}X = \mu \qquad \mathrm{var}\, X = \sigma^2$$

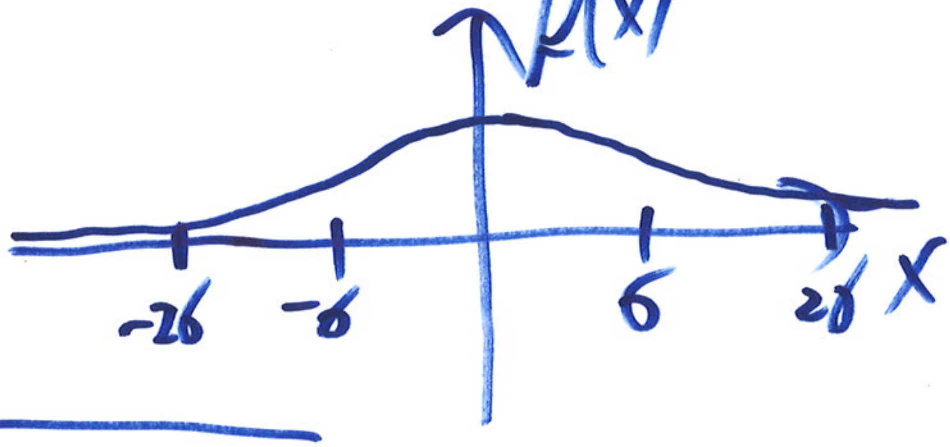
$$aX + b \sim \mathcal{N}(a\mu + b, a^2\sigma^2)$$

$$\frac{X - \mu}{\sigma} \sim \mathcal{N}(0, 1) \quad \text{STANDARD}$$

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$$

$$\Phi(x) = \int_{-\infty}^{x} f(y)\,dy = \mathbb{P}(X \le x)$$

$$\Phi(-z) = 1 - \Phi(z)$$



$X \sim \mathrm{Bin}(n, p)$ $\quad n \to \infty$

p FIXED

$$\frac{X - np}{\sqrt{np(1-p)}} \approx \mathcal{N}(0,1) \qquad \text{De Moivre-Laplace.}$$

$$\mathbb{P}\Big(\underbrace{\frac{X - np}{\sqrt{np(1-p)}}} \le a\Big) \approx \Phi(a)$$

$\mathrm{Exp}(\lambda)$: $f(x) = \lambda e^{-\lambda x}$ $\quad x > 0$

WAITING TIME

$$F(x) = 1 - e^{-\lambda x}$$

Memoryless: $\mathbb{P}(X > s+t \mid X > s) = \mathbb{P}(X > t) = e^{-\lambda t}$

$\mathbb{E}X = \frac{1}{\lambda}$ $\quad$ $\mathrm{Var}\, X = \frac{1}{\lambda^2}$

λ = rate

Gamma(α, λ) $\quad$ $\alpha > 0$

$\alpha \to$ SHAPE, $\lambda \to$ rate $\quad$ $\lambda > 0$

$$f(x) = \frac{\lambda^{\alpha} x^{\alpha-1} e^{-\lambda x}}{\Gamma(\alpha)} \qquad x > 0$$

$$\Gamma(\alpha) = \int_0^{\infty} z^{\alpha-1} e^{-z}\, dz \qquad \alpha > 0$$

$$\boxed{\alpha \Gamma(\alpha) = \Gamma(\alpha+1)}$$

$$\Gamma(n) = (n-1)!$$

$f(x)$ density

$\mathbb{P}(X \in B) = \int_B f(x)\, dx$

$\Gamma(1) = 1$

$0! = 1$

$\Gamma(2) = 1 \cdot \Gamma(1) = 1$ $\quad$ $1! = 1$

$\Gamma(3) = 2\Gamma(2) = 2$ $\quad$ $2! = 2$

$\Gamma(4) = 3\Gamma(3) = 3 \cdot 2$ $\quad$ $3! = 3 \cdot 2$

$\vdots$ $\quad$ $\vdots$

$$\mathbb{E}X = \frac{\alpha}{\lambda}\ ; \ \mathrm{Var}\, X = \frac{\alpha}{\lambda^2}$$

2/6

Transformations:

X is continuous r.v. $\rightarrow f_X(x)$

$g(X) = Y$

$f_Y(y) = ?$

$$F_Y(y) = \mathbb{P}(Y \le y) = \mathbb{P}(\underbrace{g(X) \le y}_{\text{SOLVE}})$$

if g is increasing

$$= \mathbb{P}(X \le g^{-1}(y)) = F_X(g^{-1}(y))$$

d/dy

$$f_Y(y) = f_X(g^{-1}(y)) \cdot (g^{-1}(y))'$$

---

5 $\cancel{F(a,b)}$ X, Y discrete

$$p(x,y) = \mathbb{P}(X = x, Y = y) \ge 0$$

$$\sum_{i,j} p(x_i, y_j) = 1$$

MARGINALS

$$\mathbb{P}(X=x) = p_X(x) = \sum_j p(x, y_j)$$

$$\mathbb{P}(Y=y) = p_Y(y) = \sum_i p(x_i, y)$$

$$F(x) = \mathbb{P}(X \le x)$$

$f(x)$ = density

$$= \frac{d}{dx} F(x)$$

$$F(x) = \int_{-\infty}^{x} f(y)\,dy$$

Independence:

$$p(x,y) = p_X(x) \cdot p_Y(y)$$

Convolutions:

X, Y are indep., integer.

$$\mathbb{P}(X+Y=i) = p_{X+Y}(i) =$$

$$= \sum_{k=-\infty}^{\infty} p_X(i-k) \cdot p_Y(k)$$

CONVOLUTION of p_X and p_Y

$X * Y$

$$p_{X+Y} = p_X * p_Y$$

$$f_{X+Y}(x) = \int_{-\infty}^{\infty} f_X(x-y) \cdot f_Y(y)\, dy$$

$$\text{Gamma}(2,\lambda) = \text{Exp}(\lambda) * \text{Exp}(\lambda)$$

$$\text{Gamma}(\alpha+1,\lambda) = \text{Gamma}(\alpha,\lambda) * \text{Exp}(\lambda)$$

$$\text{Binom}(n,\underline{\underline{p}}) * \text{Binom}(m,\underline{\underline{p}}) = \text{Binom}(n+m,\underline{\underline{p}})$$

$$\text{Poi}(\lambda_1) * \text{Poi}(\lambda_2) = \text{Poi}(\lambda_1+\lambda_2)$$

$$\sum_{k=-\infty}^{\infty} P_{\text{Poi}(\lambda_1)}(i-k) \cdot P_{\text{Poi}(\lambda_2)}(k) = \sum_{k=0}^{i} \frac{\lambda_1^{i-k}}{(i-k)!} e^{-\lambda_1} \cdot \frac{\lambda_2^{k}}{k!} e^{-\lambda_2} = \ldots = \frac{(\lambda_1+\lambda_2)^i}{i!} \cdot e^{-(\lambda_1+\lambda_2)}$$

$$\mathcal{N}(\mu_1,\sigma_1^2) * \mathcal{N}(\mu_2,\sigma_2^2) = \mathcal{N}(\mu_1+\mu_2, \sigma_1^2+\sigma_2^2)$$

Conditional:

$$P_{X|Y}(x|y_j) = \mathbb{P}(X = x \mid Y = y_j)$$

$$= \frac{\mathbb{P}(X = x, Y = y_j)}{\mathbb{P}(Y = y_j)} = \frac{p(x, y_j)}{p_Y(y_j)}$$

6 Expectations / variance / covariance

$$\mathbb{E}\, g(X, Y)\ \cancel{g(X,Y)} = \sum_{i,j} p(x_i, y_j)\, g(x_i, y_j)$$

$$\mathbb{E}(X + Y) = \mathbb{E}X + \mathbb{E}Y$$

If $X \leq Y \Rightarrow \mathbb{E}X \leq \mathbb{E}Y$

If indep. then $\mathbb{E}(XY) = \mathbb{E}X \cdot \mathbb{E}Y$

$$\mathbb{E}(g(X) \cdot h(Y)) = \mathbb{E}g(X)\, \mathbb{E}h(Y)$$

$$\mathrm{Cov}(X, Y) = \mathbb{E}[(X - \mathbb{E}X)(Y - \mathbb{E}Y)] = \mathbb{E}XY - \mathbb{E}X \cdot \mathbb{E}Y$$

Indep $\Rightarrow$ $\mathrm{Cov}(X,Y)=0$

$$\mathrm{Cov}(X,X)=\mathrm{Var}\,X$$

$$\mathrm{Cov}(X,Y)=\mathrm{Cov}(Y,X)$$

$$\mathrm{Cov}\Big(\sum_i a_i X_i + b,\ \sum_j c_j Y_j + d\Big) =$$

$$= \sum_{i,j} a_i c_j \,\mathrm{Cov}(X_i, Y_j)$$

$$\mathrm{Var}\sum_i X_i = \sum_i \mathrm{Var}\,X_i + 2\sum_{i<j}\mathrm{Cov}(X_i, X_j)$$