

Planar lattices do not recover from forest fires

Ioan Manolescu

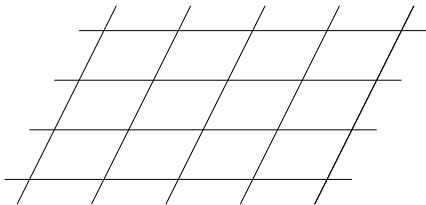
Joint work with Demeter Kiss and Vladas Sidoravicius



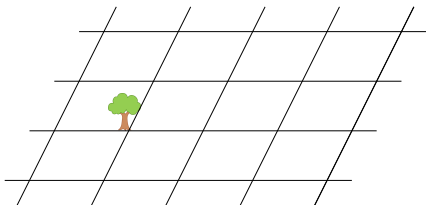
**UNIVERSITÉ
DE GENÈVE**

8 May 2015

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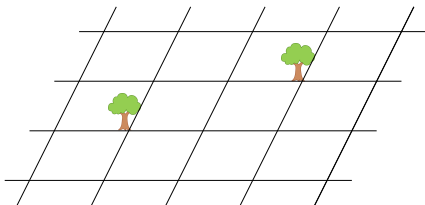


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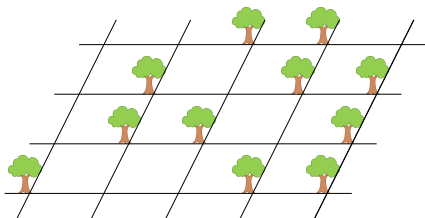
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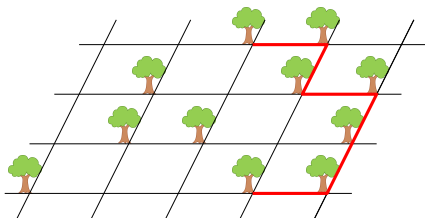
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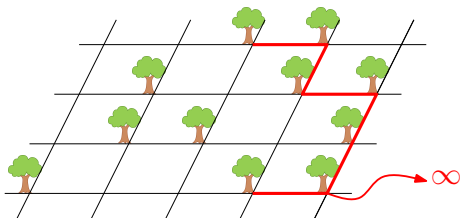
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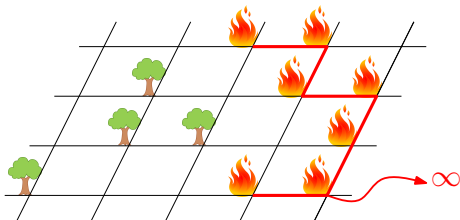
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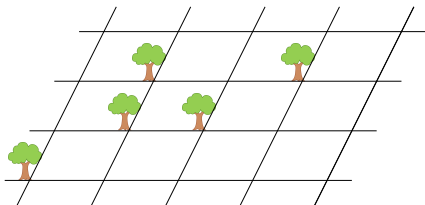
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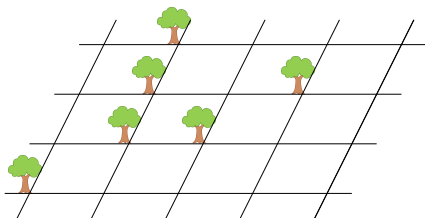
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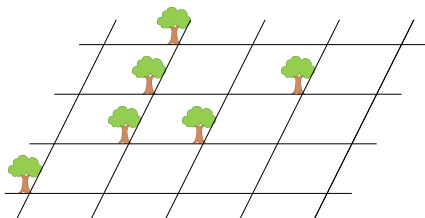
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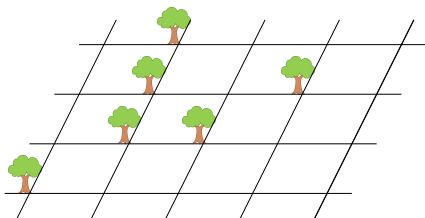
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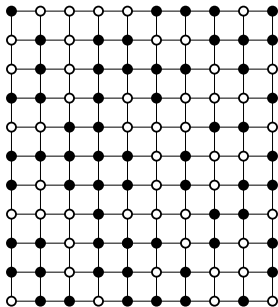
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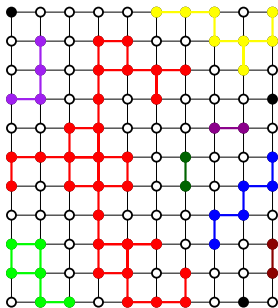
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Question: Does this make sense?

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 vertices are open with probability p , closed with probability $1 - p$, independently.

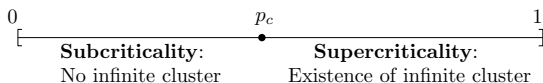


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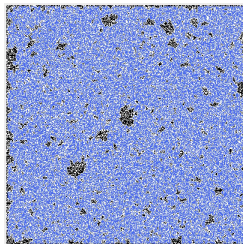
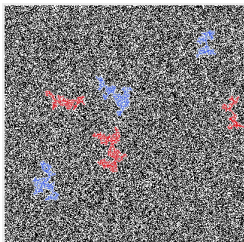
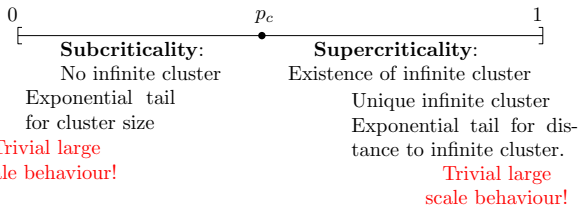
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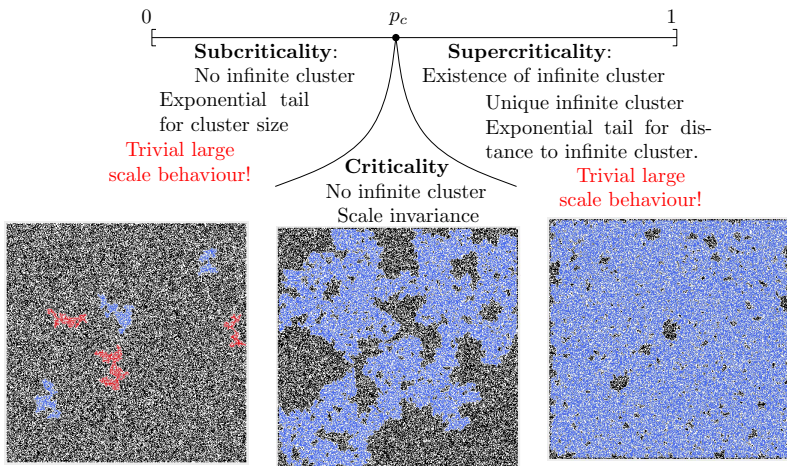
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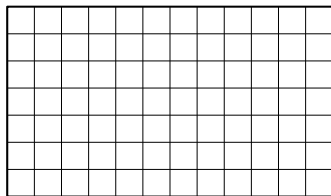
At $p_c \dots$ Crossing probabilities do not degenerate. (RSW)

$$\forall n, \mathbf{P}_{p_c} \left[\begin{array}{c} \boxed{\text{red curve}} \\ \text{width } 2n, \text{ height } n \end{array} \right] \geq \epsilon$$

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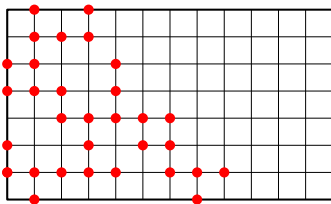
$$\forall n, \mathbf{P}_{p_c} \left[\begin{array}{c} \boxed{\text{wavy line}} \\ 2n \end{array} \right] \geq \epsilon$$



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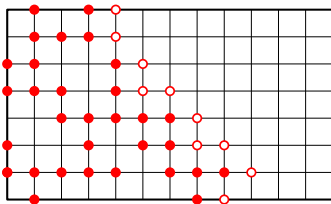
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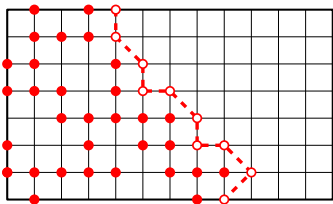
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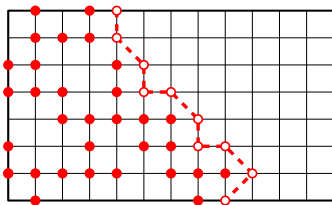
$$\forall n, \mathbf{P}_{p_c} \left[\begin{array}{|c|} \hline \text{[Diagram of a red curve crossing a square]} \\ \hline \end{array} \right] \geq \epsilon$$



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 Existence of **critical exponents** (arm exponents)

$$\forall n, \mathbf{P}_{p_c} \left[\begin{array}{c} \text{[Diagram: Solid red path across } 2n \times 2n \text{ box]} \\ 2n \end{array} \right] \geq \epsilon \qquad \mathbf{P}_{p_c} \left[\begin{array}{c} \text{[Diagram: Dashed red path across } 2n \times 2n \text{ box]} \\ 2n \end{array} \right] \geq \epsilon$$

$$\mathbf{P}_{p_c} \left[\begin{array}{c} \text{[Diagram: Solid red path from center to boundary in } n \times n \text{ box]} \\ n \end{array} \right] \leq n^{-\alpha_1} \qquad \mathbf{P}_{p_c} \left[\begin{array}{c} \text{[Diagram: Solid red path and dashed red paths from center to boundary in } n \times n \text{ box]} \\ n \end{array} \right] \leq n^{-(2+\lambda)}$$

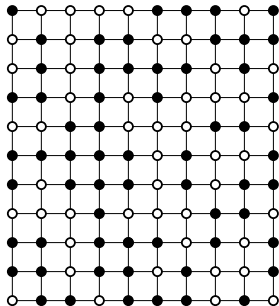
What is self-destructive percolation?

A planar lattice: here \mathbb{Z}^2 .

Let $p, \delta \in [0, 1]$.

Two (site) percolation configurations:

- ω - intensity p (measure \mathbb{P}_p).



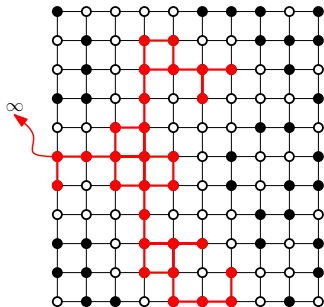
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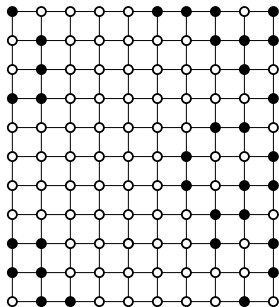
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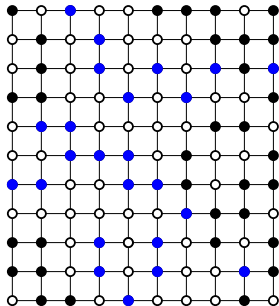
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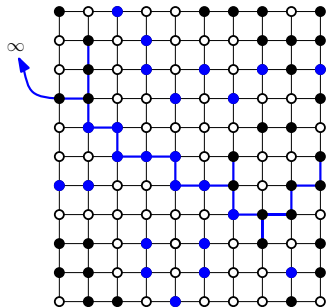
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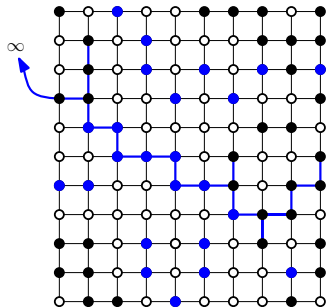
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Question: $\delta_c(p) \rightarrow 0$ as $p \searrow p_c$?

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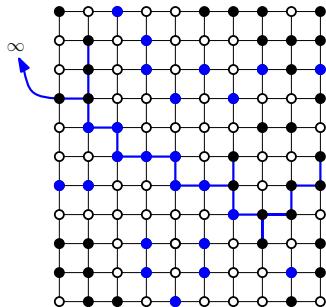
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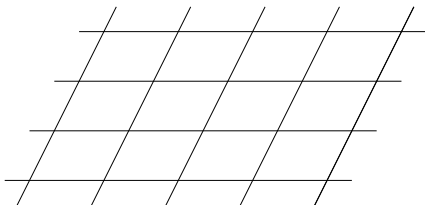


Theorem [Kiss, M., Sidoravicius] : There exists $\delta > 0$ such that, for all $p > p_c$,

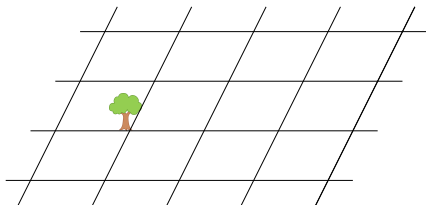
$$\mathbb{P}_{p,\delta}(\text{infinite cluster in } \bar{\omega}^\delta) = 0.$$

In particular $\lim_{p \rightarrow p_c} \delta_c(p) > 0$

Back to forest fires: non-existence

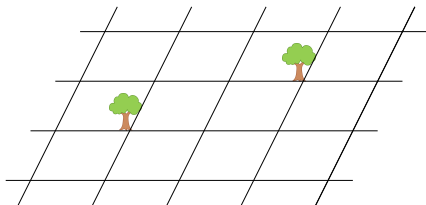


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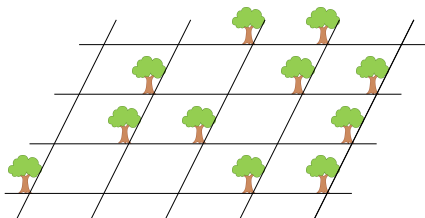
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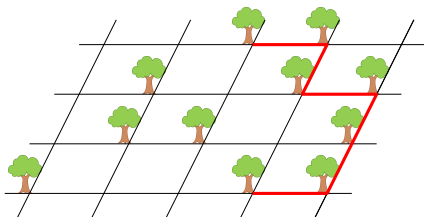
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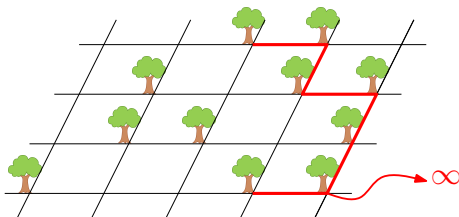
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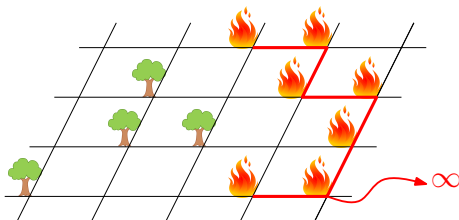
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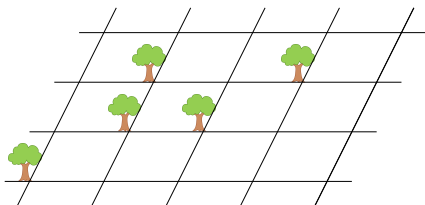
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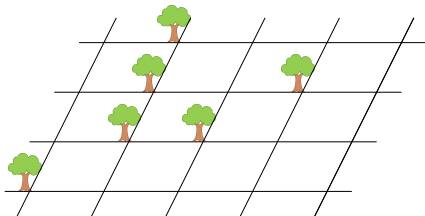
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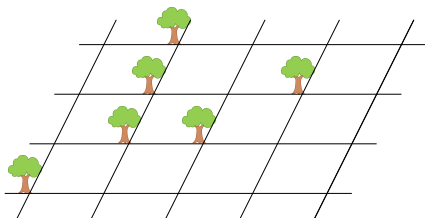
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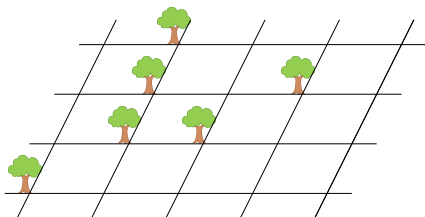
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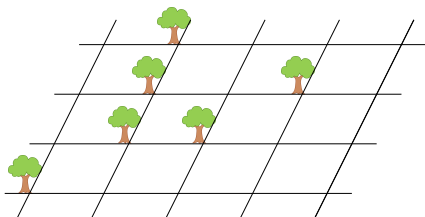
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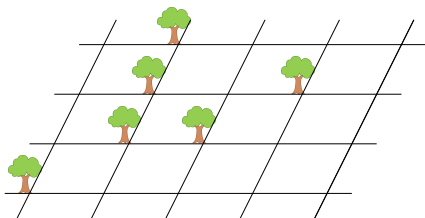


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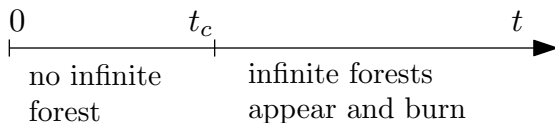


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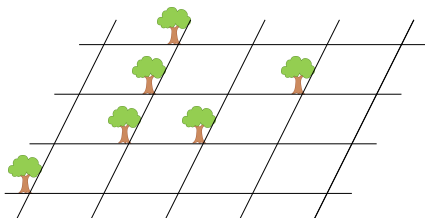


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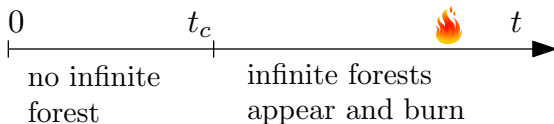


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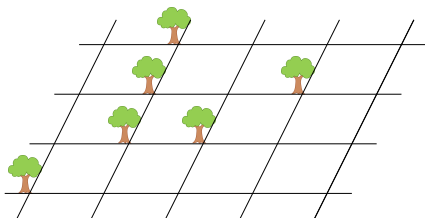


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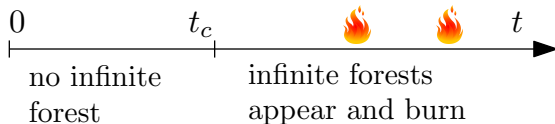


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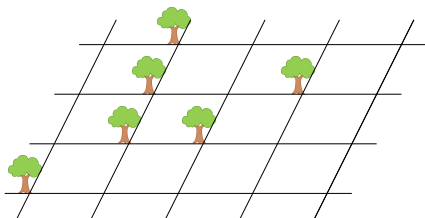


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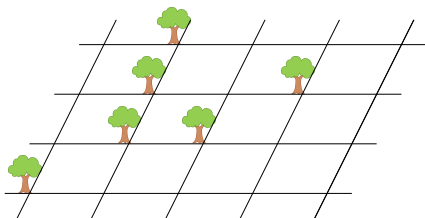


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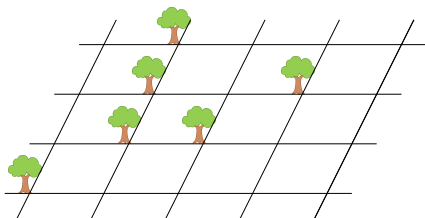


- Trees grow at i.i.d exponential times, instantly - on vacant sites;
- As soon as an infinite forest (i.e. cluster) is formed, it instantly burns
- Trees then continue to grow ... and burn

Question: Does this make sense?



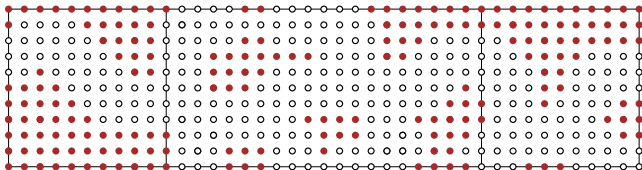
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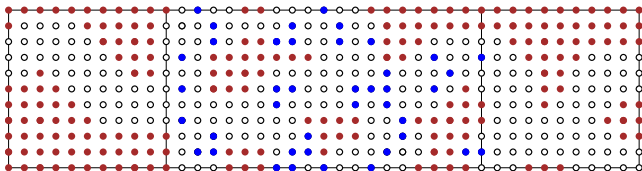


ω containing crossing $\xrightarrow{\text{delete crossing cluster}}$ $\tilde{\omega}$

Proposition

For $\delta > 0$ small enough, as $n \rightarrow \infty$,

$$\mathbb{P}_{p_c, \delta} \left(\begin{array}{c} \text{Diagram 1: A rectangle with width } 6n \text{ and height } n. \text{ The bottom edge is divided into three segments of length } n. \text{ A black curve } \omega \text{ starts at the top-left corner, goes down, then across, then up, then across, then down to the bottom-right corner.} \\ \text{Diagram 2: A rectangle with width } 6n \text{ and height } n. \text{ The bottom edge is divided into three segments of length } n. \text{ A blue curve } \tilde{\omega}^\delta \text{ starts at the top-left corner, goes down, then across, then up, then across, then down to the bottom-right corner.} \end{array} \right) \rightarrow 0$$



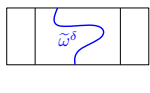
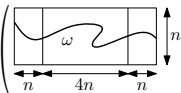
ω containing crossing $\xrightarrow{\text{delete crossing cluster}} \tilde{\omega} \xrightarrow{\text{enhancement}} \tilde{\omega}^\delta$

Proposition

For $\delta > 0$ small enough, as $n \rightarrow \infty$,

$$\mathbb{P}_{p_c, \delta} \left(\begin{array}{c} \boxed{\begin{array}{c} \text{---} \omega \text{---} \\ \text{---} \end{array}} \\ \begin{array}{c} \leftarrow n \quad 4n \quad n \rightarrow \\ \uparrow n \quad \downarrow n \end{array} \end{array} \text{ and } \begin{array}{c} \boxed{\begin{array}{c} \text{---} \tilde{\omega}^\delta \text{---} \\ \text{---} \end{array}} \\ \begin{array}{c} \leftarrow n \quad 4n \quad n \rightarrow \\ \uparrow n \quad \downarrow n \end{array} \end{array} \right) \rightarrow 0$$

For $\delta > 0$ small, $\mathbb{P}_{p_c, \delta} \left(\left(\begin{array}{c} \text{Diagram 1} \\ \text{Diagram 2} \end{array} \right) \right) \rightarrow 0$



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Diagram 1: A rectangle of height n and width $6n$ is divided into three vertical sections of width $2n$ each. A black curve ω starts at the top left, crosses the first boundary, crosses the second boundary, and ends at the top right. The width of the first section is labeled n , the middle section $4n$, and the last section n .

Diagram 2: A rectangle of height n and width $6n$ is divided into three vertical sections of width $2n$ each. A blue curve $\tilde{\omega}^\delta$ starts at the top left, crosses the first boundary, crosses the second boundary, and ends at the top right.

For $p_1 \geq p_2$ and δ_1, δ_2 such that $p_1 + (1 - p_1)\delta_1 \leq p_2 + (1 - p_2)\delta_2$,

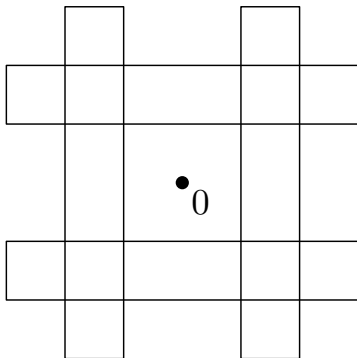
$$\mathbb{P}_{p_1, \delta_1} \leq_{\text{st}} \mathbb{P}_{p_2, \delta_2}.$$

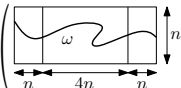

For $\delta' > 0$ small, $\mathbb{P}_{p, \delta'} \left(\left(\begin{array}{c} \text{Diagram 1} \\ \text{Diagram 2} \end{array} \right) \right) \rightarrow 0$, for $p \geq p_c$.

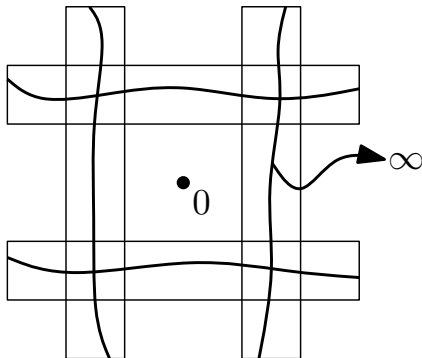
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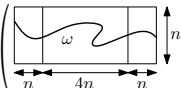

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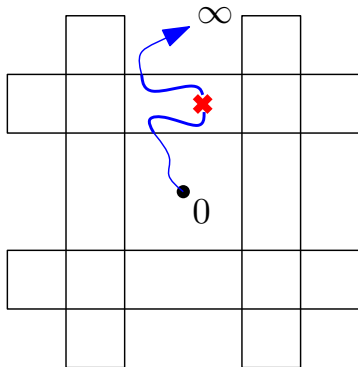
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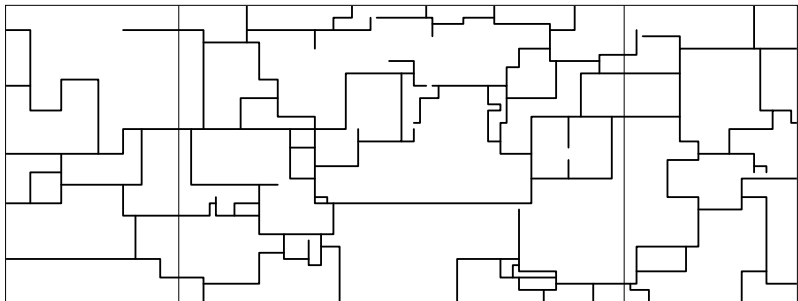
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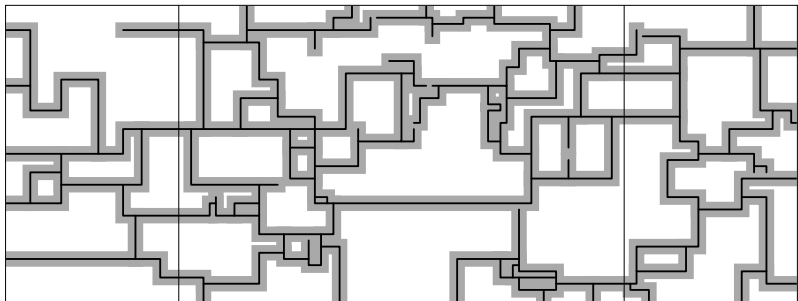


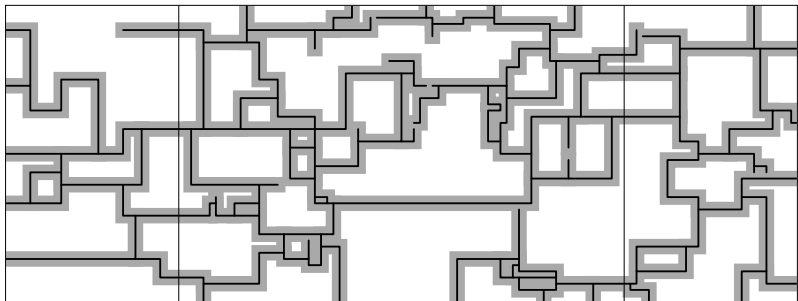
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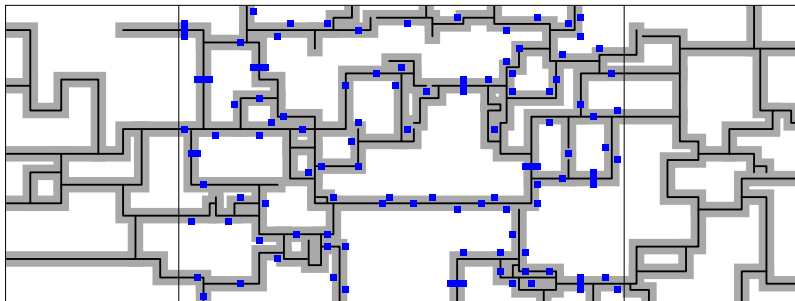


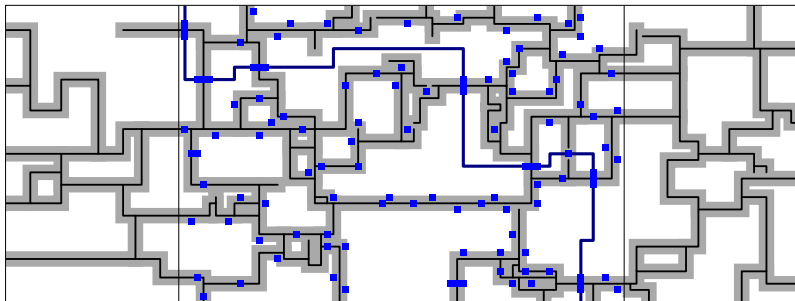
In both $\tilde{\omega}^\delta$ and $\bar{\omega}^\delta$!



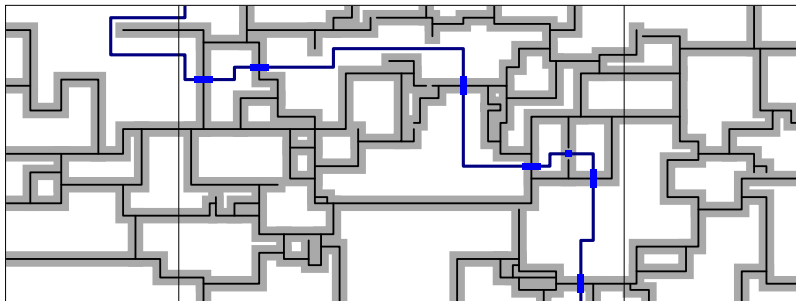








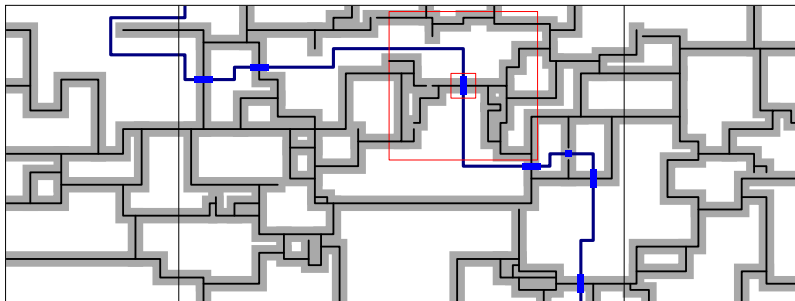
γ - vertical crossing with minimal number of enhanced points.



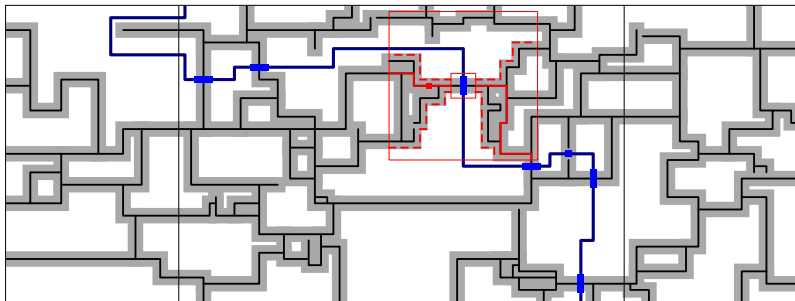
γ - vertical crossing with minimal number of enhanced points.

$\mathcal{X} = \{\text{enhanced points used by } \gamma\}$. If no crossing $\mathcal{X} = \emptyset$.

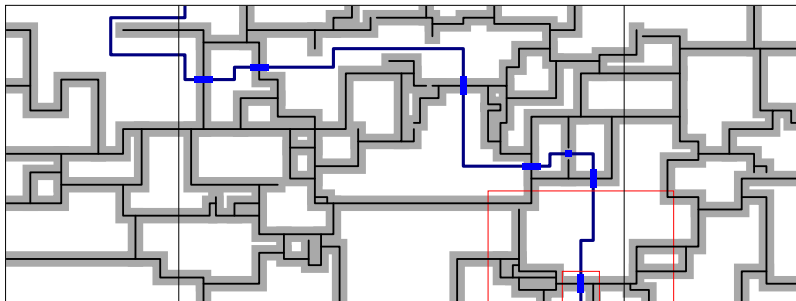
$$\mathbb{P}_{\rho_c, \delta}(\text{vertical crossing in } \tilde{\omega}^\delta) = \sum_{\mathcal{X} \neq \emptyset} \mathbb{P}_{\rho_c, \delta}(\mathcal{X} = X).$$



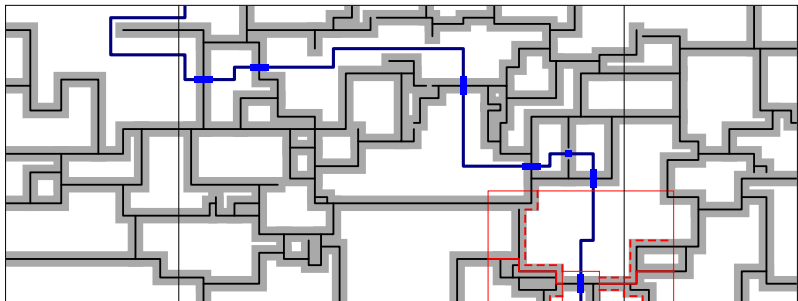
Annulus surrounding passage points but not containing passage points:
 6 arms or 4 half-plane arms in ω (possibly with one defect).



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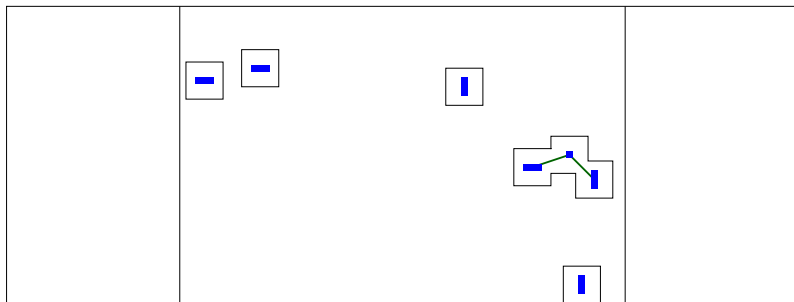


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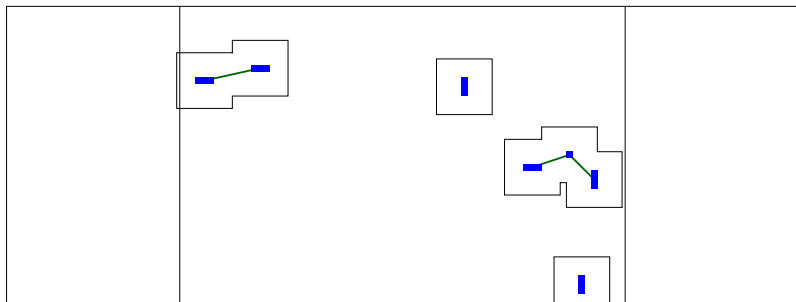


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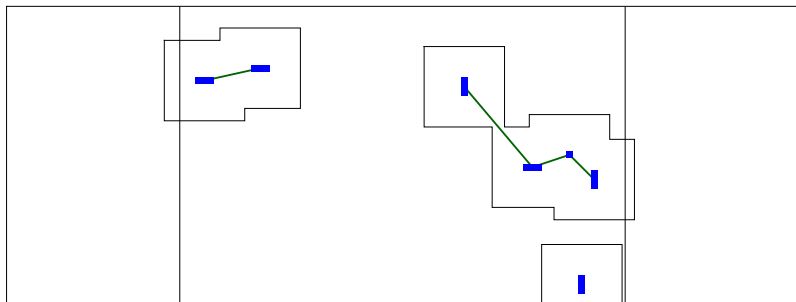
$$\mathbb{P}_{p_c} \left(\begin{array}{c} \text{Diagram 1: A square of side } R \text{ with a smaller square of side } r \text{ inside. Red dashed lines represent paths connecting the boundary of the inner square to the boundary of the outer square.} \\ \updownarrow \\ R \\ \updownarrow \\ r \end{array} \right) \leq \left(\frac{r}{R} \right)^{2+\lambda} \quad \mathbb{P}_{p_c} \left(\begin{array}{c} \text{Diagram 2: A square of side } R \text{ with a smaller square of side } r \text{ inside. Red solid lines represent paths connecting the boundary of the inner square to the boundary of the outer square.} \\ \updownarrow \\ R \\ \updownarrow \\ r \end{array} \right) \leq \left(\frac{r}{R} \right)^{2+\lambda}$$



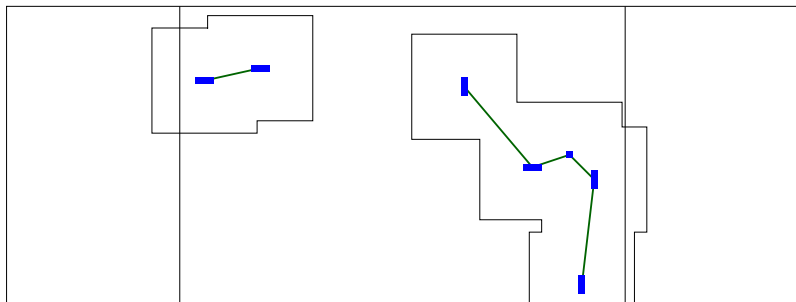
For a set X , what is $\mathbb{P}_{\rho_c, \delta}(\mathcal{X} = X)$?



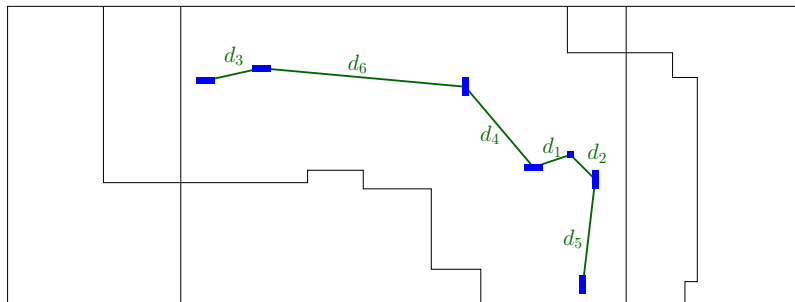
For a set X , what is $\mathbb{P}_{\rho_c, \delta}(\mathcal{X} = X)$?



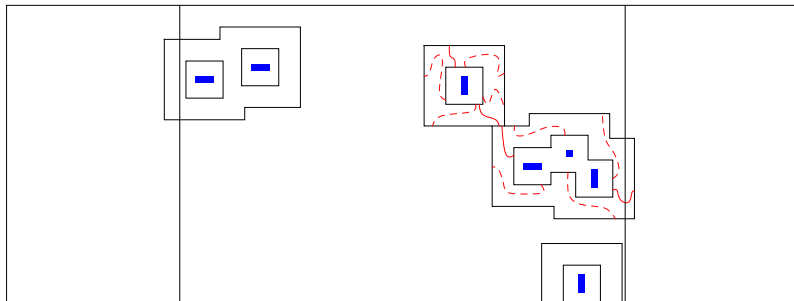
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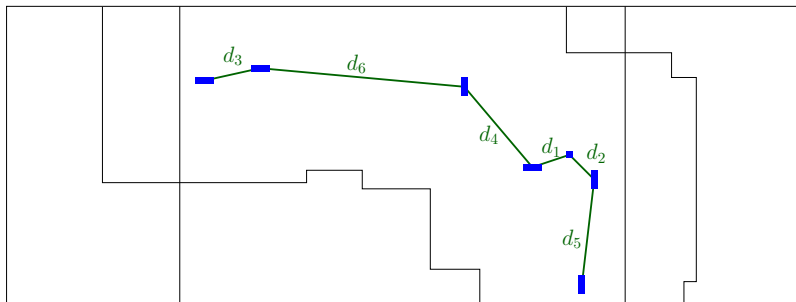
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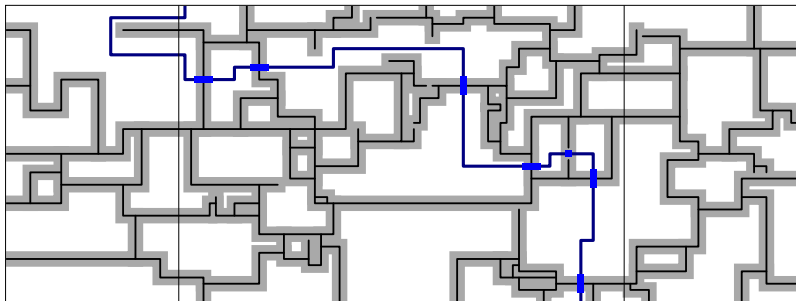
For a set X , what is $\mathbb{P}_{\rho_c, \delta}(\mathcal{X} = X)$?



$$\mathbb{P}_{p,\delta}(\mathcal{X} = X) \leq c^k n^{-2-\lambda} \prod_j d_j^{-2-\lambda} \times \delta^k,$$

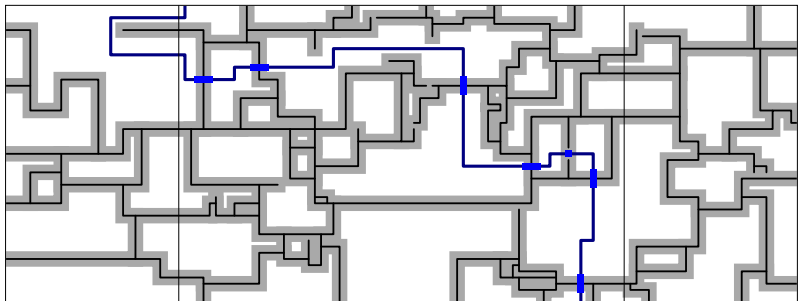
where d_1, \dots, d_k are the merger times of X .

$$\#\{X \text{ with merger times } d_1, \dots, d_k\} \leq C^k n^2 \prod_j d_j.$$



$$\begin{aligned} \mathbb{P}(\text{vertical crossing in } \tilde{\omega}^\delta) &\leq n^{-\lambda} \sum_{\mathcal{X}} \mathbb{P}_{\rho, \delta}(\mathcal{X} = X) \\ &\leq n^{-\lambda} \sum_{\substack{k \geq 1 \\ d_1, \dots, d_k}} \left(\delta^k c^k \prod_k d_k^{-1-\lambda} \right) = n^{-\lambda} \sum_{k \geq 1} \left(\delta c \sum_{d \geq 1} d^{-1-\lambda} \right)^k \rightarrow 0, \end{aligned}$$

for $\delta > 0$ small.



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Thank you!