Cooperative branching and pathwise duality for monotone systems

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Outline

- Pathwise duality for monotone systems

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- Cooperative branching

Probability kernels

For general sets S, T, let $\mathcal{F}(S, T)$ denote the set of all functions $f : S \to T$.

Let S, T be finite sets. A linear operator $A : \mathcal{F}(T, \mathbb{R}) \to \mathcal{F}(S, \mathbb{R})$ is uniquely characterized by its matrix $(A(x, y))_{x\in S,\ y\in T}$ through the formula

$$Af(x) := \sum_{y\in T} A(x, y)f(y) \qquad (x \in S).$$

A linear operator $K : \mathcal{F}(T, \mathbb{R}) \to \mathcal{F}(S, \mathbb{R})$ is a *probability kernel from S to T* if and only if

$$K(x, y) \geq 0 \quad \text{and} \quad \sum_{z\in T} K(x, z) = 1 \qquad (x \in S,\ y \in T).$$

Random mapping representations

Let K be a probability kernel from S to T.

A *random mapping representation* of K is an $\mathcal{F}(S,T)$-valued random variable M such that

$$K(x,y) = \mathbb{P}[M(x) = y] \qquad (x \in S,\ y \in T).$$

We say that K is *representable* in $\mathcal{G} \subset \mathcal{F}(S,T)$ if M can be chosen so that it takes values in $\mathcal{G}$.

Monotone probability kernels

For partially ordered sets S, T, let $\mathcal{F}_{\rm mon}(S, T)$ be the set of all monotone maps $m : S \to T$, i.e., those for which $x \leq x'$ implies $m(x) \leq m(x')$.

A probability kernel K is called *monotone* if

$$Kf \in \mathcal{F}_{\rm mon}(S, \mathbb{R}) \quad \forall f \in \mathcal{F}_{\rm mon}(T, \mathbb{R}),$$

and *monotonically representable* if K is representable in $\mathcal{F}_{\rm mon}(S, T)$.

Monotonical representability implies monotonicity:

$$f \in \mathcal{F}_{\rm mon}(T, \mathbb{R}) \quad \text{and} \quad x \leq x' \quad \Rightarrow$$
$$Kf(x) = \mathbb{E}\big[f\big(M(x)\big)\big] \leq \mathbb{E}\big[f\big(M(x')\big)\big] = Kf(x').$$

Monotone probability kernels

J.A. Fill & M. Machida (AOP 2001) (and also D.A. Ross (unpublished)) discovered that the converse does not hold. There are counterexamples with $S = T = \{0,1\}^2$.

On the positive side, Kamae, Krengel & O'Brien (1977) and Fill & Machida (2001) have shown that:

(Sufficient conditions for monotone representability)
Let S, T be finite partially ordered sets and assume that at least one of the following conditions is satisfied:

(i) *S is totally ordered.*

(ii) *T is totally ordered.*

Then any monotone probability kernel from S to T is monotonically representable.

Stochastic order

In particular, setting $S = \{1, 2\}$, this proves that if μ_1, μ_2 are probability laws on T such that

$$\mu_1 f \leq \mu_2 f \quad \forall f \in \mathcal{F}_{\rm mon}(T, \mathbb{R}),$$

then it is possible to couple random variables M_1, M_2 with laws μ_1, μ_2 such that $M_1 \leq M_2$.

Markov semigroups

Let S be finite. By definition, a *Markov semigroup* is a collection of probability kernels $(P_t)_{t\geq 0}$ on S such that

$$P_0 = \lim_{t\downarrow 0} P_t = 1 \quad \text{and} \quad P_s P_t = P_{s+t}.$$

Each Markov semigroup is of the form

$$P_t := e^{tG} = \sum_{n=0}^{\infty} \frac{1}{n!} t^n G^n \qquad (t \geq 0),$$

where the *generator* G satisfies

$$G(x,y) \geq 0 \quad (x \neq y) \quad \text{and} \quad \sum_{y\in S} G(x,y) = 0 \quad (x \in S).$$

Representability of semigroups

By definition, G is *representable in* $\mathcal{G} \subset \mathcal{F}(S,S)$ if G can be written as

$$Gf(x) = \sum_{m \in \mathcal{G}} r_m \big(f(m(x)) - f(x) \big),$$

where $(r_m)_{m \in \mathcal{G}}$ are nonnegative constants (rates).

(Representability of semigroups)
Assume that $\mathcal{G}$ is closed under composition and contains the identity map. Then the following statements are equivalent:

(i) *G can be represented in $\mathcal{G}$.*

(ii) *P_t can be represented in $\mathcal{G}$ for all $t \geq 0$.*

Stochastic flows

Proof of (i)⇒(ii) Let ω be a Poisson subset of $\mathcal{G} \times \mathbb{R}$ with local intensity $r_m \mathrm{d}t$ and let $\omega_{s,u} := \{(m,t) \in \omega : s < t \leq u\}$.
Define random maps $(\mathbf{X}_{s,u})_{s \leq u}$ by composing the maps in $\omega_{s,u}$ in the order of the time at which they occur:

$$\mathbf{X}_{s,u} := m_n \circ \cdots \circ m_1$$
$$\text{with} \quad \omega_{s,u} = \big\{(m_1, t_1), \ldots, (m_n, t_n)\big\}, \quad t_1 < \cdots < t_n.$$

The $(\mathbf{X}_{s,u})_{s \leq u}$ form a *stochastic flow:*

$$\mathbf{X}_{s,s} = 1 \quad \text{and} \quad \mathbf{X}_{t,u} \circ \mathbf{X}_{s,u} \quad (s \leq t \leq u),$$

with independent increments:

$$\mathbf{X}_{t_0,t_1}, \ldots, \mathbf{X}_{t_{n-1},t_n} \quad \text{independent for} \quad t_0 < \cdots < t_n.$$

Stochastic flows

If X_0 is independent of ω, then

$$X_t := \mathbf{X}_{0,t}(X_0) \qquad (t \geq 0)$$

defines a Markov process $(X_t)_{t\geq 0}$ with generator G, and

$$P_t(x, y) = \mathbb{P}[\mathbf{X}_{0,t}(x) = y]$$

gives the desired random mapping representation of the Markov semigroup $(P_t)_{t\geq 0}$ with generator G. ∎

We call the Poisson set ω a *graphical representation* of X.

Note: We have defined $\mathbf{X}_{s,t}$ right-continuous in s and t. As a result, $(X_t)_{t\geq 0}$ has right-continuous sample paths.

Duality

Two Markov processes X and Y with state spaces S and T are *dual* with *duality function* $\psi : S \times T \to \mathbb{R}$ iff

$$\mathbb{E}\big[\psi(X_t, Y_0)\big] = \mathbb{E}\big[\psi(X_0, Y_t)\big] \qquad (*).$$

for all deterministic initial states X_0 and Y_0.
If $(*)$ holds for deterministic initial states, then also for random initial states, provided X_t is independent of Y_0 and X_0 is independent of Y_t.

In terms of semigroups $(P_t)_{t\geq 0}, (Q_t)_{t\geq 0}$ and generators G, H, duality says

$$\begin{aligned} & P_t\psi = \psi Q_t^\dagger \qquad (t \geq 0), \\ \Leftrightarrow \quad & G\psi = \psi H^\dagger, \end{aligned}$$

where $A^\dagger$ denotes the adjoint of a matrix A.

Pathwise duality

Two maps $m : S \to S$ and $\hat{m} : T \to T$ are *dual* w.r.t. the duality function ψ iff

$$\psi\big(m(x), y\big) = \psi\big(x, \hat{m}(y)\big) \qquad (x \in S,\ y \in T).$$

Two stochastic flows $(\mathbf{X}_{s,t})_{s \leq t}$ and $(\mathbf{Y}_{s,t})_{s \leq t}$ with independent increments are *dual* w.r.t. the duality function ψ if:

(i) A.s. $\forall\ s \leq t$, the maps $\mathbf{X}_{s-,t-}$ and $\mathbf{Y}_{-t,-s}$ are dual w.r.t. ψ.

(ii) $(\mathbf{X}_{t_0-,t_1-}, \mathbf{Y}_{-t_1,-t_0}), \ldots, (\mathbf{X}_{t_{n-1},t_n}, \mathbf{Y}_{-t_n,-t_{n-1}})$ are independent for $t_0 < \cdots < t_n$.

To get a sensible definition, we have to take the left-continuous modification $\mathbf{X}_{s-,t-}$ (if $\mathbf{Y}_{s,t}$ is right-continuous as usual).

Pathwise duality

Two Markov processes X and Y are *pathwise dual* if they can be constructed from stochastic flows that are dual.
Pathwise duality implies duality:

$$\begin{aligned}\mathbb{E}\big[\psi(X_t, Y_0)\big] &= \mathbb{E}\big[\psi\big(\mathbf{X}_{0-,t-}(X_0), Y_0\big)\big] \\ &= \mathbb{E}\big[\psi\big(X_0, \mathbf{Y}_{-t,0}(Y_0)\big)\big] = \mathbb{E}\big[\psi(X_0, Y_t)\big].\end{aligned}$$

Even though pathwise duality is much stronger than duality, lots of well-known dualities can be realized as pathwise dualities.

Pathwise duality

(Pathwise duality) *If the generators G and H of X and Y have random mapping representations of the form*

$$Gf(x) = \sum_{m \in \mathcal{G}} r_m \big(f(m(x)) - f(x) \big),$$
$$Hf(x) = \sum_{m \in \mathcal{G}} r_m \big(f(\hat{m}(y)) - f(y) \big),$$

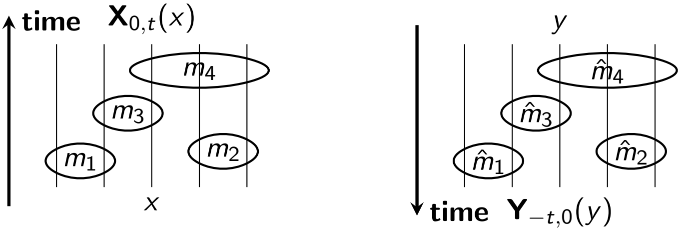
where each map $\hat{m}$ is a dual of m, then X and Y are pathwise dual.

Proof Given a graphical representation ω of X, we can define a graphical representation $\hat{\omega}$ for Y by

$$\hat{\omega} := \big\{ (\hat{m}, -t) : (m, t) \in \omega \big\}.$$

Then the stochastic flows $(\mathbf{X}_{s,t})_{s \leq t}$ and $(\mathbf{Y}_{s,t})_{s \leq t}$ associated with ω and $\hat{\omega}$ are dual. ∎

Pathwise duality



In this picture

$$\mathbf{X}_{0,t} = m_4 \circ \cdots \circ m_1 \quad \text{is dual to} \quad \mathbf{Y}_{-t,0} = \hat m_1 \circ \cdots \circ \hat m_4.$$

Invariant subspaces

Let $\mathcal{P}(S)$ be the set of all subsets of S.
Let $m^{-1} : \mathcal{P}(S) \to \mathcal{P}(S)$ denote the *inverse image map*

$$m^{-1}(A) := \{x \in S : m(x) \in A\}.$$

Observation m^{-1} is dual to m w.r.t. to the duality function

$$\psi(x, A) := 1_{\{x \in A\}}.$$

Consequence Each Markov process X with state space S (and given random mapping representation) has a pathwise dual Y with state space $\mathcal{P}(S)$ and generator

$$Hf(A) := \sum_{m \in \mathcal{G}} r_m \big(f(m^{-1}(A)) - f(A) \big)$$

In practise, this dual is not very useful since the space $\mathcal{P}(S)$ is very big. *Useful* duals are associated with *invariant subspaces* of $\mathcal{P}(S)$.

A bit of order theory

Let S be a finite partially ordered space. The "upset" and "downset" of $A \subset S$ are defined as

$$A^{\uparrow} := \{x \in S : x \geq a \text{ for some } a \in A\},$$
$$A^{\downarrow} := \{x \in S : x \leq a \text{ for some } a \in A\}.$$

A set $A \subset S$ is *increasing* (resp. *decreasing*) if $A^{\uparrow} = A$ (resp. $A^{\downarrow} = A$) and a *principal filter* (resp. *principal ideal*) if A is of the form $A = \{a\}^{\uparrow}$ (resp. $A = \{a\}^{\downarrow}$) for some $a \in S$. We let

$$\mathcal{P}_{\rm inc}(S) := \{A \subset S : A \text{ is increasing}\},$$
$$\mathcal{P}_{\rm !inc}(S) := \{A \subset S : A \text{ is a principal filter}\},$$
$$\mathcal{P}_{\rm dec}(S) := \{A \subset S : A \text{ is decreasing}\},$$
$$\mathcal{P}_{\rm !dec}(S) := \{A \subset S : A \text{ is a principal ideal}\}.$$

A bit of order theory

A partially ordered set S is *bounded from below* resp. *above* if there exists an element 0 resp. 1 such that

$$0 \leq x \quad (x \in S) \quad \text{resp. } x \leq 1 \quad (x \in S).$$

A *lattice* is a partially ordered set such that for every $x, y \in S$ there exist $x \vee y \in S$ and $x \wedge y \in S$ called the *supremum* or *join* and *infimum* or *meet* of x and y, respectively, such that

$$\{x\}^{\uparrow} \cap \{y\}^{\uparrow} = \{x \vee y\}^{\uparrow} \quad \text{and} \quad \{x\}^{\downarrow} \cap \{y\}^{\downarrow} = \{x \wedge y\}^{\downarrow}.$$

Finite lattices are bounded from below and above.

A map $m : S \to S$ is *additive* if

$$m(0) = 0 \quad \text{and} \quad m(x \vee y) = m(x) \vee m(y) \qquad (x, y \in S).$$

Monotone and additive maps

(Monotone and additive maps)
(i) *Let* S *and* T *be partially ordered sets and let* $m : S \to T$ *be a map. Then* m *is monotone if and only if*

$$m^{-1}(A) \in \mathcal{P}_{\rm dec}(S) \text{ for all } A \in \mathcal{P}_{\rm dec}(T).$$

(ii) *If* S *and* T *are finite lattices, then* m *is additive if and only if*

$$m^{-1}(A) \in \mathcal{P}_{\rm !dec}(S) \text{ for all } A \in \mathcal{P}_{\rm !dec}(S).$$

Dual spaces

Let S be a partially ordered set. A *dual* of S is a partially ordered set S' together with a bijection $S \ni x \mapsto x' \in S'$ such that

$$x \leq y \quad \text{if and only if} \quad x' \geq y'.$$

Example 1: For any partially ordered set S, we may take $S' := S$ but equipped with the reversed order, and $x \mapsto x'$ the identity map.

Example 2: If Λ is a set and $S \subset \mathcal{P}(\Lambda)$ is a set of subsets of Λ, equipped with the partial order of inclusion, then we may take for $x' := \Lambda \backslash x$ the complement of x and $S' := \{x' : x \in S\}$.

Additive systems duality

Let X be a Markov process in a finite lattice S.
Assume that the generator of X is representable in additive maps.
Then X has a pathwise dual that takes values in the invariant subspace $\mathcal{P}_{!\rm dec}(S) \subset \mathcal{P}(S)$.
A convenient way to encode an element $A \in \mathcal{P}_{!\rm dec}(S)$ is to write

$$A = \{y'\}^{\downarrow} \quad \text{with} \quad y \in S'.$$

Identifying $\mathcal{P}_{!\rm dec}(S) \cong S'$, the duality function becomes

$$\psi(x, y) = 1_{\{x \leq y'\}} = 1_{\{y \leq x'\}} \qquad (x \in S,\ y \in S').$$

(Additive duality) *A map $m : S \to S$ has a dual $m' : S' \to S'$ w.r.t. ψ if and only if m is additive. The dual map m' is unique and also an additive map.*

Siegmund's duality

Let $S = \{0, \ldots, n\}$ be totally ordered and let $S' := S$ equipped with the reversed order.
A map $m : S \to S$ is additive iff m is monotone and $m(0) = 0$.
Each such map has a dual $m' : S' \to S'$ that is monotone and satisfies $m(n) = n$.

(Siegmund's dual) *Let X be a monotone Markov process in S such that 0 is a trap. Then X has a dual Y w.r.t. to the duality function $\psi(x, y) := 1_{\{x \leq y\}}$. The dual process is also monotone and has n as a trap. Moreover, the duality can be realized in a pathwise way.*

Additive particle systems

Let $S = \mathcal{P}(\Lambda)$ with Λ a finite set, and let $x \mapsto x' \in S' := \mathcal{P}(\Lambda)$ denote the complement map $x' := \Lambda \backslash x$.

(Additive particle systems) *Let X be a Markov process in S whose generator can be represented in additive maps. Then X has a pathwise dual Y w.r.t. to the duality function $\psi(x, y) := 1_{\{x \cap y = \emptyset\}}$, and Y is also an additively representable Markov process.*

Examples: Voter model, contact process, exclusion process, systems of coalescing random walks.

Krone's duality

Steve Krone [AAP 1999] has studied a two-stage contact process, with state space of the form $S = \{0, 1, 2\}^\Lambda$.
He interprets $x(i) = 0, 1$, or 2 as an empty site, young, or adult organism, and defines maps

grow up	$a_i(x)(k) := 2$	if $k = i,\ x(i) = 1,$
give birth	$b_{ij}(x)(k) := 1$	if $k = j,\ x(i) = 2,\ x(j) = 0,$
young dies	$c_i(x)(k) := 0$	if $k = i,\ x(i) = 1,$
death	$d_i(x)(k) := 0$	if $k = i,$
grow younger	$e_i(x)(k) := 1$	if $k = i,\ x(i) = 2,$

where in all cases not mentioned, the maps have no effect.

Krone's duality

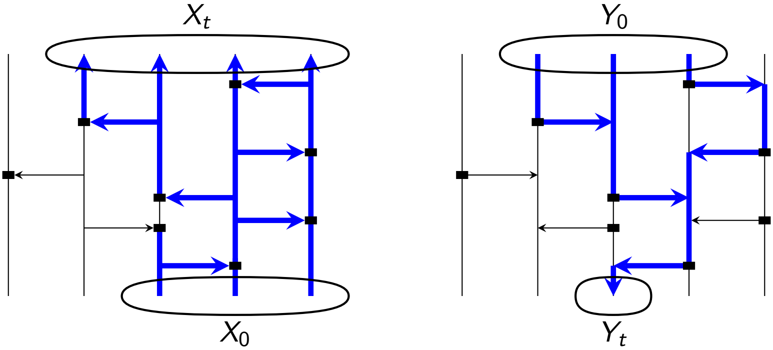
We set $S' := S$ and define $S \ni x \mapsto x' \in S'$ by $x'(i) := 2 - x(i)$. Then the duality function becomes

$$\psi(x,y) = 1_{\{x \leq y'\}} = 1_{\{x(i)+y(i) \leq 2 \ \forall \ i \in \Lambda\}}.$$

(Krone's dual) *The maps $a_i, b_{ij}, c_i, d_i, e_i$ are all additive and their duals are given by*

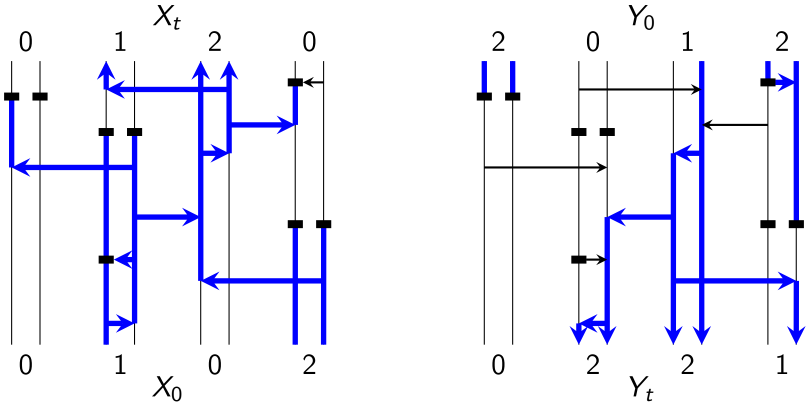
$$a'_i = a_i, \quad b'_{ij} = b_{ji}, \quad c'_i = e_i, \quad d'_i = d_i, \quad e'_i = c_i.$$

Percolation representations



Additive particle systems and their duals can be constructed in terms of open paths. In this example, X is a voter model and Y are coalescing random walks.

Percolation representations



Every additive Markov process X taking values ina finite lattice S has a percolation representation. If moreover S is a *distributive* lattice, then X and its dual Y can be represented together.

Monotone systems duality

Let X be a Markov process in a finite partially ordered set S.
Assume that the generator of X is representable in monotone maps.
Then X has a pathwise dual that takes values in the invariant subspace $\mathcal{P}_{\rm dec}(S) \subset \mathcal{P}(S)$.
A convenient way to encode an element $A \in \mathcal{P}_{\rm !dec}(S)$ is to write

$$A = \{B'\}^{\downarrow} \quad \text{with} \quad B \subset S'.$$

The duality function then becomes

$$\psi(x, B) = 1_{\{x \leq b' \text{ for some } b \in B\}}$$

For a monotone $m : S \to S$, we define $m^{\dagger} : \mathcal{P}(S') \to \mathcal{P}(S')$ and $m^* : \mathcal{P}(S') \to \mathcal{P}(S')$ by

$$m^{\dagger}(B)' := (m^{-1}(B'^{\downarrow}))_{\max} \quad \text{and} \quad m^*(B)' := \bigcup_{x \in B} (m^{-1}(\{x'\}^{\downarrow}))_{\max}.$$

Monotone systems duality

(Gray's (1986) dual) *The maps* $m^\dagger$ *and* m^* *are both dual to* m *w.r.t.* ψ*. Moreover,*

$$m^\dagger(B) = m^\dagger(B)_{\min} = m^*(B)_{\min},$$
$$m^*(B \cup C) = m^*(B) \cup m^*(C).$$

In the special case that S *is a lattice and* m *is additive,*

$$m^*(B) = m'(B) := \{m'(y) : y \in B\},$$

where m' *is the additive dual of* m*.*

Here $A_{\min} := \{x \in A : x \text{ is a minimal element of } A\}$

$$= \{x \in A : \nexists y \in A, y \neq x \text{ s.t. } y \leq x\}.$$

Cooperative branching

Let S be a finite lattice and let $m : S \to S$ be monotone. Then m is automatically *superadditive:*

$$m(x \vee y) \geq m(x) \vee m(y)$$

For monotone maps that are not additive, this inequality is strict. A good example is the *cooperative branching map*

$$\begin{aligned} 110 &\mapsto 111, \\ 100 &\mapsto 100, \\ 010 &\mapsto 010, \end{aligned}$$

which can be interpreted as two individuals cooperating to give birth to a third one.

Cooperative branching

DeMasi, Ferrari & Lebowitz [JSP 1986], *C. Noble* [AOP 1992], *R. Durrett* [JAP 1992], and *C. Neuhauser and S.W. Pacala* [AAP 1999] consider a model with cooperative branching, deaths, and fast stirring. They call this the *sexual reproduction process*.

The sexual reproduction process

$(X_t)_{t\geq 0}$ with $X_t = (X_t(i))_{i\in\mathbb{Z}}$ takes values in the space of all configurations $\ldots 101101001001 \ldots$ and evolve as:

$$
\begin{array}{lrcllr}
\text{(coop. bra.)} & 110 & \mapsto & 111 & \text{with rate} & \frac{1}{2}\lambda, \\
\text{(coop. bra.)} & 011 & \mapsto & 111 & \text{with rate} & \frac{1}{2}\lambda, \\
\text{(death)} & 1 & \mapsto & 0 & \text{with rate} & 1, \\
\text{(stirring)} & 10 & \mapsto & 01 & \text{with rate} & \varepsilon^{-1}, \\
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Interpretation:

- 'Sexual' reproduction.

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Interpretation:

- 'Sexual' reproduction.
- Competition for limited space.

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Interpretation:

- 'Sexual' reproduction.
- Competition for limited space.
- Death.

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Interpretation:

- 'Sexual' reproduction.
- Competition for limited space.
- Death.
- Migration.

A cooperative branching-coalescent

Let $(X_t)_{t\geq 0}$ with $X_t = (X_t(i))_{i\in\mathbb{Z}}$ take values in the space of all configurations $\ldots 101101001001 \ldots$ and evolve as:

(coop. bra.)	110	$\mapsto$	111	with rate	$\frac{1}{2}\lambda$,
(coop. bra.)	011	$\mapsto$	111	with rate	$\frac{1}{2}\lambda$,
(coal. RW)	10	$\mapsto$	01	with rate	$\frac{1}{2}$,
(coal. RW)	01	$\mapsto$	10	with rate	$\frac{1}{2}$,
(coal. RW)	11	$\mapsto$	01	with rate	$\frac{1}{2}$,
(coal. RW)	11	$\mapsto$	10	with rate	$\frac{1}{2}$.

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- Cooperative reproduction.

A cooperative branching-coalescent

Let $(X_t)_{t\geq 0}$ with $X_t = (X_t(i))_{i\in\mathbb{Z}}$ take values in the space of all configurations $\ldots 101101001001\ldots$ and evolve as:

(coop. bra.)	110	$\mapsto$	111	with rate	$\frac{1}{2}\lambda$,
(coop. bra.)	011	$\mapsto$	111	with rate	$\frac{1}{2}\lambda$,
(coal. RW)	10	$\mapsto$	01	with rate	$\frac{1}{2}$,
(coal. RW)	01	$\mapsto$	10	with rate	$\frac{1}{2}$,
(coal. RW)	11	$\mapsto$	01	with rate	$\frac{1}{2}$,
(coal. RW)	11	$\mapsto$	10	with rate	$\frac{1}{2}$.

Interpretation:

- Cooperative reproduction.
- Competition for limited space.

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\text{(coop. bra.)} & 011 & \mapsto & 111 & \text{with rate} & \frac{1}{2}\lambda, \\
\text{(coal. RW)} & 10 & \mapsto & 01 & \text{with rate} & \frac{1}{2}, \\
\text{(coal. RW)} & 01 & \mapsto & 10 & \text{with rate} & \frac{1}{2}, \\
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- Cooperative reproduction.
- Competition for limited space.
- Migration.

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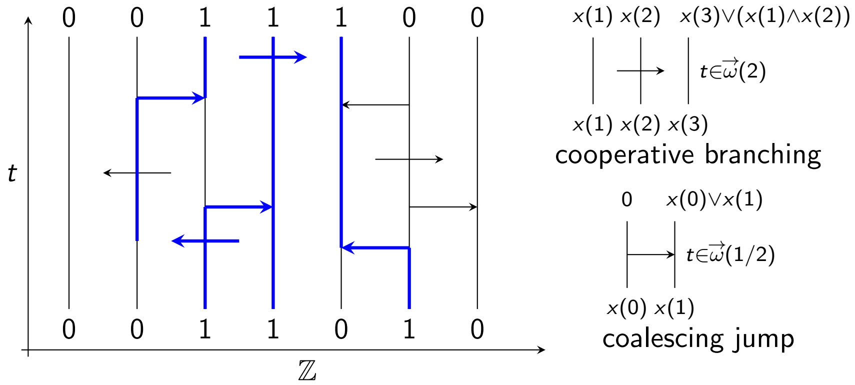
\end{array}
$$

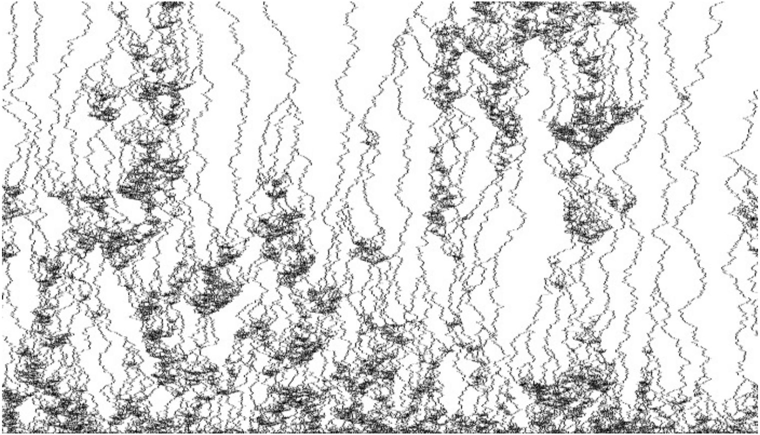
Interpretation:

- Cooperative reproduction.
- Competition for limited space.
- Migration.
- No spontaneous deaths!

A graphical representation



A cooperative branching-coalescent



Time = upwards, black = a particle, $\lambda = 2.333$.

Critical points

Define

- The process *survives* if $\mathbb{P}^x\big[|X_t| > 1\ \forall t \geq 0\big] > 0$ for some, and hence for all initial states with $1 < |x| < \infty$ particles. Note: a single particle can neither die nor reproduce!
- The process is *stable* if there exists an invariant law that is concentrated on nonzero states.

Monotonicity implies that there exist $\lambda_{\rm c}, \lambda'_{\rm c}$ such that

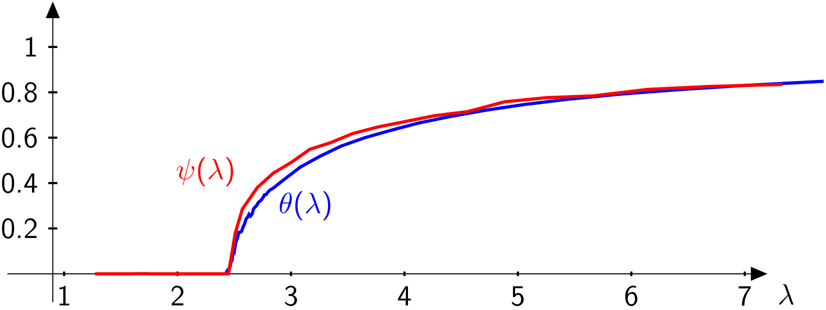
- The process survives for $\lambda > \lambda_{\rm c}$ and dies out for $\lambda < \lambda_{\rm c}$.
- The process is stable for $\lambda > \lambda'_{\rm c}$ and unstable for $\lambda < \lambda'_{\rm c}$.

[Sturm & S. '14] $1 \leq \lambda_{\rm c}, \lambda'_{\rm c} < \infty$.

Numerically: $\lambda_{\rm c} \approx \lambda'_{\rm c} \approx 2.47 \pm 0.02$.

Open problem: Prove that $\lambda_{\rm c} = \lambda'_{\rm c}$.

Critical points



$\psi(\lambda) := \mathbb{P}\big[|X_t| > 1 \ \forall t \geq 0\big]$ starting with two particles on neighboring sites.

$\theta(\lambda) := \mathbb{P}[X_\infty(0) = 1]$ where X_∞ distributed according to the upper invariant law.

The subcritical regime

Consider

$$\mathbb{P}\big[|X_t| > 1\big] \qquad \text{with} \quad X_0 = \delta_0 + \delta_1 \quad \text{(two particles)},$$
$$\mathbb{P}\big[X_t(0) = 1\big] \qquad \text{with} \quad X_0 = \underline{1} \quad \text{(fully occupied)}.$$

[Bezuidenhout & Grimmett '91] For the contact process, in the subcritical regime $\lambda < \lambda_{\rm c}$, both quantities decay exponentially fast to zero.

[Sturm & S. '14] For the cooperative branching-coalescent, both quantities decay not faster than as $t^{-1/2}$. For $\lambda \leq \frac{1}{2}$, this is the exact rate of convergence.

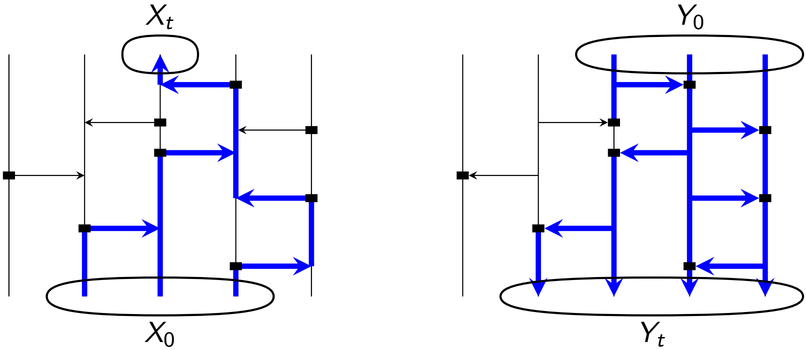
Proof of the lower bound: By monotonicity, we can estimate the cooperative branching-coalescent by a pure coalescent, for which both quantities decay like $t^{-1/2}$.

Proof of the upper bound

The generator of the process has the random mapping representation

$$Gf(x) = \lambda \sum_{i\in\mathbb{Z}} \big(\tfrac{1}{2} f(\vec{\mathrm{coop}}_i(x)) + \tfrac{1}{2} f(\overleftarrow{\mathrm{coop}}_i(x)) - f(x)\big) + \sum_{i\in\mathbb{Z}+\frac{1}{2}} \big(\tfrac{1}{2} f(\vec{\mathrm{rw}}_i(x)) + \tfrac{1}{2} f(\overleftarrow{\mathrm{rw}}_i(x)) - f(x)\big).$$

Proof of the upper bound



The coalescing random walk map $\vec{\mathrm{rw}}_i$ is dual to the voter model map $\overleftarrow{\mathrm{vot}}_i$ in the sense of additive systems duality, and likewise $\overleftarrow{\mathrm{rw}}_i$ is dual to $\vec{\mathrm{vot}}_i$.

Proof of the upper bound

The cooperative branching maps $\overrightarrow{\mathrm{coop}}_i$ and $\overleftarrow{\mathrm{coop}}_i$ are not additive, but they are still monotone, so we resort to Gray's dual map m^* and the duality function

$$\begin{aligned}\psi(x,Y) &= 1_{\{x \le y' \text{ for some } y \in Y\}}\\ &= 1_{\{x \wedge y = 0 \text{ for some } y \in Y\}},\end{aligned}$$

or equivalently,

$$\phi(x,Y) := 1 - \psi(x,Y) = 1_{\{x \wedge y \neq 0 \text{ for all } y \in Y\}}.$$

The dual process Y_t takes values in the space $\mathcal{P}_{\rm fin}\big(\{0,1\}^{\mathbb{Z}}\big)$ of all finite collections of "voter model configurations".

Proof of the upper bound

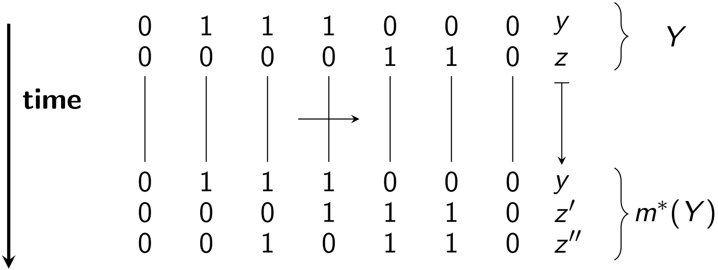
We recall that if m is an additive map, m' is its additive dual and m^* is Gray's dual map, then

$$m^*(B) = m'(B) := \{m'(y) : y \in B\}.$$

In particular, if $m = \vec{\mathrm{rw}}_i$ or $= \overleftarrow{\mathrm{rw}}_i$ is a coalescing random walk map, then m^* is a voter model map, applied to all configurations $y \in Y_t$ simultaneously.

In the absence of cooperative branching, Y_t is a collection of coupled voter models that evolve simultaneously.

Proof of the upper bound



For the cooperative branching maps $m = \overrightarrow{\mathrm{coop}}_i$ and $= \overleftarrow{\mathrm{coop}}_i$, application of Gray's dual map m^* can in some cases increase the number of elements of the set Y_t.

Proof of the upper bound

Since the full dual is (so far) too complicated to work with, we resort to a (pathwise) *subdual*, which satisfies

$$\phi\big(\mathbf{X}_{s,t}(x), y\big) \leq \phi\big(x, \mathbf{Y}_{-t,-s}(y)\big).$$

Each element of the subdual is a voter model configuration of the form

$$\cdots 00000111111000011111 \cdots$$

with exactly three *interfaces*, i.e., sites where a 0 borders a 1.

Under nearest-neighbor voter dynamics, it is known that such voter configurations survive till time t with a probability that decays as $t^{-3/2}$.

For $\lambda \leq 1/2$, the probability that an element of Y_t creates another element during its lifetime is ≤ 1 and the proof follows from comparison with subcritical branching. ∎