Ongoing simplifications for the lace expansion

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Weakly self-avoiding random walks on $\mathbb{Z}^d$: $\omega = (\omega_0 = 0, \ldots, \omega_n)$ a path in $\mathbb{Z}^d$ (or $\mathbb{R}^d$), not necessarily nearest neighbor. p a one-step distribution on $\mathbb{Z}^d$.

$$P_n^{\mathsf{RW}}(\omega) := \prod_{i=1}^{n} p(\omega_i - \omega_{i-1})$$

$$Q_{n,\beta}^{\mathsf{SAW}}(\omega) := P_n^{\mathsf{RW}}(\omega) \prod_{0 \le i < j \le N} \left(1 - \beta 1_{\omega_i = \omega_j}\right),\ 0 < \beta \le 1.$$

$$C_{n,\beta}^{\mathsf{SAW}}(x) := Q_{n,\beta}^{\mathsf{SAW}}(\omega_n = x),\ x \in \mathbb{Z}^d.$$

$$P_{n,\beta}^{\mathsf{SAW}}(\omega) := \frac{1}{c_n} Q_{n,\beta}^{\mathsf{SAW}}(\omega),$$

where $c_n := \sum_\omega Q_{n,\beta}^{\mathsf{SAW}}(\omega) = \sum_x C_{n,\beta}^{\mathsf{SAW}}(x)$.

The **lace expansion** (Brydges, Spencer): Expand the product:

$$\prod_{0\le i<j\le n}\left(1-\beta\mathbf{1}_{\omega_i=\omega_j}\right)=\sum_{\Gamma\in\mathcal{P}\left(\mathcal{B}^{(n)}\right)}U_\Gamma(\omega).$$

Γ runs over $\mathcal{P}\left(\mathcal{B}^{(n)}\right)$, the set of all subsets of $\mathcal{B}^{(n)}:=\{(i,j),\ 0\le i<j\le n\}$, and

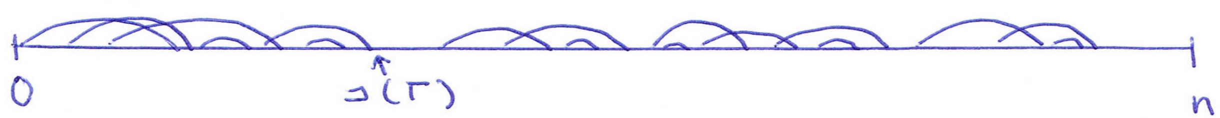
$$U_\Gamma(\omega):=\prod_{(i,j)\in\Gamma}\left(-\beta\mathbf{1}_{\omega_i=\omega_j}\right)$$

Accordingly

$$C_{n,\beta}^{\mathsf{SAW}}(x)=\sum_{\Gamma\in\mathcal{P}\left(\mathcal{B}^{(n)}\right)}\sum_{\omega:0\to x,\ |\omega|=n}P_n^{\mathsf{RW}}(\omega)\,U_\Gamma(\omega)$$

Split the set of graphs: $\mathcal{P}\left(\mathcal{B}^{(n)}\right) = \bigcup_{k=0}^{n} \mathcal{G}_k^{(n)}$, where $\mathcal{G}_0^{(n)} := \{\Gamma : i \geq 1,\ \forall (i,j) \in \Gamma\}$, $\mathcal{G}_k^{(n)} := \{\Gamma : s(\Gamma) = k\}$ for $k \geq 1$, where

$$s(\Gamma) \stackrel{\text{def}}{=} \min\{k : \nexists\, (i,j) \in \Gamma \text{ with } i < k < j\}.$$



By an abuse, call $\mathcal{C}_k \stackrel{\text{def}}{=} \mathcal{G}_k^{(k)}$ the set of *connected* graphs on $\{0, \ldots, k\}$. Putting

$$\Pi_k(x) = \sum_{\Gamma \in \mathcal{C}_k} \sum_{\omega : 0 \to k,\ |\omega| = k} P_k^{\mathrm{RW}}(\omega)\, U_\Gamma(\omega)$$

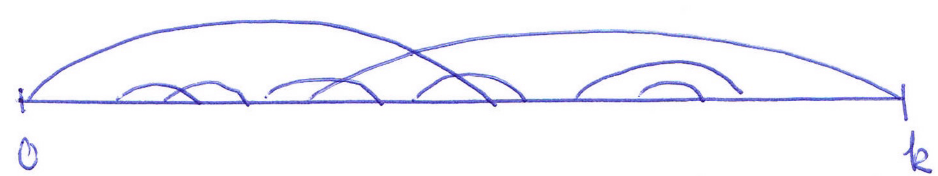
one gets by resummation after the split time:

$$C_n = p * C_{n-1} + \sum_{k=1}^{n} \Pi_k * C_{n-k}.$$

There is a further splitting of Π_k according to $m \stackrel{\text{def}}{=} |\Gamma_{\text{min}}|$, where for $\Gamma \in \mathcal{C}_k$, $\Gamma_{\text{min}} \subset \Gamma$ is a minimal subset which is still connected (Brydges and Spencer)

$$\Pi_k = \sum_{m \geq 1} \Pi_k^{(m)}.$$

Here is an example with $m = 2$:



In $\mathbb{R}^d$ (instead of $\mathbb{Z}^d$), p is a probability density, and $\mathbf{1}_{|\omega_i - \omega_j| \leq \delta}$ some δ instead of $\mathbf{1}_{\omega_i = \omega_j}$.

How "small" can the second part be not to destroy the CLT behavior. This is the wrong question: The Π_k have in leading order the same decay as the C_k. The smallness has to be expressed relative to the C's. Set

$$c_n := \sum_x C_n(x)$$

Put $\Pi_k = \beta c_k B_k$. Consider the equation for $\mathbf{C} = \{C_n\}_{n \geq 0}$, $C_0 = \delta_0$ as a function of the input sequence $\mathbf{B} = \{B_k\}_{k \geq 1}$

$$C_n = p * C_{n-1} + \beta \sum_{k=1}^{n} c_k B_k * C_{n-k},\ n \geq 1,$$

and ask: How "small" has $\mathbf{B}$ to be to get a CLT for the solution $\mathbf{C}$ of the above quadratic equation. This question is *independent* of SAWs.

Avena, B., Ritzmann (Ann. Prob., to appear): In $\mathbb{R}^d$, with $p = \phi$, the standard normal. More general rotational symmetric p need only easy modifications. (Asymmetric p are more delicate.) The B_k are assumed rotational symmetric.

Earlier: Unpublished thesis of Christine Ritzmann: Taylored for the SAW.

For SAW (or other models): $\mathbf{B}^{\mathsf{SAW}}$ has to satisfy the conditions on "smallness". This is a rather easy "circular" argument, provided one proves the right theorem about the convolution equations.

The assumptions on $\mathbf{B}$ depend on the form in which one wants a CLT in the end.

In ABR: Ass $|B_n| \leq \Gamma_n$, for suitable Γ_n. A simple example is $\Gamma_n = n^{-a}\phi_{[n/2]},\ a > 2$. This is however not appropriate for SAW: B_n^{SAW} do not satisfy $\left|B_n^{\mathsf{SAW}}\right| \leq Kn^{-a}\phi_{[n/2]}$. The one which works for SAW is

$$\Gamma_n = \text{const} \times n^{-d/2} \sum_{k=1}^{[n/2]} k^{1-d/2}\phi_k,\ d \geq 5.$$

Probably, for applications to other models (which we haven't done), other choices have to be made.

Conditions in ABR on Γ_n :

- $\Gamma_m * \Gamma_n \leq \chi_{m+n}(m)\, \Gamma_{m+n}$, where $\sup_n \sum_{s=1}^{[n/2]} s\chi_n(s) < \infty$.

- $\Gamma_s \leq \text{const} \times \Gamma_{2t}$ for $t \leq s \leq 2t$.

- With $\gamma^{(k)}(m) \overset{\text{def}}{=} \int |y|^{2k}\, \Gamma_m(y)\, dy$, $\sum_n n^{1-k}\gamma^{(k)}(n) < \infty, \ k = 0, 1, 2$.

- $$\int \phi_t(x-y)\, |y|^{2k}\, \Gamma_m(dy) \leq \text{const} \times \gamma^{(k)}(m)\, \phi_{t+m}(y)\,, \ k = 0, 1, 2.$$

Theorem In $\mathbb{R}^d$ with $p = \phi$. There exists $\delta = \delta(\mathbf{B}, \beta)$, $\delta \to 1$ for $\beta \to 0$, such that for $\varepsilon > 0$, β small enough

$$\left|\frac{C_n}{c_n} - \phi_{n\delta}\right| \le K(\varepsilon)\,\beta\left[\bar{\zeta}(n)\,\phi_{n(1+\varepsilon)} + \sum_{s=1}^{[n/2]} s\phi_{s(1+\varepsilon)} * \Gamma_{n-s}\right],$$

with some sequence $\bar{\zeta}(n) \to 0$ depending on the exact conditions.

For instance with $\Gamma_n = n^{-a}\phi_{[n/2]}$, one has $\bar{\zeta}(n) = n^{-a+2}$ for $a < 3$, $n^{-1}\log n$ for $a = 4$, and n^{-1} for $a > 5$. So in this case, one gets

$$\left|\frac{C_n}{c_n} - \phi_{n\delta}\right| \le K(\varepsilon)\,\bar{\zeta}(n)\,\phi_{n(1+\varepsilon)}.$$

Outline: It is not difficult to prove that c_n is in leading order exponential: $c_n = \mu^n a_n$, and under the above conditions $a_n \to a > 0$.

Shift of the variance: Plug in an ansatz $\mathsf{cov}\left(C_n/c_n\right) \approx \delta n I_d \Longrightarrow$

$$\delta = \frac{\mu^{-1} + \beta \sum_m a_m \bar{b}_m}{\mu^{-1} + \beta \sum_m m a_m b_m},$$

where

$$b_m := \int B_m\left(x\right), \ \int x^T x B_m\left(x\right) = \bar{b}_m I_d.$$

Key: Define an operator $\boldsymbol{\Psi}$ on sequences $\mathbf{G} = \left(G_n\right)_{n \geq 0}$ of densities by $\boldsymbol{\Psi}\left(\mathbf{G}\right)_0 = G_0$, and for $n \geq 1$

$$\Psi\left(\mathbf{G}\right)_n = G_n - \sum_{j=1}^{n} a_j \phi_{(n-j)\delta} * \left[G_j - \mu^{-1} G_{j-1} - \beta \sum_{m=1}^{j} a_m B_m * G_{j-m} \right].$$

The crucial observation is that the fixed point $\Psi\left(\mathbf{G}\right) = \mathbf{G}$, with $G_0 = \delta_0$ is exactly what we want: It is characterized by $[\cdot] = 0$ for all j, and this is satisfied exactly for

$$G_j = \mu^{-j} C_j.$$

Trying $G_n = a_n \phi_{n\delta}$, of course $\Psi\left(\mathbf{G}\right) \neq \mathbf{G}$, but it is "asymptotically" so, i.e.

$$\Psi\left(\{a_k \phi_{k\delta}\}\right)_n \approx a_n \phi_{n\delta},\ n \to \infty.$$

Ψ is contracting if β is small enough, when equipping the space of $\mathbf{G}$'s with a norm, which weights the difference $G_n - G'_n$ large for large n. Therefore, one starts with

$$G_n^{(0)} = a_n \phi_{n\delta},$$

and puts $\mathbf{G}^{(k)} = \mathbf{\Psi}^k\left(\mathbf{G}^{(0)}\right)$, then this converges towards a fixed point which never moves asymptotically away from $\{a_n\phi_{n\delta}\}$. This fixed point is $\left\{\mu^{-n}C_n\right\}$. Therefore,

$$\mu^{-n}C_n \approx a_n\phi_{n\delta},\ n \to \infty,\ \text{i.e. } C_n/c_n \approx \phi_{n\delta},\ n \to \infty.$$

Proved by nothing than careful Taylor (and the Banach fixed point theorem), everything in x-space.

The application to SAW needs some care in $\mathbb{R}^d$ (and is more tricky than on the lattice). By the lace expansion, one estimates the Π_k by the C_n: The outcome for SAW on $\mathbb{R}^d$ with standard normal one-jump distribution is that for any $\varepsilon > 0$, $0 < \rho \leq 1$, and β small enough, one has "nearly" a local CLT:

$$\lim_{K\to\infty} \limsup_{n\to\infty} n^{d/2} \sup_{x:|x|\geq K} \left| \frac{C_n^{\mathsf{SAW}}(x)}{c_n} - \phi_{n\delta}(x) \right| = 0,$$

which is qualitatively the best possible result.

On the lattice, this was already obtained by Christine Ritzmann in her thesis, and it is more precise than results obtained by other methods.

Green's function. (based on a joint paper with Gady Kozma and Remco van der Hofstad). A very simple argument for standard weakly SAW (on $\mathbb{Z}^d$) to estimate the Green's function, also based directly on x-space contraction.

$$G_{\lambda}^{\mathsf{SAW}}(x) = \sum_{n=0}^{\infty} \lambda^n C_{n,\beta}(x).$$

There is a critical value λ_{cr} such that the series converges for $\lambda < \lambda_{\mathsf{cr}}$, and diverges for $\lambda > \lambda_{\mathsf{cr}}$. The question is about $G_{\lambda_{\mathsf{cr}}}^{\mathsf{SAW}}$.

Theorem If $d \geq 5$, then there exists $\beta_0 > 0$ such that for $\beta < \beta_0$, one has

$$G_{\lambda_{\mathsf{cr}}}^{\mathsf{SAW}}(x) \leq 2G^{\mathsf{RW}}(x).$$

Simple direct proof is based on the following two lemmas:

Lemma 1. Same assumptions, and $\lambda < \lambda_{\mathsf{cr}}$: Assuming $G_\lambda^{\mathsf{SAW}} \leq 3G^{\mathsf{RW}}$ pointwise, then one can invert G_λ^{SAW} with a rapidly decaying outcome: There exists $\Delta_\lambda^{\mathsf{SAW}}$ with $\Delta_\lambda^{\mathsf{SAW}} * G_\lambda^{\mathsf{SAW}} = \delta_0$, and with

- Invariance under lattice isometries

- $\sum_x \Delta_\lambda^{\mathsf{SAW}}(x) \geq 0$

- For some $C_1(d)$ and $\mu(d, \beta) \in [0, 1/2d]$:

$$\left|\Delta_\lambda^{\mathsf{SAW}}(x) - \Delta_\mu^{\mathsf{RW}}\right| \leq C_1 \beta (1 + |x|)^{-d-4},$$

  where

$$\Delta_\mu^{\mathsf{RW}}(x) = \delta_0(x) - \mu 1_{\{|x|=1\}}.$$

This part is standard, and comes out from the lace expansion easily. One puts

$$\Delta_\lambda^{\mathsf{SAW}} = \Delta_\lambda^{\mathsf{RW}} - \sum_n \lambda^n \Pi_n.$$

The convolution equations proves that $\Delta_\lambda^{\mathsf{SAW}} = \left(G_\lambda^{\mathsf{SAW}}\right)^{-1}$, and the bound on G_λ^{SAW} allows in a straightforward way to estimate the laces, and prove the decay property. (This is always the easy part in lace expansion business).

The crucial new ingredient is

Lemma 2 Assume $d > 2$ and that Δ is any function that satisfies above. Then, for β small enough, one can invert Δ with $\left|\Delta^{-1}\right| \leq 2G^{\mathsf{RW}}$. (This part has no longer anything to do with SAWs).

The two lemmas imply the theorem by a continuity argument. One starts with $\lambda = 0$ where $G_0^{\mathsf{SAW}} = G^{\mathsf{RW}} \leq 3G^{\mathsf{RW}}$, and then increases λ up to λ_{cr}, but clearly G_λ^{SAW} has to stay beyond $2G^{\mathsf{RW}}$.

To prove Lemma 2, one introduces the following Banach algebra norm on functions on $\mathbb{Z}^d$:

$$\|f\| := C_d \,\mathsf{max}\left\{\sum\nolimits_x |f(x)|\,, \mathsf{sup}_x\, |x|^d\, |f(x)|\right\}.$$

Furthermore, a rather elementary estimate (using however precise estimates of the RW-Green's function) gives that if $\rho : \mathbb{Z}^d \to \mathbb{R}$ is symmetric, satisfies $\sum_x \rho(x) = 0$ and $|\rho(x)| \leq |x|^{-d-4}$, then

$$\left\|\rho * G^{\mathsf{RW}}\right\| \leq \mathsf{const}$$

The next point is to generalize that slightly: Define

$$\Delta_\mu^{\mathsf{RW}}(x) = \begin{cases} 1 & \text{if } x = 0 \\ -\mu & \text{if } |x| = 1 \\ 0 & \text{otherwise} \end{cases},$$

$$G_\mu^{\mathsf{RW}} = \left(\Delta_\mu^{\mathsf{RW}}\right)^{-1} = \sum_{n=0}^{\infty} (2d\mu)^n \, p_n,$$

then for $\mu \in [0, 2d]$, and ρ as above

$$\begin{aligned} \left\|\rho * G_\mu^{\mathsf{RW}}\right\| &= \left\|\rho * G_{1/2d}^{\mathsf{RW}} * \Delta^{\mathsf{RW}} * G_\mu^{\mathsf{RW}}\right\| \\ &\leq \left\|\rho * G_{1/2d}^{\mathsf{RW}}\right\| \left\|\Delta^{\mathsf{RW}} * G_\mu^{\mathsf{RW}}\right\| \leq C \end{aligned}$$

where C is independent of μ. That $\left\|\Delta^{\mathsf{RW}} * G_\mu^{\mathsf{RW}}\right\| \leq C$ is checked directly.

Proof of Lemma 2: First one finds μ such that

$$\sum_x \left(\Delta(x) - \Delta_\mu^{\mathsf{RW}}(x)\right) = 0.$$

It is easily checked that for small enough β one has $0 \leq \mu \leq 2d$.

Define

$$\rho = \frac{1}{C_1 \beta}\left(\Delta - \Delta_\mu^{\mathsf{RW}}\right)$$

which satisfies our conditions on ρ.

$$\Longrightarrow \left\|\rho * G_\mu^{\mathsf{RW}}\right\| \leq C \Longrightarrow \left\|\left(\boldsymbol{\Delta} - \boldsymbol{\Delta}_\mu^{\mathsf{RW}}\right) * G_\mu^{\mathsf{RW}}\right\| = \left\|\boldsymbol{\Delta} * G_\mu^{\mathsf{RW}} - \delta_0\right\| \leq C\beta.$$

This implies that $\boldsymbol{\Delta} * G_\mu^{\mathsf{RW}}$ is invertible with $\left(\boldsymbol{\Delta} * G_\mu^{\mathsf{RW}}\right)^{-1} = \delta_0 + E,\ \|E\| \leq C\beta$. Then

$$G := \left(\boldsymbol{\Delta} * G_\mu^{\mathsf{RW}}\right)^{-1} G_\mu^{\mathsf{RW}} = G_\mu^{\mathsf{RW}} + E * G_\mu^{\mathsf{RW}}$$

is the desired inverse of $\boldsymbol{\Delta}$.

It finally remains to prove $|G(x)| \leq 2G^{\mathsf{RW}}(x),\ \forall x$, which comes from a straightforward computation using the bound on $\|E\|$. Actually one gets $|G| \leq (1 + O(\beta))\, G^{\mathsf{RW}}$.