Duality theory, via examples

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joint work with
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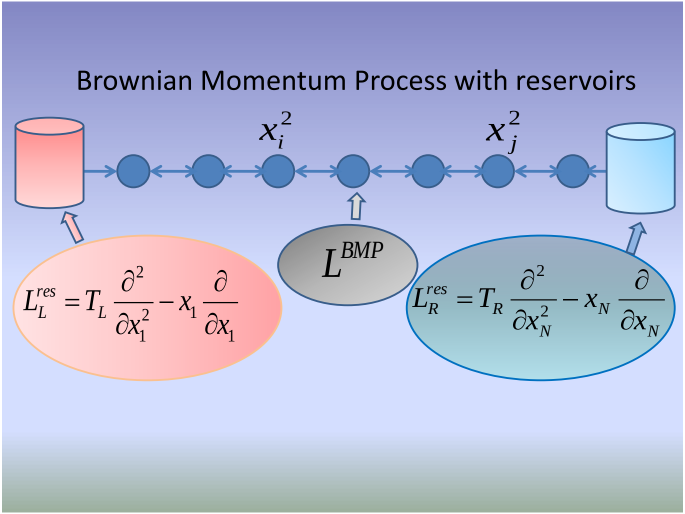
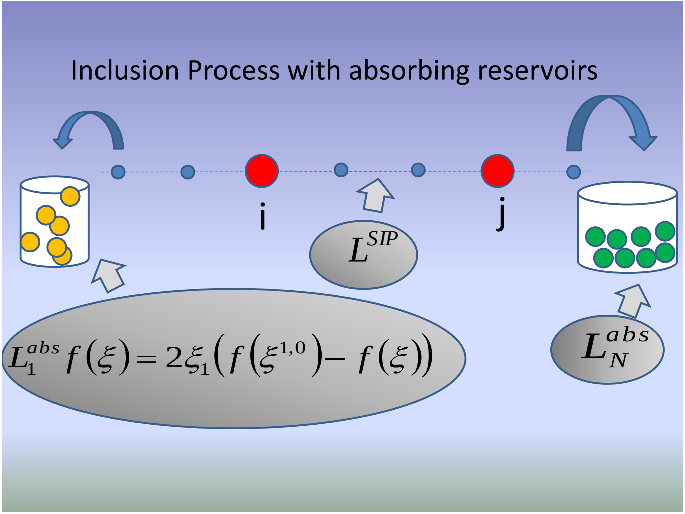
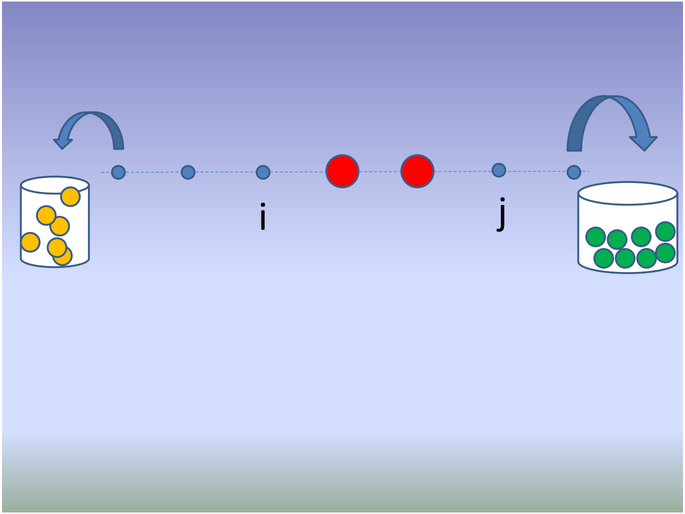
Context, outline

Duality is a powerful technique connecting two Markov processes (a process and its dual) via a duality function. It enables to simplify the study of the original process via e.g.

- Going from continuous to discrete variables.
- Going from many to a few particles (individuals).
- Replacing boundary reservoirs by absorbing states.
- Proof of existence.


Brownian Momentum Process with reservoirs
x_i^2
x_j^2
L^{BMP}
$L_L^{res} = T_L \frac{\partial^2}{\partial x_1^2} - x_1 \frac{\partial}{\partial x_1}$
$L_R^{res} = T_R \frac{\partial^2}{\partial x_N^2} - x_N \frac{\partial}{\partial x_N}$


Inclusion Process with absorbing reservoirs
i
j
L^{SIP}
$L_1^{abs} f(\xi) = 2\xi_1 \left(f(\xi^{1,0}) - f(\xi) \right)$
L_N^{abs}


i
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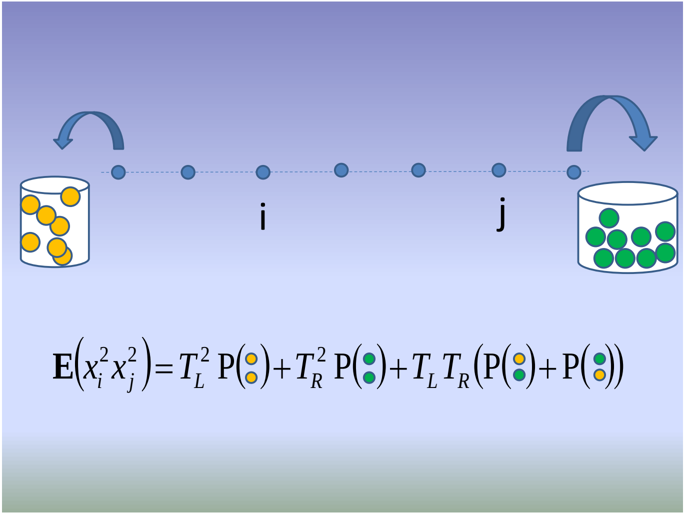

i
j

Basic definitions

Two Markov processes on state spaces Ω, $\hat{\Omega}$ are dual with duality function $D : \Omega \times \hat{\Omega} \to \mathbb{R}$ if for all $(x, y) \in \Omega \times \hat{\Omega}$

$$\mathbb{E}_x D(x_t, y) = \hat{\mathbb{E}}_y D(x, y_t)$$

In semigroup notation: for all $(x, y) \in \Omega \times \hat{\Omega}$

$$S_t^l D(x, y) = \hat{S}_t^r D(x, y)$$

with S_t semigroup of x_t, $\hat{S}_t$ semigroup of y_t.
For the generators: for all $(x, y) \in \Omega \times \hat{\Omega}$

$$L^l D(x, y) = \hat{L}^r D(x, y)$$

If the processes are the same, we speak about self-duality and a self-duality function.

Two operators $K, \hat{K}$ are in duality with duality function D if

$$K^l D(x,y) = \hat{K}^r D(x,y)$$

We denote this property by

$$K \rightarrow^D \hat{K},$$

This notation suggests that "duality with duality function D" is a relation between operators, and naturally (by what will follow) between algebras of operators.

Basic properties of the relation $\to^D$

- **Sums and products**: $K_1 \to^D \hat{K}_1, K_2 \to^D \hat{K}_2$ implies

$$aK_1 + bK_2 \to^D a\hat{K}_1 + b\hat{K}_2$$

and

$$K_1 K_2 \to^D \hat{K}_2 \hat{K}_1.$$

To see this:

$$\begin{aligned} K_1^l K_2^l D &= K_1^l (K_2^l D) \\ &= K_1^l (\hat{K}_2^r D) \\ &= \hat{K}_2^r (K_1^l D) \\ &= \hat{K}_2^r (\hat{K}_1^r D) \\ &= \hat{K}_2^r \hat{K}_1^r D \end{aligned}$$

As a consequence every element of the algebra generated by K_1, K_2 is dual to a corresponding element of the algebra generated by $\hat{K}_2, \hat{K}_1$.
Order of multiplications are reversed so commutators change sign (cf. matrix transposition). I.e., an algebra is mapped to its dual algebra (where order of multiplication is reversed, or right representation goes to left representation).

Duality functions and symmetries

- **New duality function from symmetries**: If $K \to^D \hat{K}$ and S is a symmetry of K, i.e., $[K, S] = KS - SK = 0$ then $K \to^{S^l D} \hat{K}$. Indeed,

$$\begin{aligned} K^l S^l D &= S^l K^l D \\ &= S^l \hat{K}^r D \\ &= \hat{K}^r S^l D \end{aligned}$$

Similarly, if $K \to^D \hat{K}$ and $\hat{S}$ is a symmetry of $\hat{K}$, then $K \to^{\hat{S}^r D} \hat{K}$.

- **Trivial self-duality function**: if $\mu(x)$ is a reversible measure for a generator L, then

$$D(x,y) = \frac{1}{\mu(x)}\delta_{x,y}$$

is a self-duality function, i.e. $L \to^D L$.
Indeed: the detailed balance relation

$$\mu(x)L(x,y) = \mu(y)L(y,x)$$

gives

$$\begin{aligned}
L^l D(x,y) &= \sum_{x'} L(x,x')\frac{1}{\mu(x')}\delta_{x',y} \\
&= L(x,y)\frac{1}{\mu(y)} \\
&= L(y,x)\frac{1}{\mu(x)} \\
&= \sum_{y'} L(y,y')\frac{1}{\mu(y')}\delta_{x,y'} = L^r D(x,y)
\end{aligned}$$

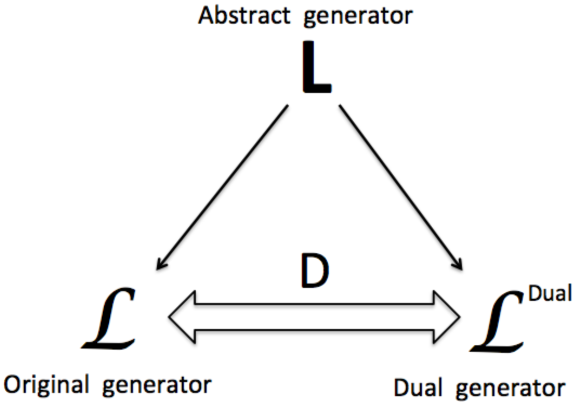
The use of duality is not always possible. Natural questions:

- Why do some processes have (nice) duals and others not?
- Can we constructively find dual processes , and corresponding duality functions starting from a given process?
- Can we construct processes with nice duals?
- Can we construct processes which are self-dual?

Lie algebra approach

Approach started in 2009 paper with Giardinà, Kurchan and Vafayi in the context of models of non-equilibrium statistical mechanics.
Basic ideas:

- **Step 1:** The generator is an element of a Lie algebra: identify this (these) algebra(s). Alternatively: start from a known Lie algebra and construct a generator.
- **Step 2:** The Lie algebra has different (left and right) representations. These representations are related via "inter-twiners". Writing the generator in two different (one left and one right) representation leads to two processes interrelated by duality. The duality function is the intertwiner .
- **Step 3:** Identify the symmetries of the generator, i.e., the operators that commute with it. These are in (one-to-one) correspondence with self-duality functions.


Abstract generator
L
D
Dual
Original generator
Dual generator

Acronym	Process Description	Algebra	Dual Process
KMP	Kipnis Marchioro Presutti	$\mathfrak{su}(1,1)$	Dual-KMP
BMP(m)	Brownian Momentum with diffusion rate m	$\mathfrak{su}(1,1)$	SIP(m)
BEP(m)	Brownian Energy with diffusion rate m	$\mathfrak{su}(1,1)$	SIP(m)
ABEP(σ,m) *	Asymmetric Brownian Energy with asymmetry parameter σ	$\mathfrak{su}_q(1,1)$	
SIP(m)	Symmetric Inclusion with diffusion rate m	$\mathfrak{su}(1,1)$	Self-Dual
WASIP(m) *	Weakly Asymmetric Inclusion	$\mathfrak{su}(1,1)$	
ASIP(q,m) *	Asymmetric Inclusion with asymmetry parameter q	$\mathfrak{su}_q(1,1)$	Self-Dual
SEP(j)	Symmetric Exclusion with at most $2j$ particles per site	$\mathfrak{su}(2)$	Self-Dual
WASEP(j) *	Weakly Asymmetric Exclusion	$\mathfrak{su}(2)$	
ASEP(q,j) *	Asymmetric Simple Exclusion with asymmetry parameter q	$\mathfrak{su}_q(2)$	Self-Dual

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Simple example: Heisenberg algebra

Start from a pair of creation and annihilation operators (generators of Heisenberg algebra) $A^\dagger$, A satisfying

$$[A, A^\dagger] = 1$$

The Heisenberg algebra is the algebra generated by $A, A^\dagger$. Most common representation:

$$A^\dagger f(x) = xf(x), Af(x) = f'(x).$$

Another well-known discrete **left** representation (Doi Peliti):

$$\begin{aligned} af(n) &= nf(n-1) \\ a^\dagger f(n) &= f(n+1) \end{aligned}$$

for functions $f : \mathbb{N}_0 \to \mathbb{R}$. These operators satisfy the "dual commutation relation" $[a, a^\dagger] = -1$.

Duality function for Heisenberg case

It is natural to look then for a function $D : \mathbb{R} \times \mathbb{N}_0 \to \mathbb{R}$ linking the $A, A^\dagger$ with $a, a^\dagger$. This goes as follows.
Suppose $D(x, 0)$ satisfies

$$A^l D(x, 0) = 0 \tag{1}$$

(vacuum state condition)
then, defining, for $x \in \mathbb{R}, n \in \mathbb{N}$

$$D(x, n) = ((a^\dagger)^n)^{right} D(x, 0) = ((A^\dagger)^n)^{left} D(x, 0) \tag{2}$$

we have

$$A, A^\dagger \to^D a, a^\dagger$$

Indeed: (1), (2) imply $A^\dagger \rightarrow^D a^\dagger$, so we have to prove that also $A \rightarrow^D a$

$$
\begin{aligned}
A^{left} D(x,n) &= A^{left}((A^\dagger)^n)^{left} D(x,0) \\
&= ((A^\dagger)^n)^{left} A^{left} D(x,0) + n((A^\dagger)^{n-1})^{left} D(x,0) \\
&= n((A^\dagger)^{n-1})^{left} D(x,0) \\
&= n((a^\dagger)^{n-1})^{right} D(x,0) = nD(x,n-1) \\
&= a^{right} D(x,n)
\end{aligned}
$$

Examples of dualities from Heisenberg algebra

1. Wright-Fisher diffusion and Kingman's coalescent.
   $A = d/dx$, $A^\dagger = x$ then
   $AD(x,0) = 0$ gives $D(x,0) = 1$ and,
   $D(x,n) = (A^\dagger)^n D(x,0) = x^n$. As a consequence

$$A^\dagger(1 - A^\dagger)A^2 \to^D a^2 a^\dagger(1 - a^\dagger)$$

Now

$$A^\dagger(1 - A^\dagger)A^2 = x(1 - x)\frac{d^2}{dx^2}$$

is the generator of the Wright-Fisher diffusion and

$$a^2 a^\dagger(1 - a^\dagger)f(n) = n(n-1)(f(n-1) - f(n))$$

is the generator of the Kingman's coalescent (block-counting process).

2) Generalization: $A = c_1 x + c_2 d/dx$, $A^\dagger = c_3 x + c_4 d/dx$. Then $A, A^\dagger \to^D a, a^\dagger$ and the duality function is

$$D(x,n) = (c_3 x + c_4 d/dx)^n D(x,0)$$

with

$$D(x,0) = e^{-\frac{c_1}{2c_2}x^2}$$

3) Laplace duality: Choosing $A = x$, $A^\dagger = d/dx$ and $a = d/dy$, $a^\dagger = y$ we find $A, A^\dagger \to^D a, a^\dagger$ with

$$D(x,y) = e^{xy}$$

e.g. $x^2 \frac{d}{dx^2} \to^D y^2 \frac{d}{dy^2}$, $x \frac{d}{dx^2} \to^D y^2 \frac{d}{dy}$ etc.

How to construct generators with many symmetries

- Start (from a representation of) a Lie algebra $\mathcal{A}$.
- Consider an element in the centre of $\mathcal{A}$, i.e., a non-trivial element commuting with all the elements of $\mathcal{A}$. A common example is the Casimir element C,
- Apply a co-product to it, i.e., a algebra homomorphism from

$$\Delta : \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$$

  in order to produce an operator acting on two variables. This operator $H = \Delta(C)$ will then automatically commute with all elements of the form $\Delta(X)$, $X \in \mathcal{A}$ (because $[\Delta(X), \Delta(C)] = \Delta([X, C]) = \Delta(0) = 0$).
- Produce (if possible) a generator out of H by omitting constants, elements in the centre, and -if necessary- a ground state transformation.

Illustration for $\mathfrak{U}(SU(1,1))$: the symmetric inclusion process

The Lie algebra we start from is $\mathfrak{U}(SU(1,1))$. It has three generators $\mathbf{K}^+, \mathbf{K}^-, \mathbf{K^o}$ satisfying the commutation relations

$$[\mathbf{K^o}, \mathbf{K}^\pm] = \pm \mathbf{K}^\pm \,, \qquad\qquad [\mathbf{K}^-, \mathbf{K}^+] = \mathbf{2K^o} \,, \tag{3}$$

A well known discrete **left** representation is given by

$$\begin{aligned} K^+ f(n) &= \left(\frac{m}{2}+n\right) f(n+1)\,, \\ K^- f(n) &= n f(n-1)\,, \\ K^0 f(n) &= \left(\frac{m}{4}+n\right) f(n)\,. \end{aligned} \tag{4}$$

i.e., these operators satisfy the dual commutation relations (with opposite sign).

- Start from the Casimir element of $\mathfrak{U}(SU(1,1))$ which is given by

$$C = \frac{1}{2}(K^+K^- + K^-K^+) - (K^0)^2$$

This element commutes with $K^\pm, K^0$. In the discrete representation it acts as $(m/4)(1-m/4)I$.

- Apply the coproduct given by

$$\Delta(K^\alpha) = K^\alpha \otimes I + I \otimes K^\alpha = K_1^\alpha + K_2^\alpha$$

for $\alpha \in \{\pm, 0\}$. Then we find

$$\begin{aligned} \Delta(C) &= \frac{1}{2}(K_1^+ + K_2^+)(K_1^- + K_2^-) \\ &+ \frac{1}{2}(K_1^- + K_2^-)(K_1^+ + K_2^+) \\ &- (K_1^0 + K_2^0)^2 \end{aligned}$$

$$\Delta(C) = K_1^+ K_2^- + K_1^- K_2^+ - 2K_1^0 K_2^0 + C_1 + C_2$$

Using now the discrete representation we find

$$\Delta(C) = L + \frac{m}{2}(1 - \frac{m}{2})I$$

where

$$\begin{aligned} Lf(\eta_1, \eta_2) &= \eta_1 \left(\frac{m}{2} + \eta_2\right)(f(\eta_1 - 1, \eta_2 + 1) - f(\eta_1, \eta_2)) \\ &+ \eta_2 \left(\frac{m}{2} + \eta_1\right)(f(\eta_1 + 1, \eta_2 - 1) - f(\eta_1, \eta_2)) \end{aligned}$$

is the generator of the "symmetric inclusion process" $SIP(m/2)$

As a consequence of the construction, L commutes with

$$\Delta(K^\alpha) = K_1^\alpha + K_2^\alpha$$

and also with

$$S^\alpha = \exp(K_1^\alpha + K_2^\alpha), \ \alpha \in \{\pm, 0\}$$

A trivial self-duality function is given via the reversible product measures, which are products of discrete Γ distributions:

$$\mu_\lambda(\eta) = \prod_{i=1}^{2} (1-\lambda)^{m/2} \frac{\lambda^{\eta_i}}{\eta_i!} \frac{\Gamma\left(\frac{m}{2} + \eta_i\right)}{\Gamma(m/2)}$$

Applying S^+ to the trivial self-duality function gives a non-trivial self-duality function for the $SIP(m/2)$:

$$\mathcal{D}(\xi, \eta) = \prod_{i=1}^{2} d(\xi_i, \eta_i)$$

$$d(k, n) = \frac{n!}{(n-k)!} \frac{\Gamma(m/2)}{\Gamma\left(\frac{m}{2} + k\right)}$$

Other representation of $SU(1,1)$.

The "abstract operator"

$$\mathbf{L} = \mathbf{K}_1^+\mathbf{K}_2^- + \mathbf{K}_1^-\mathbf{K}_2^+ - 2\mathbf{K}_1^0\mathbf{K}_2^0$$

can now also be looked at in different representations. An important example is the differential operator representation

$$\begin{aligned} \mathcal{K}^+ &= z\,, \\ \mathcal{K}^- &= z\frac{d^2}{dz^2} + \frac{m}{2}\frac{d}{dz}\,, \\ \mathcal{K}^0 &= z\frac{d}{dz} + \frac{m}{4}\,, \end{aligned} \tag{5}$$

The duality function D relating this representation with the discrete one is then easily found and given by

$$D(z,n) = \frac{z^n}{\frac{m}{2}\left(\frac{m}{2}+1\right)\dots\left(\frac{m}{2}+n-1\right)} = \frac{z^n \Gamma\left(\frac{m}{2}\right)}{\Gamma\left(\frac{m}{2}+n\right)}\,. \qquad (6)$$

I.e.,

$$(\mathcal{K}^\alpha)^{left} D(z,n) = K^\alpha D(z,n)$$

$\alpha \in \{\pm, 0\}$. Example

$$\begin{aligned}
\mathcal{K}^+ D(z,n) &= zD(z,n) \\
&= \frac{z^{n+1}\Gamma\left(\frac{m}{2}\right)}{\Gamma\left(\frac{m}{2}+n\right)} \\
&= \left(\frac{m}{2}+n\right)\frac{z^{n+1}\Gamma\left(\frac{m}{2}\right)}{\Gamma\left(\frac{m}{2}+n+1\right)} \\
&= \left(\frac{m}{2}+n\right) D(z,n+1) \\
&= K^+ D(z,n)
\end{aligned}$$

The abstract operator in this representation is a diffusion process

$$\begin{aligned}\mathcal{L}^{BEP(m)}f(z) &= \frac{1}{2}z_1z_2\left(\frac{\partial}{\partial z_1}-\frac{\partial}{\partial z_2}\right)^2 f(z)\\ &- \frac{m}{4}(z_1-z_2)\left(\frac{\partial}{\partial z_1}-\frac{\partial}{\partial z_2}\right) f(z)\,. \end{aligned} \tag{7}$$

So we immediately find that this interacting diffusion process $BEP(m)$ is dual to $SIP(m/2)$.

Thermalization of *BEP* and *SIP*

The energy redistribution model $KMP(m)$ is defined by "thermalizing" BEP generator

$$
\begin{aligned}
L_{12}^{KMP(m)} f(x_1,x_2) = \left(\lim_{t\to\infty} e^{tL^{BEP(m)}} f(x_1,x_2)\right) - f(x_1,x_2) \\
= \lim_{t\to\infty}\left(\mathbb{E}^{BEP(m)}_{x_1,x_2}\left(f(x_1(t),x_2(t))\right) - f(x_1,x_2)\right) \\
= \int_0^1 f(\epsilon(x_1+x_2),(1-\epsilon)(x_1+x_2))\nu_m(d\epsilon) - f(x_1,x_2)
\end{aligned}
$$

I.e., on Poisson event times, x_1, x_2 is replaced by a sample of its unique (micro canonical) stationary distribution under the process with generator L_{12} starting from (x_1, x_2). For $m = 2$ this is the KMP (Kipnis, Marchioro, Presutti) process. By construction, this process is then dual to the discrete redistribution model obtained from applying the same "thermalization" to the $SIP(m/2)$.

Remarks

- The same construction for the Lie algebra $\mathfrak{U}(SU(2))$ yields, in a discrete representation the $2j$-exclusion process, with generator

$$
\begin{aligned}
Lf(\eta) &= \eta_1(2j-\eta_2)(f(\eta_1-1,\eta+1)-f(\eta_1,\eta_2)) \\
&+ \eta_2(2j-\eta_1)(f(\eta_1+1,\eta-1)-f(\eta_1,\eta_2))
\end{aligned}
$$

  with stationary measures products of binomials $2j,\rho$.
- For this Lie algebra, the differential operator representation does not give a Markov process generator.

Asymmetric processes

- The processes constructed from a classical Lie algebra in the way described above are always "symmetric" (reversible).
- The "obvious" asymmetric versions fail to have self-duality properties, with the ASEP as an important exception.
- We want to find the "good" asymmetric extensions of the processes which we discussed, i.e., such that a self-duality property similar to that of the ASEP can be established.
- The way to do this is by considering "deformations" of the corresponding Lie algebras, and applying the same construction.

Quantum $SU(1,1)$, and asymmetric processes of SIP type

- $SU_q(1,1)$ is the q-deformation of $SU(1,1)$ generated by K^+, K^-, K^0 with commutation relations

$$[K^+, K^-] = -[2K^0]_q,\ [K^0, K^\pm] = \pm K^\pm$$

  with

$$[2K^0]_q = \frac{q^{2K^0} - q^{-2K^0}}{q - q^{-1}}$$

- The Casimir element is given by

$$[K^0]_q[(K^0 - 1)]_q - K^+K^-$$

- The coproduct is given by

$$\Delta(K^{\pm}) = K_1^{\pm} \otimes q^{-K_2^0} + q^{K_1^0} \otimes K_2^{\pm}$$

$$\Delta(K^0) = K_1^0 \otimes I_2 + I_1 \otimes K_2^0$$

- The coproduct applied to the Casimir in a discrete representation gives, after a suitable ground-state transformation exactly the generator of a new asymmetric interacting particle system which we called $ASIP(q, 2k)$

The discrete representation to construct the $ASIP(q, 2k)$ is given by

$$\begin{cases} K^+|n\rangle &= \sqrt{[\eta + 2k]_q[\eta + 1]_q}\,|n+1\rangle \\ K^-|n\rangle &= \sqrt{[\eta]_q[\eta + 2k - 1]_q}\,|n-1\rangle \\ K^0|n\rangle &= (\eta + k)\,|n\rangle\,. \end{cases}$$

Generator of $ASIP(q, 2k)$

$$
\begin{aligned}
& L_{12}^{ASIP(q,2k)} f(\eta_1, \eta_2) \\
= \quad & q^{\eta_1 - \eta_2 + (2k-1)} [\eta_1]_q [2k + \eta_2]_q (f(\eta_1 - 1, \eta_2 + 1) - f(\eta_1, \eta_2)) \\
+ \quad & q^{\eta_1 - \eta_2 - (2k-1)} [\eta_2]_q [2k + \eta_1]_q (f(\eta_1 + 1, \eta_2 - 1) - f(\eta_1, \eta_2))
\end{aligned}
$$

with $0 < q < 1$ and with

$$[k]_q = (q^k - q^{-k})/(q - q^{-1})$$

the k-th q number. This process reduces for $q = 1$ (undeformed case) to the old process $SIP(2k)$.

Generator of $ASEP(q, 2j)$

The same procedure in the context of $SU_q(2)$ leads to a generalized asymmetric exclusion process

$$\begin{aligned} & L_{12}^{ASEP(q,2j)} f(\eta_1, \eta_2) \\ = \quad & q^{\eta_1 - \eta_2 - (2j+1)} [\eta_1]_q [2j - \eta_2]_q (f(\eta_1 - 1, \eta_2 + 1) - f(\eta_1, \eta_2)) \\ + \quad & q^{\eta_1 - \eta_2 + (2j+1)} [\eta_2]_q [2j - \eta_1]_q (f(\eta_1 + 1, \eta_2 - 1) - f(\eta_1, \eta_2)) \end{aligned}$$

where j is a positive half integer. For $j = 1/2$ this is the standard ASEP.

Groundstate transformation

If the procedure does not immediately give a generator, one can use the following. If L is a Markov generator, then so is

$$L_f(g) = e^{-f} L(e^f g) - (e^{-f} L(e^f))g$$

Suppose the procedure gives an operator of "Schödinger" type

$$H(g) = L(g) - \psi g$$

with L a Markov generator. Then

$$(e^{-f} L(e^f)) = \psi$$

is equivalent with

$$H(e^f) = 0$$

If such a f can be found, then

$$\mathcal{L} g := L_f(g) = e^{-f} L(e^f g) - (e^{-f} L(e^f))g = e^{-f} H(e^f g)$$

is a generator, and every symmetry of H translates in a symmetry of $\mathcal{L}$.

This gives that the $ASIP(q, 2k)$ process on N sites, with generator

$$\sum_{i=1}^{N} L_{i,i+1}^{ASIP(q,2k)}$$

is self-dual with self-duality function

$$\mathcal{D}(\xi, \eta) = \prod_{i=1}^{N} \frac{\binom{\eta_i}{\xi_i}_q}{\binom{\xi_i+2k-1}{\xi_i}_q} q^{(\eta_i-\xi_i)(2\sum_{m=1}^{i-1}\xi_m+\xi_i)-4ki\xi_i} 1_{\xi_i \le \eta_i}$$

In particular, if there is only one dual particle $\xi = \delta_i$ then

$$\mathcal{D}(\xi, \eta) = \frac{q^{-4ki+1}}{q^{2k} - q^{-2k}}(q^{2N_i(\eta)} - q^{2N_{i+1}(\eta)})$$

Where

$$N_i(\eta) = \sum_{i \leq j \leq N} \eta_j$$

The duality relation applied to this gives that

$$\begin{aligned} & \mathbb{E}_\eta^{ASIP(q,2k)} \mathcal{D}(\xi, \eta_t) \\ = \quad & \mathbb{E}_\eta^{ASIP(q,2k)} \frac{q^{-4ki+1}}{q^{2k} - q^{-2k}}(q^{2N_i(\eta_t)} - q^{2N_{i+1}(\eta_t)}) \\ = \quad & \widehat{\mathbb{E}}_i \frac{q^{-4kX_t+1}}{q^{2k} - q^{-2k}}(q^{2N_{X_t}(\eta)} - q^{2N_{X_t+1}(\eta)}) \end{aligned}$$

with X_t a single continuous-time random walk making jumps to the right with rate $q^{2k}2k$ and to the left with rate $q^{-2k}2k$.

Thanks for your attention !