Biased activated random walks

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LAGA (Université Paris 13)

Probability seminar
University of Bristol — 23 April 2016

Plan of the talk

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1. Introduction of the model, results
   - ARW: Definition of the model
   - Motivations
   - Results
2. Elements of proofs
   - Fixation for $\mu \leq \frac{\lambda}{1+\lambda}$
   - Non-fixation with $\mu < 1$ in case of bias: statement
   - Mixed use of site-wise and particle-wise constructions
   - Particle fixation
   - Application to biased walks on $\mathbb{Z}^d$
   - Existence
3. Conclusion

Activated Random Walks (ARW) — Quick presentation

Dynamics: Particles evolve in continuous time on $\mathbb{Z}^d$, and can be either

- active, in **state A**: jump at rate 1 according to law $p(\cdot)$, independently from each other *(random walks)*;
- passive (sleeping), in **state S**: do not move.

Two kinds of mutations/interactions happen:

- $A \to S$ at rate λ: each particle gets asleep at rate λ (independently);
- $A + S \to 2A$ immediately: active particles awake the others on same site.

NB. Mutations $A \to S$ are only effective when the particle A is alone
$\Rightarrow$ *On each site, there is either nothing, one S, or any number of A particles.*

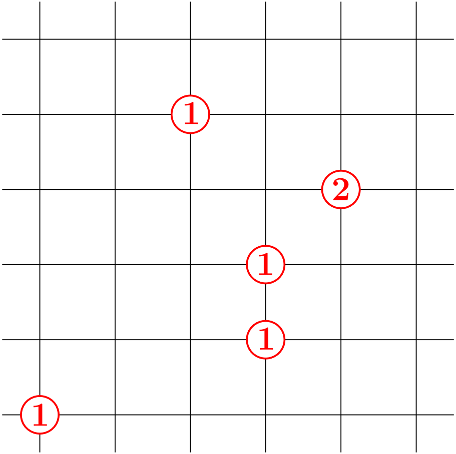
Dynamics for a finite number of particles

At each time, a uniformly chosen active particle may

$\to$ Jump, with prob. $\frac{1}{1+\lambda}$, with law $p(\cdot)$, (and wake up hit particles)

or

$\to$ Attempt to deactivate, with prob. $\frac{\lambda}{1+\lambda}$.



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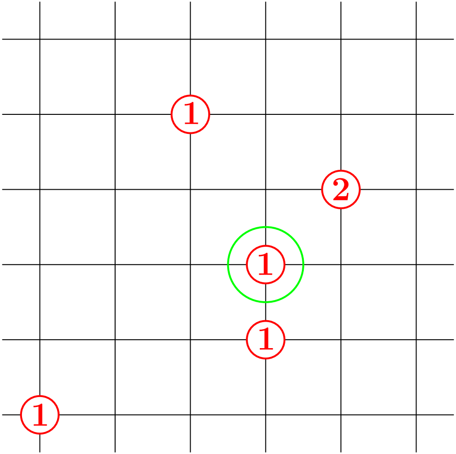

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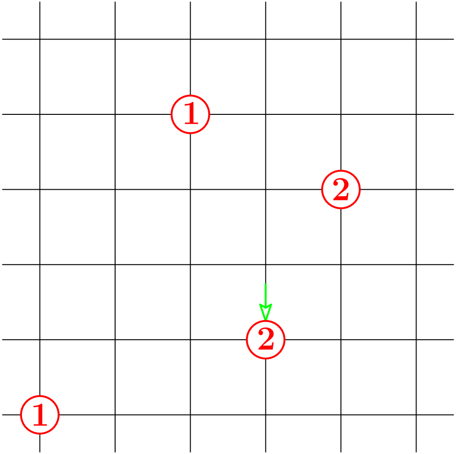

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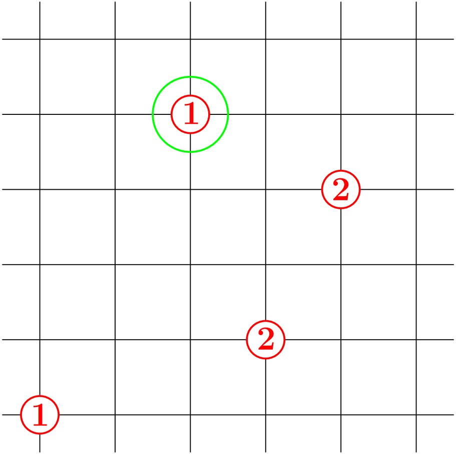

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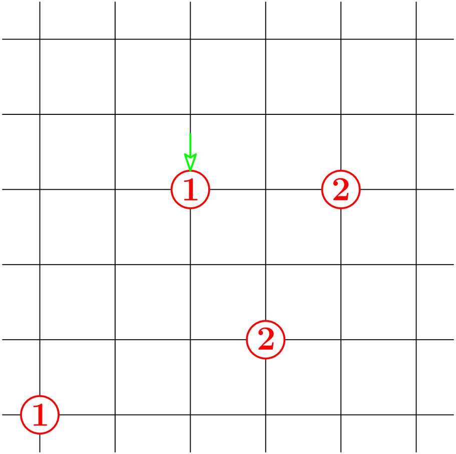

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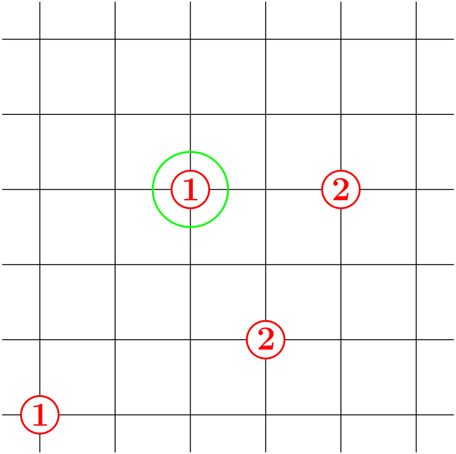

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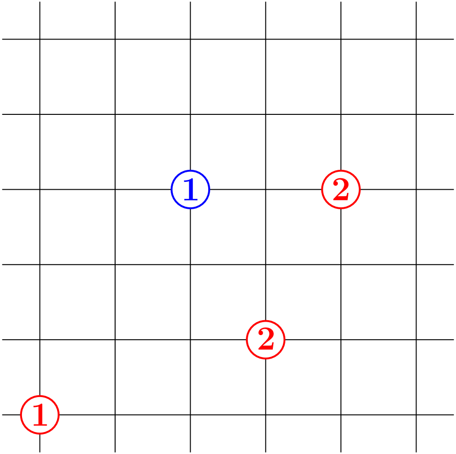

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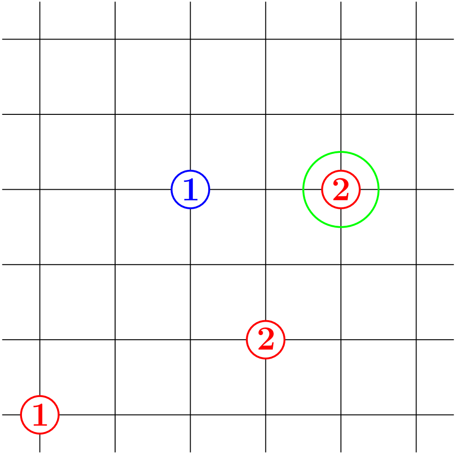

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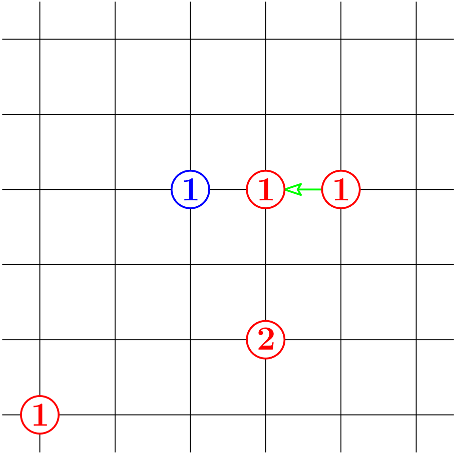

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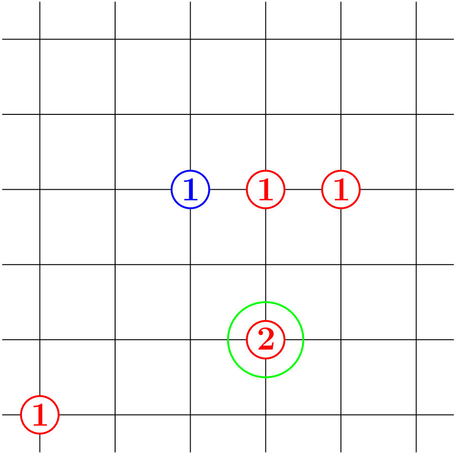

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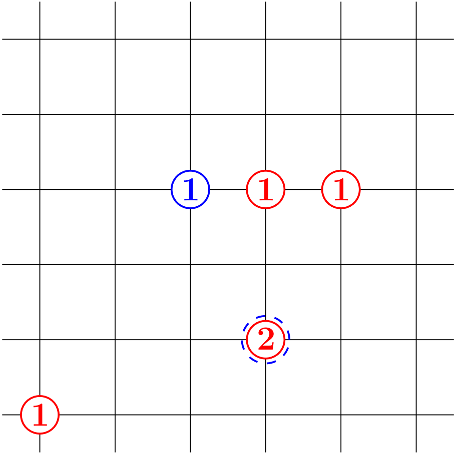

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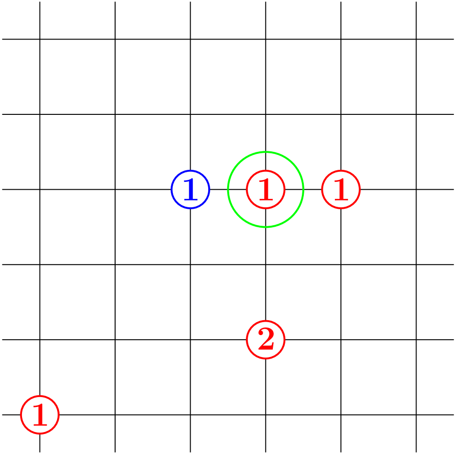

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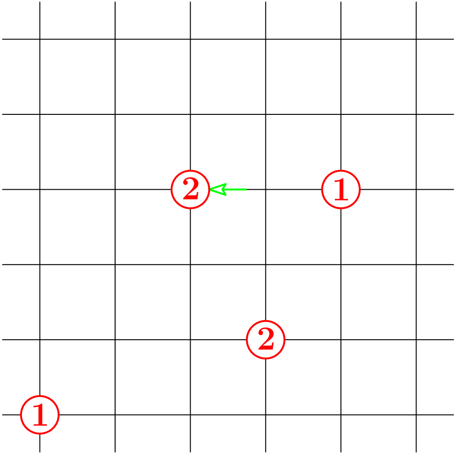

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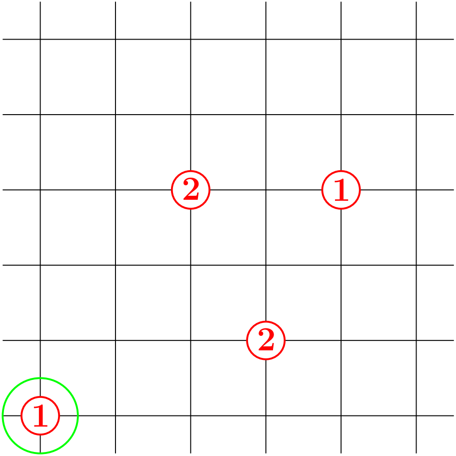

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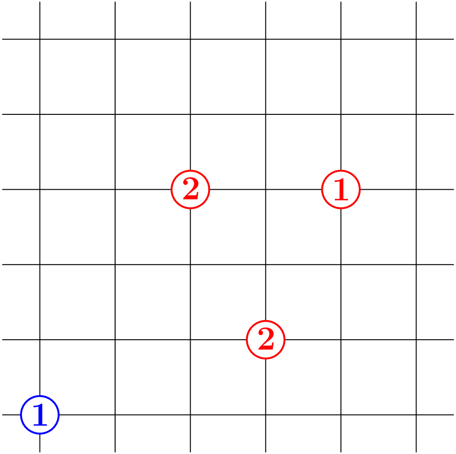

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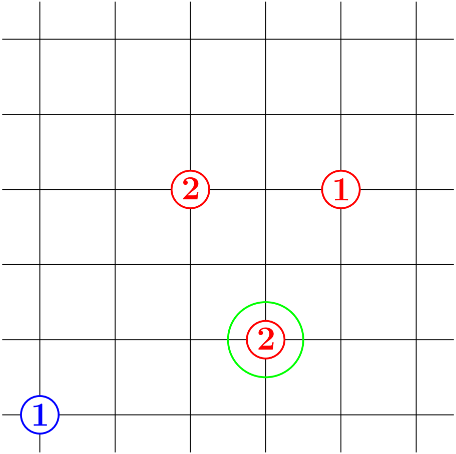

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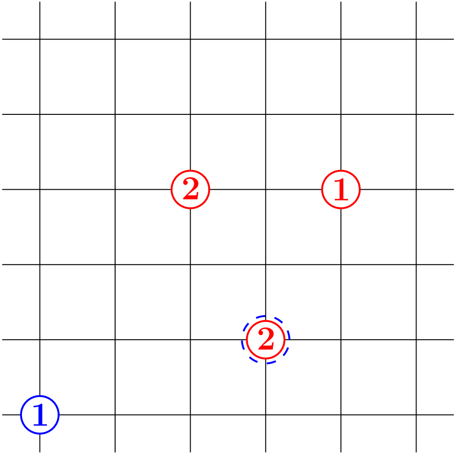

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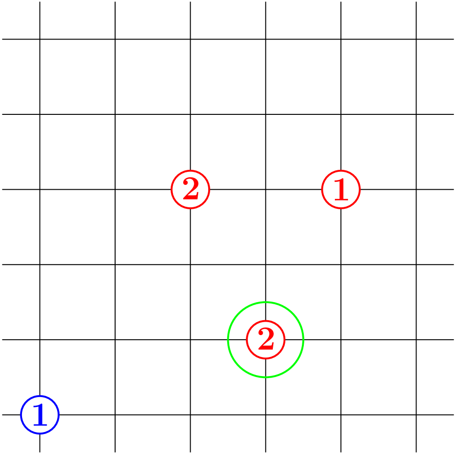

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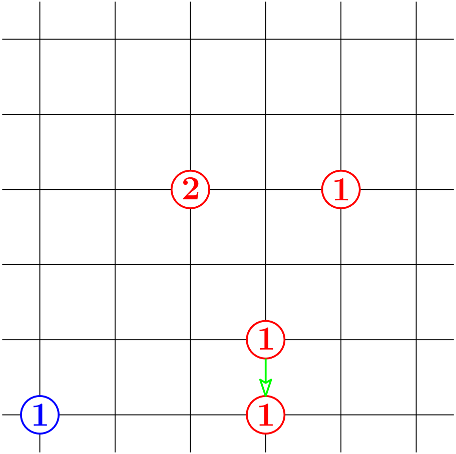

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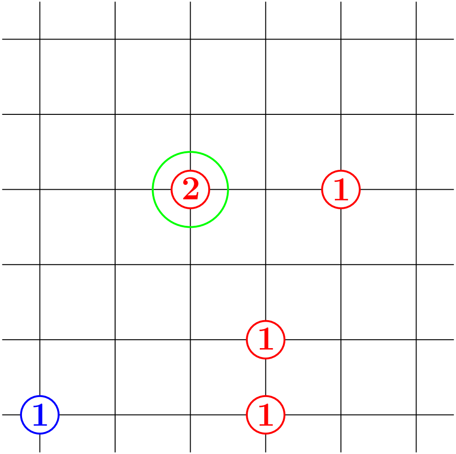

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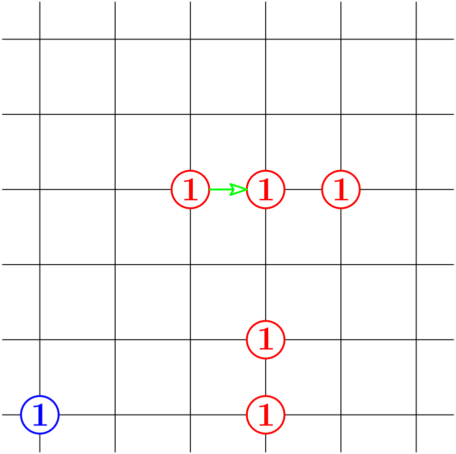

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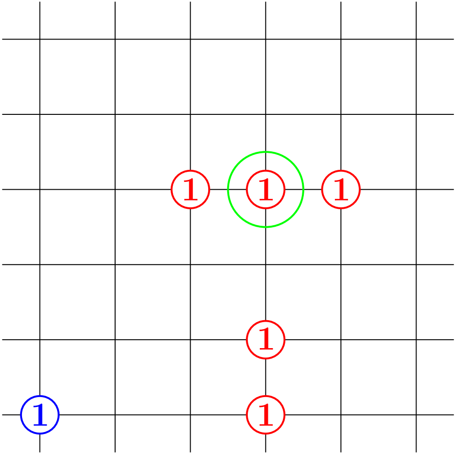

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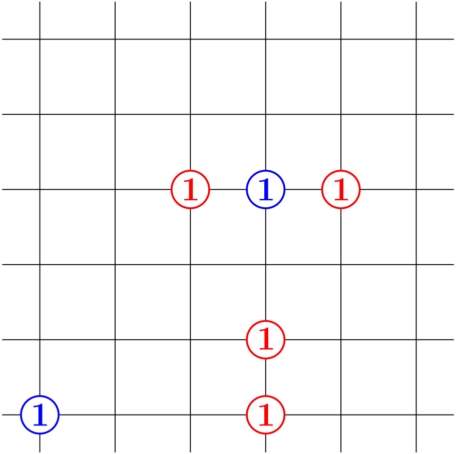

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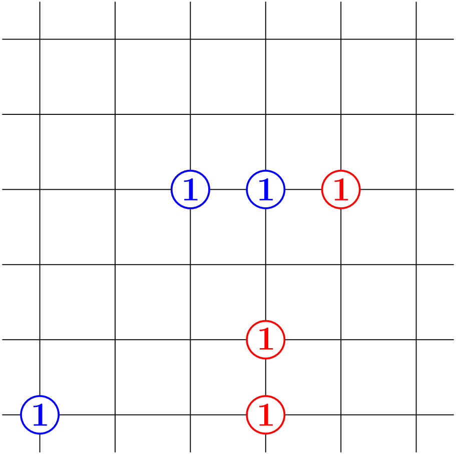

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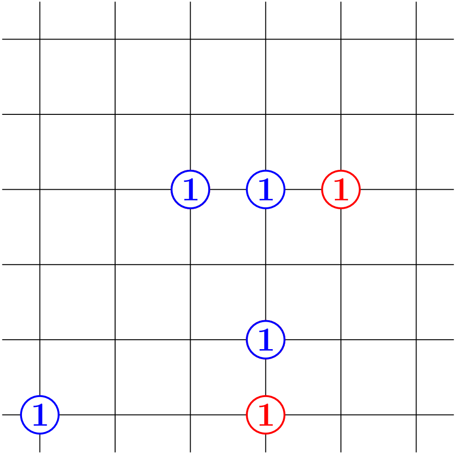

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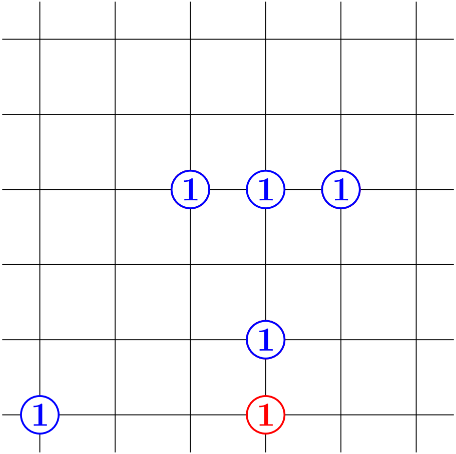

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After a while, every particule is passive: configuration is *stable*.

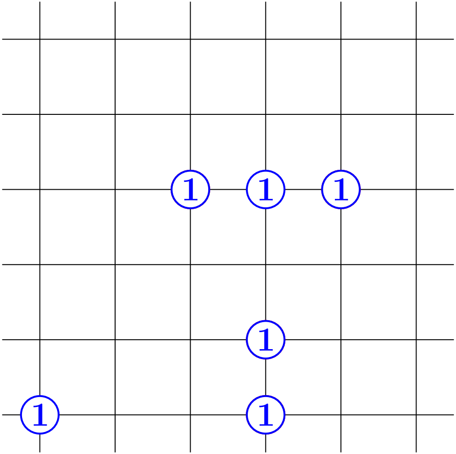
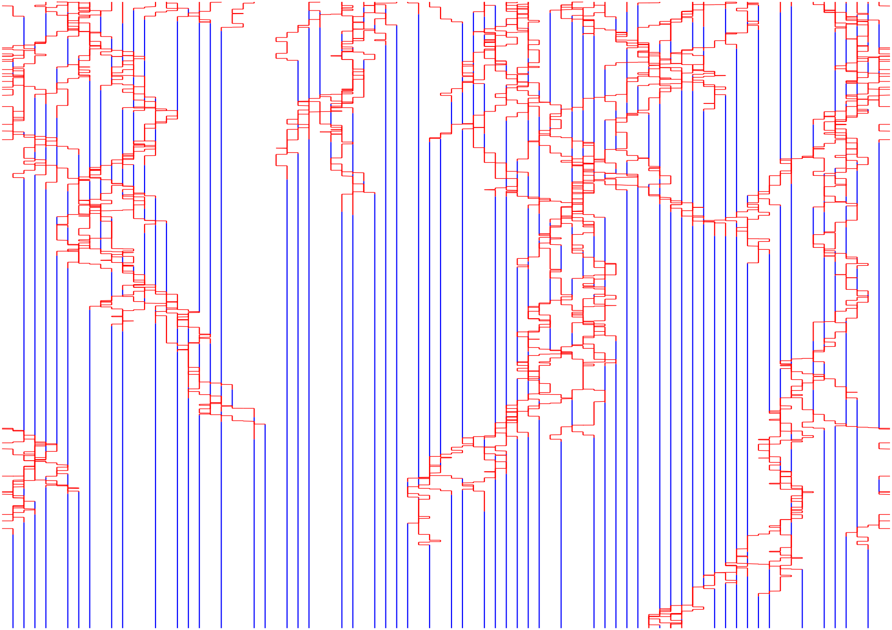


Illustration of the 1D case: space-time representation



Définition of the model

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- active, in **state A**: jump at rate 1 according to law $p(\cdot)$, independently from each other *(random walks)*;
- passive (sleeping), in **state S**: do not move.

Two kinds of mutations/interactions happen:

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- $A + S \to 2A$ immediately: active particles awake the others on same site.

NB. Mutations $A \to S$ are only effective when the particle A is alone
$\Rightarrow$ *On each site, there is either nothing, one S, or any number of A particles.*

Parameters:

- jump distribution $p(\cdot)$ on $\mathbb{Z}^d$
- sleeping rate $\lambda \in (0, \infty)$
- initial configuration of A particles (finite support, or i.i.d. in general).

Behaviors of interest:

- **fixation**: in any finite box, activity vanishes eventually;
- **non-fixation**: in any finite box, activity goes on forever.

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3. Conclusion

Let μ denote the initial density of particles (for i.i.d. initial configuration).

A **phase transition** is *expected* to happen: $\exists \mu_c(\lambda) \in (0,1)$ s.t.

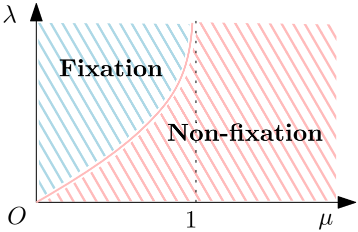
- for $\mu < \mu_c(\lambda)$, a.s. fixation;
- for $\mu > \mu_c(\lambda)$, a.s. non-fixation.

Furthermore, $\lambda \mapsto \mu_c(\lambda)$ is increasing, $\mu_c(0^+) = 0$ and $\mu_c(\infty) = 1$.

Or also: for $\mu \geq 1$ then a.s. fixation and, for $\mu < 1$, $\exists \lambda_c(\mu) \in (0,\infty)$ s.t.

- for $\lambda > \lambda_c(\mu)$, a.s. fixation;
- for $\lambda < \lambda_c(\mu)$, a.s. non-fixation.

Furthermore, $\mu \mapsto \lambda_c(\mu)$ is increasing, $\lambda_c(0^+) = 0$ and $\lambda_c(1^-) = +\infty$.



- Existence of μ_c and λ_c by monotonicity, within each increasing family of initial distributions (Poisson(μ), or Bernoulli(μ) for example).

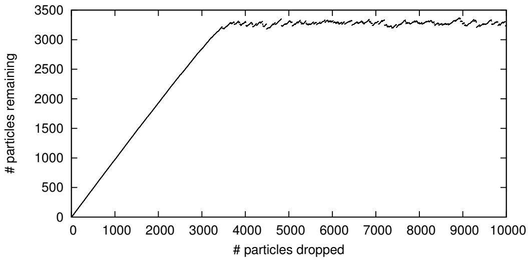
- More difficult, and partly conjectural: nontrivial bounds, limits, and universality of μ_c, λ_c with respect to initial distribution (but not with respect to $p(\cdot)$)

Motivations: 2. Self-Organized Criticality (physics)

ARW relate to the **sandspile** model, and have also been introduced as an instance of **self-organized criticality**: spontaneous (i.e. without tuning a parameter) critical behavior (polynomial decrease of correlations...). To obtain this, one "progressively augments the initial configutation":
Assume λ fixed. Inside a *finite* box,

- Drop a new particle at random,
- Stabilize the configuration by running the dynamics inside the box and by freezing particles that exit,

and repeat.



$\hookrightarrow$ Dynamics reach a stationary regime which, in large volume, should satisfy SOC.

Abelian sandpile model

Integer-valued configuration $\eta : \mathbb{Z}^d \to \mathbb{N}$ (number of grains of sand)
Deterministic rule of toppling at $x \in \mathbb{Z}^d$:
if $\eta(x) \geq 2d$ (η is said to be "unstable at x"), one defines

$$T_x\eta = \eta - 2d\delta_x + \sum_{y\sim x} \delta_y.$$

If $\eta(x) < 2d$ for all $x \in \mathbb{Z}^d$, η is called *stable*.

Abelian property

- For all $x \neq y$ where η is unstable, $T_xT_y\eta = T_yT_x\eta$.
- If a sequence of topplings leads to a stable configuration, then the latter does not depend on the toppling sequence, and neither does the number of topplings.
- Monotonicity: if $\eta \leq \eta'$, stabilizing η' requires more topplings (at each site) than stabilizing η.

Variant: *stochastic* sandpiles, where each grain is sent with law $p(\cdot)$.
$\rightarrow$ abelian property is preserved **provided** a stack of instructions is given at each site, to be used by any particle sitting there (*Diaconis-Fulton* construction, cf. DLA)

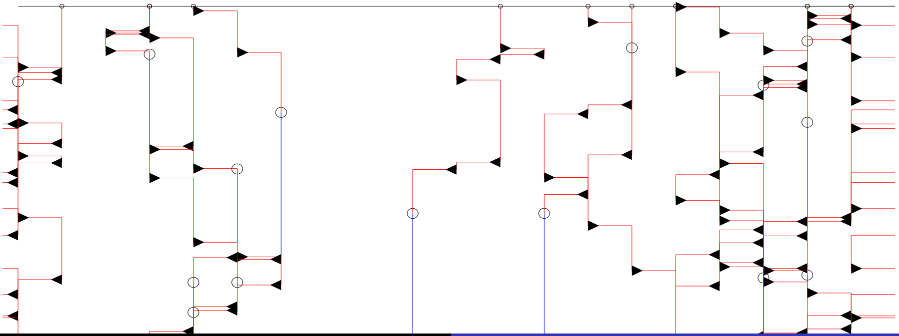
Diaconis-Fulton coupling for ARW

Let $\eta_0 \in \mathbb{N}^{\mathbb{Z}^d}$ be a finitely supported configuration.
Let $\mathscr{P} = (\mathscr{P}_x)_{x\in\mathbb{Z}^d}$ be i.i.d. Poisson processes of intensity 1 (*clocks*).
Let $\mathscr{I} = (I_{x,n}\,;\,n\in\mathbb{N})_{x\in\mathbb{Z}^d}$ be i.i.d. *stacks* of i.i.d. *instructions* in $\mathbb{Z}^d \uplus \{S\}$, s.t.

$$I_{x,n} = \begin{cases} y & \text{with probability } \frac{p(y)}{1+\lambda}, \text{ for all } y\in\mathbb{Z}^d; \qquad \textit{(jump to } x+y\textit{)} \\ S & \text{with probability } \frac{\lambda}{1+\lambda}. \qquad \textit{(attempt to sleep)} \end{cases}$$

"Diaconis-Fulton coupling" is a construction of the ARW from η_0 using $\mathscr{I}$ and $\mathscr{P}$:

- Clocks run at speed equal to number of *active* particles on each site;
- When the clock at x rings, we apply to x the first unused instruction at x.



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This actually defines two processes: $(\eta_t)_{t\geq 0}$ (ARW) and $(h_t)_{t\geq 0}$ (odometer) where
- $\eta_t(x)$ is the number of particles at x at time t (or 'S' if one sleepy particle);
- $h_t(x)$ counts how many instructions have been read at x by time t.
$\rightsquigarrow$ η and h are functions of $(\eta_0, \mathscr{I}, \mathscr{P})$ and such that:

Lemma

1. *(Abelian property) The final state η_∞, h_∞ does not depend on $\mathscr{P}$.*
2. *(Monotonicity by addition of particles) For all $t \geq 0$, h_t increases with η_0.*
3. *(Monotonicity by removal of S) For all $t \geq 0$, h_t increases when some 'S' instructions are replaced by neutral instructions (i.e. 0).*

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$$I_{x,n} = \begin{cases} y & \text{with probability } \frac{p(y)}{1+\lambda}, \text{ for all } y\in\mathbb{Z}^d; \qquad \textit{(jump to } x+y\textit{)} \\ S & \text{with probability } \frac{\lambda}{1+\lambda}. \qquad \textit{(attempt to sleep)} \end{cases}$$

"Diaconis-Fulton coupling" is a construction of the ARW from η_0 using $\mathscr{I}$ and $\mathscr{P}$:

- Clocks run at speed equal to number of *active* particles on each site;
- When the clock at x rings, we apply to x the first unused instruction at x.

This actually defines two processes: $(\eta_t)_{t\geq 0}$ (ARW) and $(h_t)_{t\geq 0}$ (odometer) where
- $\eta_t(x)$ is the number of particles at x at time t (or 'S' if one sleepy particle);
- $h_t(x)$ counts how many instructions have been read at x by time t.
$\rightsquigarrow$ η and h are functions of $(\eta_0, \mathscr{I}, \mathscr{P})$ and such that:

Lemma

1. *(Abelian property) The final state η_∞, h_∞ does not depend on $\mathscr{P}$.*
2. *(Monotonicity by addition of particles) For all $t\geq 0$, h_t increases with η_0.*
3. *(Monotonicity by removal of S) For all $t\geq 0$, h_t increases when some 'S' instructions are replaced by neutral instructions (i.e. 0).*

For i.i.d. η_0, $P(\text{non-fixation}) = P\big(h_\infty(0;\eta_0\cdot\mathbf{1}_V,\mathscr{P},\mathscr{I}) \nearrow +\infty \text{ as } V\nearrow\mathbb{Z}^d\big) \in \{0,1\}$

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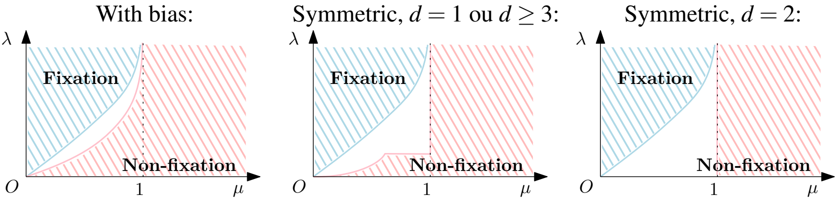
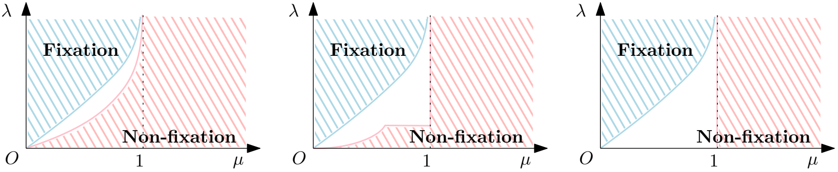
In a summary,

- $\mu_c \leq 1$ in very great generality;
- $\mu_c \geq \frac{\lambda}{1+\lambda} > 0$ in great generality (amenable graphs);
- $\mu_c(0^+) = 0$, and $\mu_c < 1$ for all λ in the case of **biased random walks**;
- $\mu_c(0^+) = 0$ (and thus $\mu_c < 1$ for small λ) if $d = 1$ or $d \geq 3$ (or transient graph).



With bias: Symmetric, $d = 1$ ou $d \geq 3$: Symmetric, $d = 2$:



Rolla–Sidoravicius '09 $d = 1$

To nearest-neighbours, $\mu_c(\lambda) \geq \frac{\lambda}{1+\lambda}$. For any jump law, $\mu_c(\lambda) > 0$.

Shellef '10, Amir–Gurel-Gurevitch '10 $\forall d \geq 1$

For any jump law, $\mu_c(\lambda) \leq 1$.

Sidoravicius–Teixeira '14 $\forall d \geq 1$

For the simple symmetric random walk, $\mu_c(\lambda) > 0$.

Taggi '15 $d = 1$

For biaised walks, $\mu_c \leq 1 - F(\lambda)$ where $F(\cdot) > 0$ et $F(0^+) = 1$.

+ *non-fixation criterion if $d \geq 2$, depending on the law of η_0.*

Rolla–T. '15 $d \geq 2$

For biased walks, $\mu_c \leq 1 - F(\lambda)$ where $F(\cdot) > 0$ and $F(0^+) = 1$.

Basu–Ganguly–Hoffman '15 $d = 1$

For the simple symmetric random walk: for all $\mu > 0$, $\lambda_c(\mu) > 0$.

Stauffer–Taggi '15 $\forall d \geq 1$

$\mu_c(\lambda) \geq \frac{\lambda}{1+\lambda}$, and $\mu_c(0^+) = 0$ if transient ($d \geq 3$).

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Theorem (Stauffer–Taggi '15)

On any amenable graph, $\mu_c(\lambda) \geq \frac{\lambda}{1+\lambda}$.

Proof. Let $V \subset \mathbb{Z}^d$, finite. Let $\mathbb{P}_V$ denote the law of ARW starting from $\eta_0 \cdot \mathbf{1}_V$, and $\theta_V(x) = \mathbb{P}(\text{after stabilization of } V\text{, there remains a particle at } x) = \mathbb{P}_V(\eta_\infty(x) = S)$.

$\rightsquigarrow \frac{1}{|V|}\sum_{x\in V}\theta_V(x) =$ "mean density of particles in V after stabilization" $\leq \mu$

Lemme

$\theta_V(x) \geq \frac{\lambda}{1+\lambda}\mathbb{P}(x \textit{ is visiting while stabilizing } V) = \frac{\lambda}{1+\lambda}\mathbb{P}_V(h_\infty(x) \geq 1)$

Proof of lemma: stabilize V while keeping for last the reading of the (maybe) last instruction at x (if x is ever visited). If x was indeed visited, and the remaining instruction is S (probability $\frac{\lambda}{1+\lambda}$), then we did stabilize V and there remains one particle at x.

Conclusion: *If* μ *is supercritical*, then for $x \in B(0,n) \setminus B(0,n-\log n)$,

$$\theta_{B(0,n)}(x) \geq \frac{\lambda}{1+\lambda}\mathbb{P}_{B(0,n)}(h_\infty(x) \geq 1) \geq \frac{\lambda}{1+\lambda}\mathbb{P}_{B(x,\log n)}(h_\infty(x) \geq 1) \to \frac{\lambda}{1+\lambda},$$

hence, averaging over $B(0,n)$, $\mu \geq \frac{\lambda}{1+\lambda}$.

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Assume the jump distribution $p(\cdot)$ has a **bias**: for the simple random walk X with jump distribution $p(\cdot)$, for some direction ℓ we have $X_n \cdot \ell \to +\infty$, a.s..

For $\lambda > 0$, $v \in \mathbb{R}^d \setminus \{0\}$, if T_v is the time spent by X in $\{x \in \mathbb{Z}^d : x \cdot v \leq 0\}$,

let
$$F_v(\lambda) = E\left[\frac{1}{(1+\lambda)^{T_v}}\right]$$
$$= P(\text{a walk killed at rate } \lambda \text{ in } \{x \cdot v \leq 0\} \text{ survives forever})$$

NB. If $v \cdot \ell > 0$, then $0 < F_v(\lambda) \longrightarrow 1$ as $\lambda \to 0^+$.

Theorem (Taggi '14)

- *Assume* $d = 1$. *Then* $\mu > 1 - F_1(\lambda) \Rightarrow$ *non-fixation a.s.*
- *Assume* $d \geq 2$. *Then* $\mu F_v(\lambda) > \mathbb{P}(\eta_0(0) = 0) \Rightarrow$ *non-fixation a.s.*

Theorem (Rolla-T. '15)

- *Assume* $d \geq 2$. *Then* $\mu > 1 - F_v(\lambda) \Rightarrow$ *non-fixation a.s.*

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Site-wise vs. particle-wise

The **site-wise viewpoint** *(this is "Diaconis-Fulton" construction)* attaches randomness to *sites*: from finite initial configuration,

- each **site** contains a random stack of i.i.d. *instructions* ("jump to y", or "sleep"), and a Poisson clock;
- when a clock rings at a site, apply the top instruction to a particle there;
- each clock runs at speed equal to the number of active particles present at its site (as if each particle read an instruction at rate 1).

$\hookrightarrow$ we don't distinguish particles at a site, and get $\eta_t(x) \in \{0, S, 1, 2, \dots\}$.
Crucial properties: abelianness and monotonicity.

The **particle-wise viewpoint** attaches randomness to *particles*:

- each **particle** (x,i) (i-th particle starting at x) has a "life plan" $(X_t^{x,i})_{t\geq 0}$ (that is a continuous-time RW, jumping at rate 1), and a Poisson clock with rate λ;
- particles move according to their life plan,
- when the clock of a particle rings, if it is alone then its gets asleep, and in this case its clock stops;
- when a particle is awoken, its clock resumes ticking.

$\hookrightarrow$ we get a whole family of paths $(Y_t^{x,i})_{t\geq 0}$, which carries more information.
Properties: Not the above, but a control on the effect of adding one particle.

Plan of the talk

Site fixation vs. particle fixation

Definition

Site fixation *occurs when, at each site, there is eventually no active particle.*
Particle fixation *occurs when each particle is eventually sleeping.*

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Example of use. Assume particles fixate a.s., then

$$\begin{aligned}\mu &= \mathbb{E}[\#\text{ particles initially at }0]\\ &= \mathbb{E}[\#\text{ sites where a particle initially at 0 settles}]\\ &= \sum_v \mathbb{P}(\text{some particle initially at 0 settles at } v)\\ &= \sum_v \mathbb{P}(\text{some particle initially at } -v \text{ settles at } 0)\\ &= \mathbb{E}[\#\text{ particles settling at }0] \leq 1.\end{aligned}$$

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Theorem (Amir–Gurel-Gurevich '10)

Site fixation implies particle fixation. Thus, they are equivalent. And $\mu_c \le 1$.

(for i.i.d. initial conditions, 0-1 laws hold for site and particle fixation)

A non-fixation condition

• Direct technique for proving non-fixation: proving that arbitrarily many particles *visit precisely* the site o.

• In fact, proving that a positive density of particles *exit a box* is sufficient.

For $n \in \mathbb{N}$, let $V_n = \{-n, \ldots, n\}^d$, denote $\mathbb{P}_{[V_n]}$ the law of the ARW restricted to V_n (i.e. particles freeze outside), and M_n the number of particles exiting V_n.

Proposition

$$\limsup_n \frac{\mathbb{E}_{[V_n]}[M_n]}{|V_n|} > 0 \quad \Rightarrow \quad \textit{(particle) non-fixation, a.s.}$$

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Idea of proof:

Let $\widetilde{V}_n = V_{n-\log n}$. Then, if $\eta_0(x) \le K$ a.s. (to simplify)

$$\begin{aligned} \mathbb{E}[M_n] &\le \mu|V_n \setminus \widetilde{V}_n| + \mathbb{E}[\text{number of particles of } \widetilde{V}_n \text{ that quit } V_n] \\ &\le o(|V_n|) + |\widetilde{V}_n| K\,\mathbb{P}(\text{particle } Y^{0,1} \text{ reaches distance } \log n) \\ &\sim |V_n| K \mathbb{P}(\text{particle } Y^{(0,1)} \text{ doesn't fixate}) \end{aligned}$$

by using translation invariance under $\mathbb{P}$. Hence

$$\mathbb{P}(Y^{0,1} \text{ does not fixate}) \ge \frac{1}{K} \limsup_n \frac{\mathbb{E}[M_n]}{|V_n|}$$

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$\rightsquigarrow$ it remains to justify $\mathbb{E}[M_n] \ge \mathbb{E}_{[V_n]}[M_n]$, which needs an extension of monotonicity.

Plan of the talk

Non-fixation for biased ARW on $\mathbb{Z}^d$

Let $\boldsymbol{v} \in \mathbb{R}^d$ and assume $\mu > 1 - F_{\boldsymbol{v}}(\lambda)$.

Consider ARW restricted to V_n (particles freeze outside), with site-wise construction. Let us devise a **toppling strategy** (i.e. a choice of clocks) that throws a positive density of particles outside of V_n, i.e., we describe the order of sites in which we read instructions – which is irrelevant for the value of M_n, by **abelianness**.

Preliminary step: levelling

Topple sites in V_n until all particles are either alone or outside V_n.

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Label $V_n = \{x_1, \dots, x_r\}$ so that $x_1 \cdot \boldsymbol{v} \leq \cdots \leq x_r \cdot \boldsymbol{v}$.

Main step

For $i = 1, \dots, r$, if there is a particle in x_i, then topple this particle, and topple it again, and so on until either

- it exits V_n,
- it reaches an empty site in $\{x_{i+1}, \dots, x_r\}$, or
- it falls asleep on $\{x_1, \dots, x_i\}$.

NB. By induction, there is always at most one particle at x_i.

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NB. By induction, there is always at most one particle at x_i.

The probability of the last case is $\leq 1 - F_{\boldsymbol{v}}(\lambda)$, hence in the end (for $i = r$), this procedure has 'left behind' at most $|V_n|(1 - F_{\boldsymbol{v}}(\lambda))$ particles in average:

$$\mathbb{E}_{[V_n]}[M_n] \geq \mu|V_n| - (1 - F_{\boldsymbol{v}}(\lambda))|V_n|.$$

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Construction of particle-wise process in infinite volume

Problem: **existence** of ARW with infinitely many particles?

$\rightarrow$ for the usual process $(\eta_t(\cdot))_{t\geq 0}$ on $\{0, S, 1, \ldots\}^{\mathbb{Z}^d}$, the standard theory from particle systems adapt (cf. Liggett, and Andjel on Zero-Range-Process)

$\rightarrow$ for particle-wise process, non standard. (Amir and Gurel-Gurevich *assume* it)

Actually, we show that the previous particle-wise construction has a limit as more and more particles are introduced, and its law is translation invariant.

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Actually, we show that the previous particle-wise construction has a limit as more and more particles are introduced, and its law is translation invariant.
Principle: follow and control spread of influence.

For η_0, X, $\mathscr{P}$, particle (x,i) **has an influence on** $z\in\mathbb{Z}^d$ **during** $[0,t]$ if removing that particle changes the process $\overline{\eta}|_{[0,t]\times\{z\}}(\eta_0,X,\mathscr{P})$.

Lemme

Let $Z_t^{x,i}(\eta_0,X,\mathscr{P})$ be the set of sites influenced by (x,i) before t.
There exists a branching random walk $\widetilde{Z}$ on $\mathbb{Z}^d$ such that, for any finite config. π,

$$Z_t^{x,i}(\pi,X,\mathscr{P})\subset_{\text{st.}} x+\widetilde{Z}_t,$$

et $E[|\widetilde{Z}_t|]\leq e^{ct}$.

Assume $\sup_x \mathbb{E}[\eta_0(x)]<\infty$. Then the construction of ARW by addition of particles is a.s. well-defined, and translation invariant.

Extensions of parts of the proof, of possible independent interest:

- The non-fixation condition naturally extends to amenable graphs.
- The particle-wise construction extends to transitive graphs for which the mass transport principle holds (unimodular graphs).

Most striking open questions:

- in the symmetric case, for $d = 2$, non-fixation for some $\lambda > 0$ and $\mu < 1$?
- And for large λ and some $\mu < 1$ in $d = 1$ or $d \geq 3$?
- Study of critical case (non-fixation?), link with self-organized criticality,...