Percolation and isoperimetric inequalities

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Percolation

Physical phenomenon:

(i) Models how fluid can spread through a medium;

(ii) Models how certain epidemics can spread through a network;

(iii) Many other motivational examples!

Introduced by *Broadbent and Hammersley* in '57 (independent percolation).

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Ingredients:

(i) A graph $G = (V, E)$ (e.g., $\mathbb{Z}^d$);

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Two types of percolation: *bond* (edges) and *site* (vertices) percolation.

Today we focus on **SITE** percolation.

Example of Percolation

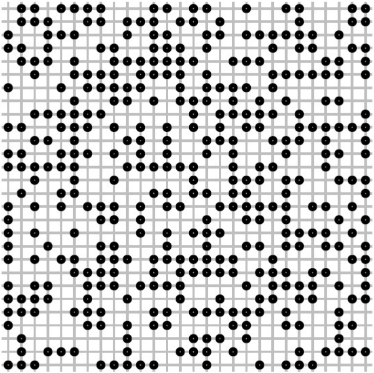

Figure: $\mathbb{Z}^2$ with $p = 0.5$.

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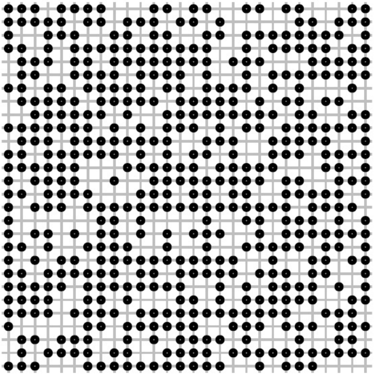

Figure: $\mathbb{Z}^2$ with $p = 0.7$.

Questions

(i) Connectivity properties of the black (random) subgraph?

(ii) Phase transitions? (Typically interested in **INFINITE** graphs: is a certain vertex connected to infinity?)

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(ii) Phase transitions? (Typically interested in **INFINITE** graphs: is a certain vertex connected to infinity?)

Define

$$\theta(p) := \mathbb{P}_p[\text{vertex } o \text{ is connected to infinity}].$$

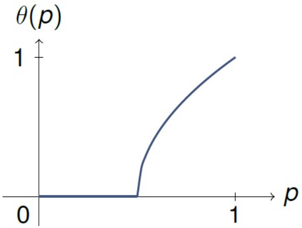

Figure: The function $\theta(p)$.

A critical value

From the previous picture it is then natural to define

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When is $p_c \in (0,1)$?

d-dimensional lattices

On $\mathbb{Z}^d$ we know several things, for example:

- If $d \geq 2$ we know that $p_c \in (0,1)$; moreover, θ is smooth for all $p \geq p_c$;
- If $d = 2$ or $d \geq 11$ (or so), then we know that $\theta(p_c) = 0$.

We still don't know what happens in the intermediate range of d's.

Other graphs

In general, for *independent percolation*, it is true that

If the degree of the graph G is at most Δ, then $p_c(G) \geq \frac{1}{\Delta} > 0$.

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If the degree of the graph G is at most Δ, then $p_c(G) \geq \frac{1}{\Delta} > 0$.

We do not have such an easy way to investigate upper-bounds for p_c.

The first step in a study of percolation on other graphs [...] will be to prove that the critical probability on these graphs is smaller than one.

Benjamini and Schramm

Other graphs

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- Cayley graph of the Grigorchuck group (example of a graph with intermediate growth) [Muchnik–Pak].

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Vertex-transitive graphs with polynomial growth.

The proof of this fact involves Gromov's theorem (a very difficult and powerful result from group theory) and combinatorial techniques developed by Babson and Benjamini, and later on simplified by Timar.

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Isoperimetric inequalities

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Does the dimension play a role for $p_c(G) < 1$?
How important?

Isoperimetric inequalities

For every finite set $A \subset V(G)$, define the vertex-boundary as

$$\partial A := \{ y \in V(G) \ : \ \{x, y\} \in E(G),\ x \in A,\ y \notin A \}.$$

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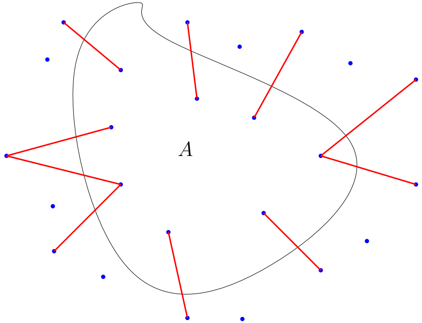

Figure: Constructing the boundary ∂A.

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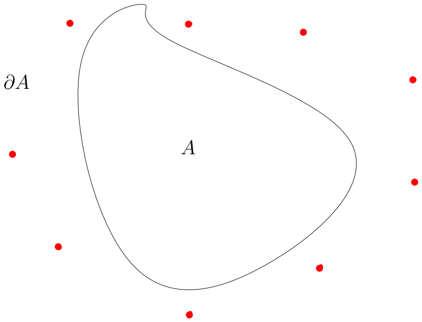

Figure: The boundary ∂A.

Isoperimetric inequalities (dimension)

Define the (isoperimetric) **dimension** of G as follows: we say that $\dim(G) = d > 1$

if and only if d is the largest value for which

there is a constant $c > 0$ such that

$$\inf_{A \subset V(G),\ A \text{ finite}} \frac{|\partial A|}{|A|^{(d-1)/d}} \geq c.$$

Isoperimetric inequalities (remarks)

Remark: for every $d \geq 2$, $\mathbb{Z}^d$ has isoperimetric dimension d.

Remark: If G has isoperimetric dimension $d > 1$, then we can say that it satisfies IS_d (d-isoperimetric inequality).

Question (Benjamini and Schramm '96)

Is it true that $\dim(G) > 1$ implies that $p_c(G) < 1$?

Some results

- If G is *planar*, has *polynomial growth* and *no accumulation points* then $\dim(G) > 1 \;\Rightarrow\; p_c(G) < 1.$ [Kozma]

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  then $\dim(G) > 1 \;\Rightarrow\; p_c(G) < 1.$ [Kozma]

- If G satisfies a *stronger* condition than the isoperimetric inequality (called *local isoperimetric inequality*), and has *polynomial growth*

  then $\dim_\ell(G) > 1 \;\Rightarrow\; p_c(G) < 1.$ [Teixeira]

Our results

Definition: A measure $\mathbb{P}$ satisfies the *decoupling inequality* $\mathcal{D}(\alpha, c_\alpha)$ (where $\alpha > 0$ is a fix parameter) if for all $r \geq 1$ and any two decreasing events $\mathcal{G}$ and $\mathcal{G}'$ such that

$$\mathcal{G} \in \sigma\big(Y_z, z \in B(o,r)\big) \qquad \text{and} \qquad \mathcal{G}' \in \sigma\big(Y_w, w \notin B(o,2r)\big),$$

we have

$$\mathbb{P}(\mathcal{G} \cap \mathcal{G}') \leq \big(\mathbb{P}(\mathcal{G}) + c_\alpha r^{-\alpha}\big)\mathbb{P}(\mathcal{G}').$$

In other words: we admit dependencies, as long as they decay fast enough in the distance.

Our results

With a completely **probabilistic approach** we showed:

Theorem [C. and Teixeira]: Let G be a transitive graph of polynomial growth, and let $\mathbb{P}$ satisfy $\mathcal{D}(\alpha, c_\alpha)$ with α "large enough". Then

(i) There exists a $p_* < 1$, such that if $\inf_{x \in V} \mathbb{P}[Y_x = 1] > p_*$, then the graph contains almost surely a unique infinite open cluster.

(ii) Moreover, fixed any value $\theta > 0$, we have

$$\lim_{v \to \infty} v^\theta \mathbb{P}[v < |\mathcal{C}_o| < \infty] = 0,$$

where $\mathcal{C}_o$ = open connected component containing the origin.

Our results

Moreover, in the dependent case we also need to show that:

Theorem [C. and Teixeira]: Let G be a transitive graph of polynomial growth, and let $\mathbb{P}$ satisfy $\mathcal{D}(\alpha, c_\alpha)$ with α "large enough". Then

(i) There exists a $p_{**} > 0$, such that if $\sup_{x \in V} \mathbb{P}[Y_x = 1] < p_{**}$, then the graph contains almost surely **NO** infinite open cluster.

(ii) Moreover, fixed any value $\theta > 0$, we have

$$\lim_{v \to \infty} v^\theta \mathbb{P}[v < |\mathcal{C}_o|] = 0,$$

where $\mathcal{C}_o$ = open connected component containing the origin.

Our results (Remark)

We always assume α to be *large enough*.
Although we don't have sharp bounds on its critical value, if α is too small, there are *counterexamples*!

Our results

Definition: Two metric spaces (X_1, d_1) and (X_2, d_2) are *roughly isometric* (sometimes called "quasi-isometric") if there is a map $\varphi : X_1 \to X_2$ s.t.:

(i) There are $A \geq 1$, $B \geq 0$ such that for all $x, y \in X_1$

$$A^{-1} d_1(x,y) - B \leq d_2\big(\varphi(x), \varphi(y)\big) \leq A d_1(x,y) + B.$$

(ii) There is $C \geq 0$ such that for all $z \in X_2$ there is $x \in X_1$ s.t.

$$d_2(z, \varphi(x)) \leq C.$$

Our results (Remarks)

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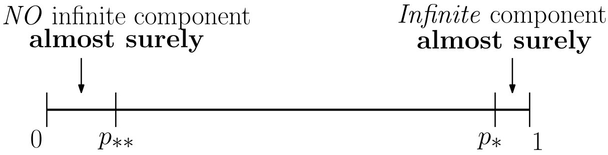
ROUGHLY TRANSITIVE $\neq$ ROUGHLY ISOMETRIC TO TRANSITIVE!

Our results

Our proof also works when G is a *roughly transitive graph*:

Theorem [C. and Teixeira]: Let G be a *roughly-transitive graph* of polynomial growth, and $\mathbb{P}$ satisfy $\mathcal{D}(\alpha, c_\alpha)$ with α "large enough". Then:



and, for every $\theta > 0$,

$$\begin{cases} \lim_{v\to\infty} v^\theta \mathbb{P}[v < |\mathcal{C}_\infty|] = 0 & \text{if } p < p_{**} \\ \lim_{v\to\infty} v^\theta \mathbb{P}_p[v < |\mathcal{C}_\infty| < \infty] = 0 & \text{if } p > p_*. \end{cases}$$

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- Repeat until you get to a *scale* where you can handle the computations:
  1. Show that assuming that a *bad event* occurs at some scale, then it must occur many times in the previous (smaller) scale.
  2. Show that in the smallest scale $\mathbb{P}(\text{bad event}) \ll 1$.

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- Define some *BAD EVENTS* A_k occurring at scale k;
- Show that $\mathbb{P}(A_k)$ is tiny (decaying exponentially fast in k);
- Iteratively, show that this implies that the probability of the same event occurring at a larger scale $k+1$ is tiny too!

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More precisely:

If the previous steps are verified for some $p_* \leq c < 1$, then OK.

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If **not** $\Rightarrow$ we obtain a contradiction!

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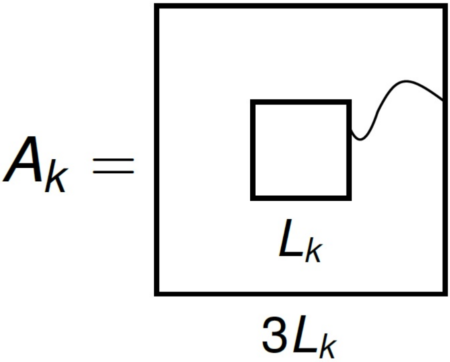

Figure: Bad event occurring at scale k.

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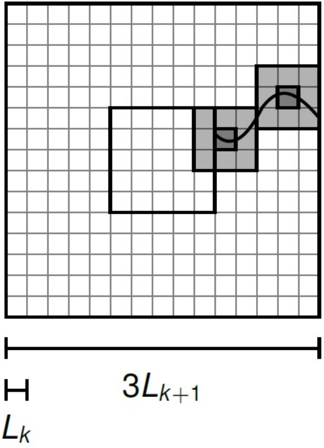

Figure: Bad event occurring at scale $k+1$.

Main hypothesis I: *polynomial growth*

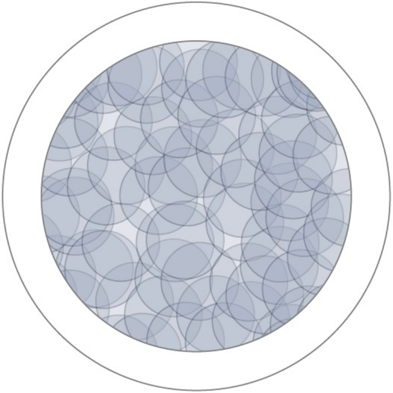

Figure: Polynomial growth allows us to split the graph into cells.

Main hypothesis II: *Isoperimetric inequality*

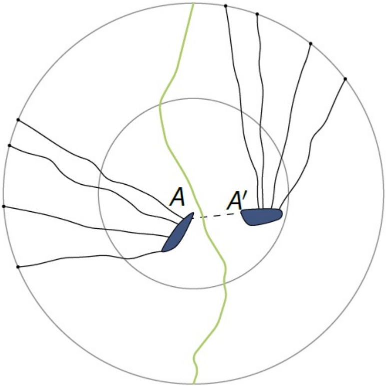

Figure: Isoperimetric inequality implies that there are lots of paths between large connected sets and infinity.

Main hypothesis III: *transitivity* (or rough-transitivity)

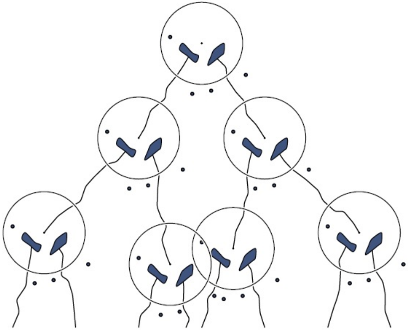

Figure: Transitivity allows us to repeat the same reasoning in different areas of the graph...

Proof

If G satisfies conditions I, II, and III (i.e., polynomial growth, isoperimetric dimension > 1, rough transitivity), then

assuming $p_c(G) = 1$

$$\Downarrow$$

it is possible to construct a binary tree inside G.

CONTRADICTION with polynomial growth of G! □

E.Candellero and A.Teixeira, *Percolation and isoperimetry on roughly transitive graphs*, `http://arxiv.org/abs/1507.07765`.

Thank you for your attention!