A random string near a wall

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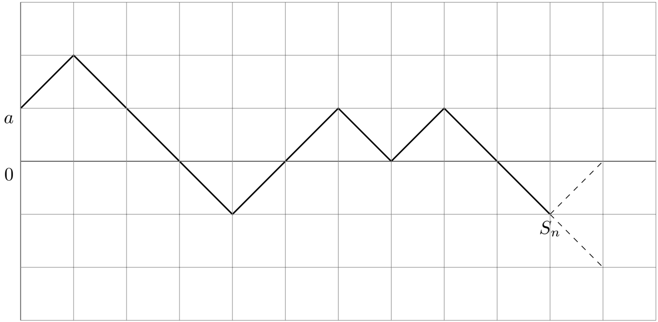
13th November 2015, Bristol

Symmetric simple random walk

$(Y_i)_{i\geq 1}$ i.i.d. sequence with $\mathbb{P}(Y_i = 1) = \mathbb{P}(Y_i = -1) = 1/2$

$$S_0 := a \in \mathbb{Z}, \qquad S_{n+1} = S_n + Y_{n+1},$$

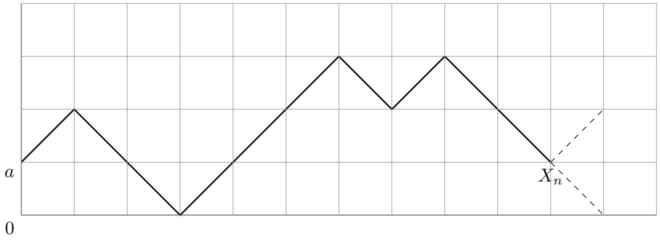
$(S_n)_{n\geq 0}$ defines a Markov chain with valued in $\mathbb{Z}$.



Symmetric simple random walk with reflection

$$X_{n+1} := \begin{cases} X_n + Y_{n+1} & \text{if } X_n > 0 \\ 1 & \text{if } X_n = 0. \end{cases}$$

This defines a Markov chain with values in $\mathbb{Z}_+ = \{0, 1, \ldots\}$.



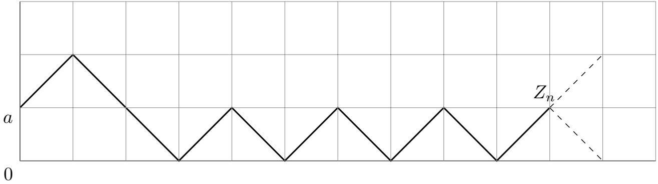
Absolute value of the SSRW

Let $Z_n := |S_n|$, with $S_0 = a \geq 0$. Although this is not obvious at first sight, $(Z_n)_{n\geq 0}$ is also a Markov chain:

$$Z_{n+1} = \begin{cases} Z_n + W_{n+1} & \text{if } Z_n > 0 \\ 1 & \text{if } Z_n = 0, \end{cases}$$

where $W_{n+1} := Y_{n+1}(\mathbb{1}_{S_n \geq 0} - \mathbb{1}_{S_n < 0})$.

It is easy to see that $(W_i)_{i\geq 1}$ is a copy of $(Y_i)_{i\geq 1}$ and then $(Z_n)_{n\geq 0}$ and $(X_n)_{n\geq 0}$ (for the same fixed a) have the same law.



We define now a Markov chain with values in discrete paths. We fix $N \in \mathbb{N}$ and we define the state space

$$E_N := \{w \in \mathbb{Z}^{2N} : w(0) = w(2N) = 0, \\ |w(i) - w(i-1)| = 1, \ \forall\, i = 1, \ldots, 2N\}.$$

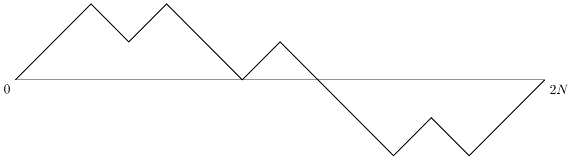
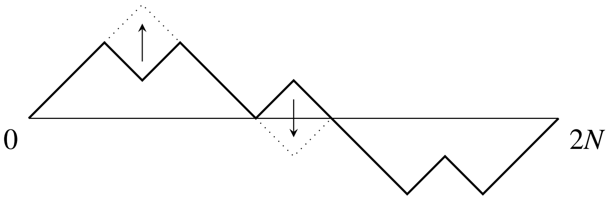


Figure: A typical path in E_N

The free evolution

Then we define a Markov chain with values in E_N as follows: we define a map $F : E_N \times \{1, \ldots, 2N-1\} \to E_N$, $\quad F(w, j) = \hat{w} \in E_N$, where

$$\hat{w}_i = \begin{cases} w_i + 2 & \text{if } i = j \text{ and } w_{i-1} = w_{i+1} > w_i \\ w_i - 2 & \text{if } i = j \text{ and } w_{i-1} = w_{i+1} < w_i \\ w_i & \text{otherwise.} \end{cases}$$



We let $(U_n)_{n \geq 1}$ be an i.i.d. sequence of uniform random variables on $\{1, \ldots, 2N-1\}$; then

$$e_{n+1} := F(e_n, U_{n+1}), \qquad e_0 \in E_N.$$

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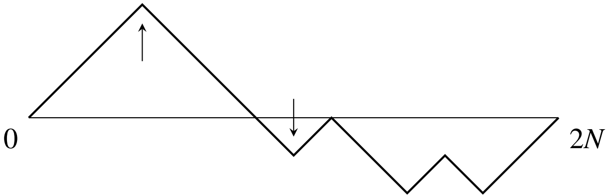

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$$e_{n+1} := F(e_n, U_{n+1}), \qquad e_0 \in E_N.$$

The free evolution

We denote the transition matrix of $(e_n)_{n\geq 0}$ by

$$P(x,y) = \mathbb{P}(F(x,U_1) = y).$$

It is easy to see that $P(x,y) = P(y,x)$ and therefore the uniform measure on E_N is invariant and reversible for P.

This is in fact the unique probability invariant measure of $(e_n)_{n\geq 0}$ by the following

Lemma

The Markov chain $(e_n)_{n\geq 0}$ is aperiodic and irreducible.

The reflected interface

Let us now add reflection to our discrete interface. We set

$$E_N^+ := \{w \in E_N : w(i) \geq 0, \ \forall\, i = 0, \dots, 2N\}.$$

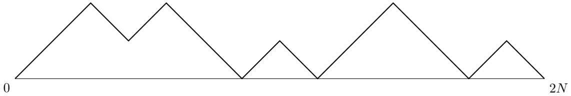
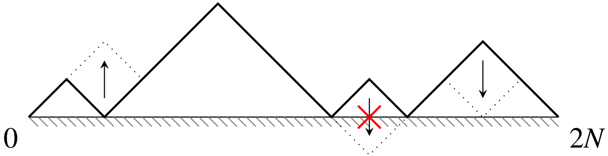


Figure: A typical path in E_N^+

Reflection means now suppression of transitions which would let e_n^+ take negative values:



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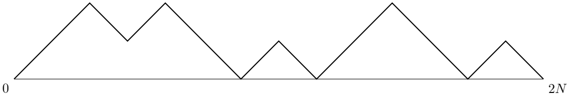

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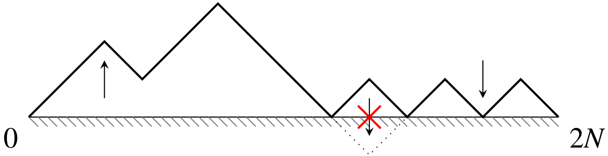


The reflected interface

The Markov evolution in E_N^+ is defined as follows:
$F^+ : E_N^+ \times \{1, \dots, 2N-1\} \to E_N^+$

$$F^+(w,j) = \begin{cases} F(w,j) & \text{if } F(w,j) \in E_N^+ \\ w & \text{otherwise.} \end{cases}$$

Then our E_N^+-valued Markov chain $(e_n^+)_{n\geq 0}$ is defined by

$$e_{n+1}^+ := F^+(e_n^+, U_{n+1}), \qquad e_0^+ \in E_N^+.$$

Lemma

The Markov chain $(e_n^+)_{n\geq 0}$ has a unique invariant probability measure, the uniform probability measure on E_N^+, which is furthermore reversible for $(e_n^+)_{n\geq 0}$.

We see that the reflection for the dynamics is equivalent to a conditioning for the invariant measure.

An important remark

Let $|e_n|$ be the absolute value of the free interface $(e_n)_{n\geq 0}$. If $(e_n)_{n\geq 0}$ is stationary then the distribution of $|e_0|$ is a probability measure on E_N^+

$$\mathbb{P}(|e_0| = w) \propto \#\{w' \in E_N : |w'| = w\} = 2^{L(w)},$$

$$L(w) := \sum_{i=1}^{2N} \mathbb{1}_{(w_i=0)}, \qquad w \in E_N^+.$$

In other words $L(w)$ is the number of excursions of w.

On the other hand, if $(e_n^+)_{n\geq 0}$ is stationary then the law of e_0^+ is uniform on E_N^+, so that e_0^+ and $|e_0|$ have different laws.

Moreover $(|e_n|)_{n\geq 0}$ is not Markovian.

Scaling limits

Let $(S_n)_{n\geq 0}$ be the SSRW with $S_0 := 0$ and

$$B_t^N := \frac{1}{\sqrt{N}} S_{\lfloor Nt \rfloor}, \qquad t \geq 0.$$

By Donsker's theorem $(B_t^N)_{t\geq 0} \Longrightarrow (B_t)_{t\geq 0}$ as $N \to +\infty$.

Under the same scaling the reflecting SSRW $(X_n)_{n\geq 0}$ converges to the reflecting Brownian motion $(\rho_t)_{t\geq 0}$.

This process is given by a stochastic differential equation

$$\mathrm{d}\rho_t = \mathrm{d}B_t + \mathrm{d}\ell_t, \qquad \rho_t \geq 0, \quad \mathrm{d}\ell_t \geq 0, \quad \int_0^\infty \rho_t \, \mathrm{d}\ell_t = 0,$$

- $t \mapsto (\rho_t, \ell_t)$ is continuous, ρ_t is non-negative
- $\ell_0 = 0$, $\ell_s \leq \ell_t$ for $s \leq t$
- $\mathrm{supp}(\mathrm{d}\ell_t) \subset \{t \geq 0 : \rho_t = 0\}$.

The measure $\mathrm{d}\ell_t$ is the reflection term.

Let us first consider the stationary version of $(e_n)_{n\geq 0}$ and define

$$v_N(t,x) = \frac{1}{\sqrt{2N}}\, e_{\lfloor 4N^2 t\rfloor}(\lfloor 2Nx\rfloor), \qquad t \geq 0,\ x \in [0,1].$$

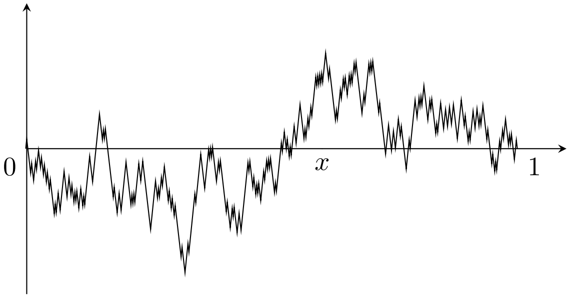


Figure: A typical path of $v_N(t,\cdot)$ for any $t\geq 0$ when N is large

The stochastic heat equation

As $N \to +\infty$, $v_N \Longrightarrow (v(t,x), t \geq 0, x \in [0,1])$, stationary solution to a stochastic partial differential equation (SPDE)

$$\begin{cases} \dfrac{\partial v}{\partial t} = \dfrac{1}{2}\dfrac{\partial^2 v}{\partial x^2} + W, \\ v(t,0) = v(t,1) = 0, \qquad t \geq 0, \\ v(0,x) = v_0(x), \qquad x \in [0,1]. \end{cases}$$

Here W is a space-time white noise.

Scaling of the reflected interface

Let us now consider the stationary version of $(e_n^+)_{n\geq 0}$ and define

$$u_N(t,x) = \frac{1}{\sqrt{2N}}\, e^+_{\lfloor 4N^2 t\rfloor}(\lfloor 2Nx\rfloor), \qquad t\geq 0,\ x\in[0,1].$$

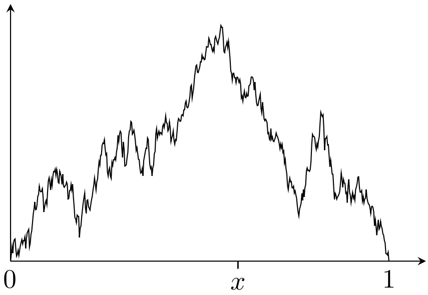


Figure: A typical path of $u_N(t,\cdot)$ for any $t\geq 0$ when N is large

A SPDE with reflection

As $N \to +\infty$, $u_N \Longrightarrow (u(t,x), t \geq 0, x \in [0,1])$, t, that we are going to study in detail from chapter 5 on and is a stationary solution to a SPDE with reflection (Funaki-Olla, Z., Etheridge-Labbé)

$$\begin{cases} \dfrac{\partial u}{\partial t} = \dfrac{1}{2}\dfrac{\partial^2 u}{\partial x^2} + W + \eta \\ u(0,x) = u_0(x), \quad u(t,0) = u(t,1) = 0 \\ u \geq 0,\ \mathrm{d}\eta \geq 0,\ \int u\,\mathrm{d}\eta = 0. \end{cases}$$

Here (u, η) is a random pair that consists of

- a continuous non-negative functions $u(t,x) \geq 0$
- a Radon measure η on $]0,+\infty[\times]0,1[$,

such that the support of η is contained in $\{(t,x) : u(t,x) = 0\}$.

The contact set

For all $t \geq 0$ the typical profile of $u(t, \cdot)$ is positive on $]0, 1[$. Where does the reflection act?

The contact set

For all $t \geq 0$ the typical profile of $u(t, \cdot)$ is positive on $]0, 1[$. Where does the reflection act?

This apparent paradox is solved if we formulate the sentence more precisely: the correct result is that for all $t \geq 0$, a.s. $u(t, \cdot) > 0$ on $]0, 1[$:

$$\forall\, t > 0, \qquad \mathbb{P}(\exists\, x \in \,]0, 1[\,: \; u(t, x) = 0) = 0.$$

However this does not exclude the existence, with positive probability, of exceptional times $t \geq 0$ and $x \in \,]0, 1[$ such that $u(t, x) = 0$:

$$\mathbb{P}(\exists\, t > 0, \; x \in \,]0, 1[\,: \; u(t, x) = 0) > 0.$$

The contact set

The next question is: what can be said about the contact set

$$\mathscr{Z} := \{(t,x) : t > 0,\ x \in]0,1[,\ u(t,x) = 0\}.$$

After proving that with positive probability u visits 0, one can ask:

- what is the typical behavior at exceptional times $t \geq 0$?
- That is, if $t > 0$ is such that there exists $x \in]0,1[$ so that $u(t,x) = 0$, then how many such points x exist?

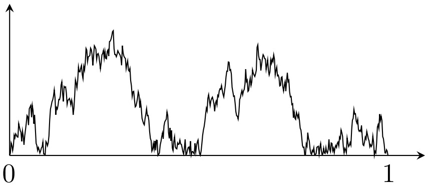


Figure: How many x such that $u(t,x) = 0$: infinitely many?

The contact set

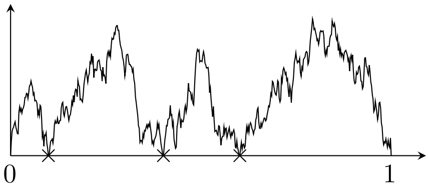


Figure: How many x such that $u(t,x) = 0$: finitely many?

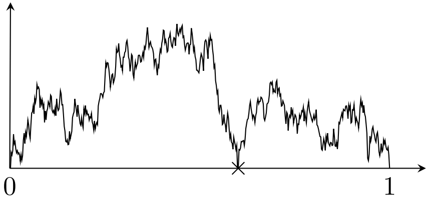


Figure: How many x such that $u(t,x) = 0$: just one? or two? or three?

SDEs with reflection

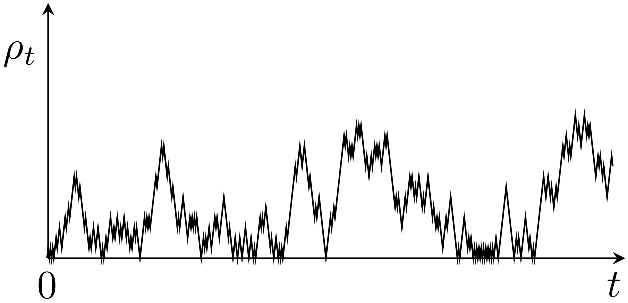
Proposition

Let $(B_t)_{t\geq 0}$ be a standard BM and $x \geq 0$. Then there exists a unique couple $(\rho_t, \ell_t)_{t\geq 0}$ of continuous real processes such that

$$\begin{cases} \rho_t = x + B_t + \int_0^t f(\rho_s)\,\mathrm{d}s + \ell_t, \quad t \geq 0 \\ \ell_0 = 0, \\ \rho_t \geq 0, \quad \mathrm{d}\ell_t \geq 0, \quad \int_0^\infty \rho_t\,\mathrm{d}\ell_t = 0. \end{cases} \tag{1}$$

If $f \equiv 0$ we call ρ the reflecting BM.

The reflecting BM



Penalisation

Let $n \geq 1$, $x \geq 0$ and

$$\rho_t^n = x + B_t + n \int_0^t (\rho_s^n)^- \, \mathrm{d}s + \int_0^t f(\rho_s^n) \, \mathrm{d}s, \qquad t \in [0,T],$$

$$\text{where} \qquad r^- = (r)^- := \max\{-r, 0\}, \qquad r \in \mathbb{R}.$$

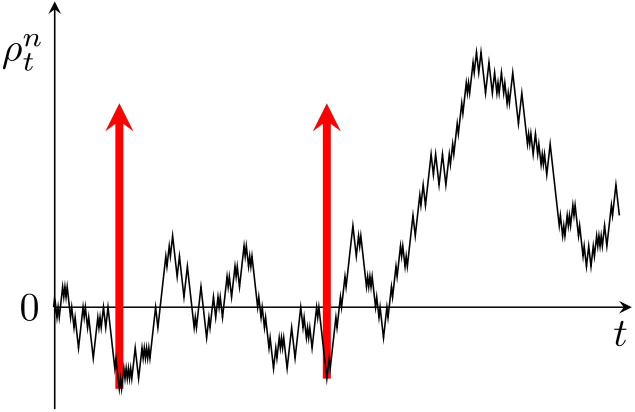
Additive noise and Lipschitz drift, so clearly pathwise uniqueness and existence of solutions by a Picard iteration.

Proposition

1. *if* $n \leq m$ *then* $\rho_t^n \leq \rho_t^m$ *for all* $t \in [0,T]$.
2. $\rho^n \uparrow \rho$ *uniformly on* $[0,T]$ *as* $n \uparrow +\infty$, *where* $(\rho_t, \ell_t)_{t \geq 0}$ *is the unique solution to the equation with reflection* (1). *Moreover*

$$\lim_{n \uparrow +\infty} n \int_0^t (\rho_s^n)^- \, \mathrm{d}s = \ell_t, \qquad t \in [0,T].$$

Penalisation



The penalised SDE

For $x \in \mathbb{R}$ and B a standard BM we set

$$\rho_t^n(x) = x + B_t + n \int_0^t (\rho_s^n(x))^- \, \mathrm{d}s + \int_0^t f(\rho_s^n(x)) \, \mathrm{d}s, \qquad t \geq 0,$$

The infinitesimal generator of ρ^n is for $\varphi \in C_c^2(\mathbb{R})$

$$L^n \varphi(x) := \frac{1}{2} \varphi''(x) + (nx^- + f(x)) \, \varphi'(x), \qquad x \in \mathbb{R}.$$

Moreover ρ^n admits as reversible invariant measure

$$\mu_n(\mathrm{d}x) = e^{-n\,(x^-)^2 + 2F(x)} \, \mathrm{d}x$$

where $F : \mathbb{R} \mapsto \mathbb{R}$ is any function such that

$$F'(x) = f(x), \qquad x \in \mathbb{R}.$$

Note that $\mu_n(]-\infty, 0]) < +\infty$ for n large, but $\mu_n([0, +\infty[) \leq +\infty$ in general.

The penalised SDE

Lemma

The measure μ is invariant and reversible for $(\rho_t)_{t\geq 0}$ and

$$\lim_{n\to+\infty} \int_{\mathbb{R}} \varphi \, \mathrm{d}\mu_n = \int_{\mathbb{R}} \varphi \, \mathrm{d}\mu, \qquad \forall \, \varphi \in C_c(\mathbb{R}),$$

where

$$\mu(\mathrm{d}x) := \mathbb{1}_{(x\geq 0)} \, e^{2F(x)} \, \mathrm{d}x.$$

Here is an important message, that we have already noticed for discrete interfaces:

Remark

A reflection for the dynamics means a conditioning for the invariant measure.

Bessel processes

δ-Bessel processes are solutions $(\rho_t)_{t\geq 0}$ to the SDE with $\delta > 1$

$$\rho_t = x + \frac{\delta - 1}{2} \int_0^t \rho_s^{-1}\, \mathrm{d}s + B_t, \qquad \rho_t \geq 0, \quad t \geq 0, \tag{2}$$

where $(B_t)_{t\geq 0}$ is a BM. If $\delta \downarrow 1$ the equation becomes

$$\rho_t = x + \ell_t + B_t, \quad \ell_0 = 0, \quad \mathrm{d}\ell \geq 0, \quad \int_0^t \rho_s \, \mathrm{d}\ell_s = 0, \tag{3}$$

i.e. the reflecting Brownian motion.

Bessel processes have the same scaling invariance of BM.

It is well known that a δ-Bessel process visits 0 with positive probability iff $\delta < 2$.

White noise

$W : L^2(\mathbb{R}^2_+) \to L^2(\Omega)$ isometry

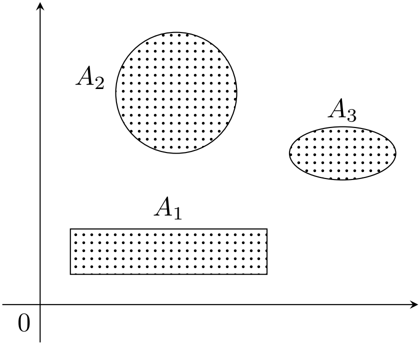


Figure: The family $(W(A_i))_i$ is Gaussian and independent since the sets A_i are pairwise disjoint. $W(A_i) \sim \mathcal{N}(0, m(A_i))$ and $W(\cup_i A_i) \sim \mathcal{N}(0, \sum_i m(A_i))$.

For all measurable set A, $W(A)$ is the amount of noise contained in A.

Space-time white noise and Cylindrical Brownian motion

Lemma

Let $(e_i)_i$ be a complete orthonormal system in $L^2([0,+\infty[)$. Then

1. *Let $w_t^i := W(\mathbb{1}_{[0,t]} \otimes e_i)$, $t \geq 0$, $i \in \mathbb{N}$. Then $(w^i)_i$ is an iid sequence of standard Brownian motions.*
2. *For all $h \in L^2([0,+\infty[)$ and $t \geq 0$*

$$W(\mathbb{1}_{[0,t]} \otimes h) = \sum_i w_t^i \langle e_i, h \rangle$$

where the equality is in $L^2(\Omega)$.

A cylindrical Brownian motion in a separable Hilbert space H is

$$\langle W_t, h \rangle := \sum_i B_t^i \langle e_i, h \rangle, \qquad t \geq 0,$$

where $(e_i)_i$ is a complete orthonormal system in H and $(B^i)_i$ is an iid sequence of Brownian motions.

We want to study the stochastic PDE

$$\begin{cases} \dfrac{\partial v}{\partial t} = \dfrac{1}{2}\,\dfrac{\partial^2 v}{\partial x^2} + W, \\ v(t,0) = v(t,1) = 0, \qquad t \geq 0, \\ v(0,x) = v_0(x), \qquad x \in [0,1] \end{cases} \tag{4}$$

where $W(t,x)$ is a space-time white-noise over $[0,+\infty[\times[0,1]$.

This SPDE is interpreted in the PDE-weak sense: for all $h \in C^2_c(0,1)$ and $t \geq 0$

$$\langle v_t, h\rangle = \langle v_0, h\rangle + \frac{1}{2}\int_0^t \langle v_s, h''\rangle\,\mathrm{d}s + \int_0^t \int_0^1 h(x)\,W(\mathrm{d}s,\mathrm{d}x).$$

Fourier decomposition

We set for all $k \geq 1$:

$$e_k(x) := \sqrt{2}\,\sin(k\pi x), \qquad x \in [0,1]. \tag{5}$$

Note that $\{e_k\}_{k\geq 1}$ is a complete basis of eigenvectors of the second derivative with homogeneous Dirichlet boundary conditions:

$$\frac{d^2}{dx^2} e_k = -(\pi k)^2 e_k, \qquad e_k(0) = e_k(1) = 0, \qquad k \geq 1.$$

Setting $v_t^k := \langle v(t,\cdot), e_k\rangle$ we obtain

$$\mathrm{d}v_t^k = -\frac{(k\pi)^2}{2}\, v_t^k\, \mathrm{d}t + \mathrm{d}B_t^k, \qquad v_0^k = \langle v_0, e_k\rangle$$

$$B_t^k := \int_{[0,t]\times[0,1]} e_k(x)\, W(\mathrm{d}s, \mathrm{d}x) = W\left(\mathbb{1}_{[0,t]} \otimes e_k\right).$$

Fourier decomposition

We proved in Lemma 7 that $(B^k_t, t \geq 0)_{k\geq 1}$ is an independent sequence of Brownian motions. Then $(v^k_\cdot)_{k\geq 1}$ is an independent family of O-U processes of respective parameter $\frac{(\pi k)^2}{2} > 0$, and

$$v(t,x) = \sum_k \left(e^{-\frac{(\pi k)^2}{2} t} v^k_0 + \int_0^t e^{-\frac{(\pi k)^2}{2}(t-s)} \, \mathrm{d}B^k_s \right) e_k(x).$$

An important remark is the following:

$$\sum_k \frac{2}{(\pi k)^2} < +\infty \qquad (d = 1).$$

Proposition

There exists a continuous modification of v s.t.

$$\sup_{x,y\in[0,1],\, t,s\in[0,T]} \frac{|v(t,x) - v(s,y)|}{|t-s|^{\frac{1-\varepsilon}{4}} + |x-y|^{\frac{1-\varepsilon}{2}}} < +\infty.$$

The invariant measure

If we let $t \to +\infty$ in

$$v(t,\cdot) = \sum_k \left(e^{-\frac{(\pi k)^2}{2} t} v_0^k + \int_0^t e^{-\frac{(\pi k)^2}{2}(t-s)} \, \mathrm{d}B_s^k \right) e_k$$

we obtain that the invariant measure of v is the law of

$$\beta := \sum_{k=1}^{+\infty} \frac{1}{\pi k} Z_k \, e_k,$$

where $(Z_k)_{k\geq 1}$ is an i.i.d. sequence of $\mathcal{N}(0,1)$.

Proposition

$\beta = (\beta(x), x \in [0,1])$ *is a Brownian bridge.*

Obstacle problems

Let $a \geq 0$. We fix a space-time white noise W on $[0, +\infty[\times[0, 1]$. We study the following SPDE with reflection:

$$\begin{cases} \dfrac{\partial u}{\partial t} = \dfrac{1}{2}\dfrac{\partial^2 u}{\partial x^2} + f(u) + W + \eta \\ u(0,x) = u_0(x),\ u(t,0) = u(t,1) = a \\ u \geq 0,\ \mathrm{d}\eta \geq 0,\ \int u\, \mathrm{d}\eta = 0 \end{cases} \tag{6}$$

where we assume that:

1. $u_0 : [0, 1] \mapsto \mathbb{R}$ is continuous and $u_0 \geq 0$.
2. $f : \mathbb{R} \mapsto \mathbb{R}$ is Lipschitz and bounded.

Reduction to a PDE with random obstacle

Let $a \geq 0$ and v be the unique solution to

$$\begin{cases} \dfrac{\partial v}{\partial t} = \dfrac{1}{2}\dfrac{\partial^2 v}{\partial x^2} + W, \\ v(t,0) = v(t,1) = a, \quad v(0,x) = u_0(x). \end{cases} \tag{7}$$

Then the function $z := u - v$ solves

$$\begin{cases} \dfrac{\partial z}{\partial t} = \dfrac{1}{2}\dfrac{\partial^2 z}{\partial x^2} + f(z+v) + \eta \\ z(0,x) = 0, \ z(t,0) = z(t,1) = 0 \\ z \geq -v, \ \mathrm{d}\eta \geq 0, \ \int (z+v)\,\mathrm{d}\eta = 0. \end{cases} \tag{8}$$

The important remark here is that equation (8) is a PDE (rather than a SPDE) with random obstacle $-v$.

The Nualart-Pardoux equation

Theorem (Nualart-Pardoux, 1992)

Let $\mathrm{w} \in C([0,T] \times [0,1])$ *with* $\mathrm{w}(0,\cdot) \geq 0$, $\mathrm{w}(\cdot,0) \geq 0$, $\mathrm{w}(\cdot,1) \geq 0$.
Then there exists a unique pair (z,η) *such that*

- $z \in C([0,T] \times [0,1]), \quad z(0,\cdot) = 0, \quad z(\cdot,0) = z(\cdot,1) = 0$
- $\eta(\mathrm{d}t,\mathrm{d}x)$ *is a measure on* $]0,T]\times]0,1[$ *such that* $\eta(]0,T] \times [\delta, 1-\delta]) < +\infty$ *for all* $\delta > 0$
- *For all* $t \in [0,T]$ *and* $h \in C_c^\infty(0,1)$

$$\begin{aligned} \langle z_t, h\rangle &= \frac{1}{2}\int_0^t \langle z_s, h''\rangle \,\mathrm{d}s + \int_0^t \langle f(z_s + \mathrm{w}_s), h\rangle \,\mathrm{d}s \\ &+ \int_0^t \int_0^1 h(x)\,\eta(\mathrm{d}s,\mathrm{d}x) \end{aligned} \tag{9}$$

- $z \geq -\mathrm{w}, \quad \int (z+\mathrm{w})\,\mathrm{d}\eta = 0.$

Existence: penalisation

We introduce the following approximating problem:

$$\begin{cases} \dfrac{\partial u^\varepsilon}{\partial t} = \dfrac{1}{2}\dfrac{\partial^2 u^\varepsilon}{\partial x^2} + f(u^\varepsilon) + \dfrac{(u^\varepsilon)^-}{\varepsilon} + W \\ u^\varepsilon(0,\cdot) = u_0, \quad u^\varepsilon(t,0) = u^\varepsilon(t,1) = a. \end{cases}$$

Proposition

The pair (u,η) is the limit of the pair $(u^\varepsilon, \eta^\varepsilon)$ where

$$\eta^\varepsilon(\mathrm{d}t, \mathrm{d}x) := \frac{(u^\varepsilon(t,x))^-}{\varepsilon}\,\mathrm{d}t\,\mathrm{d}x.$$

The invariant measure

Let us consider a Brownian motion $(B_t^{(d)})_{t\geq 0}$ in $\mathbb{R}^d$ where $B^{(d)} = (B^1, \ldots, B^d)$ and the B^i's are iid standard BMs.

Let $V : \mathbb{R}^d \to \mathbb{R}$ be a smooth function and let

$$\mathrm{d}X_t = \nabla V(X_t)\,\mathrm{d}t + \mathrm{d}B_t^{(d)}, \qquad X_0 = x \in \mathbb{R}^d.$$

It is a classical fact that an invariant measure for $(X_t)_{t\geq 0}$ is given by

$$\exp(2V(x))\,\mathrm{d}x.$$

If this measure is finite on $\mathbb{R}^d$, we obtain an invariant probability measure.

The penalised invariant measure

We consider now the penalised SPDE

$$\begin{cases} \dfrac{\partial u^\varepsilon}{\partial t} = \dfrac{1}{2}\dfrac{\partial^2 u^\varepsilon}{\partial x^2} + f(u^\varepsilon) + \dfrac{(u^\varepsilon)^-}{\varepsilon} + W \\ u^\varepsilon(0,\cdot) = u_0, \quad u^\varepsilon(t,0) = u^\varepsilon(t,1) = a. \end{cases}$$

The invariant measure is

$$\nu_\varepsilon^a(\mathrm{d}\zeta) := \frac{1}{Z_\varepsilon^a} \exp\left(2\langle F_\varepsilon(\zeta), 1\rangle\right) \mathbf{W}_{a,a}(\mathrm{d}\zeta),$$

where $\mathbf{W}_{a,a}$ is the law of $a+\beta$ and F_ε satisfies

$$F_\varepsilon(0) = 0, \qquad F'_\varepsilon(y) := f(y) + \frac{y^-}{\varepsilon} = f_\varepsilon(y).$$

The penalised invariant measure

Note that

$$\frac{d}{dr}\left(r^-\right)^2 = -2r^- .$$

Then for $a > 0$

$$\nu_\varepsilon^a(\mathrm{d}\zeta) \;=\; \frac{1}{Z_\varepsilon^a}\,\exp\left(2\langle F(\zeta),1\rangle - \frac{1}{\varepsilon}\langle\left(\zeta^-\right)^2,1\rangle\right)\mathbf{W}_{a,a}(\mathrm{d}\zeta),$$

converges as $\varepsilon \downarrow 0$ to

$$\nu^a(\mathrm{d}\zeta) \;:=\; \frac{1}{Z^a}\,\exp\left(2\langle F(\zeta),1\rangle\right)\mathbb{1}_K(\zeta)\,\mathbf{W}_{a,a}(\mathrm{d}\zeta),$$

where $K := \{u_0 : [0,1] \to \mathbb{R} :\ u_0 \in L^2(0,1),\ u_0 \geq 0\}$.

The Brownian excursion, or the 3-Bessel bridge

It turns out that

$$\mathbf{P}^3_{a,a} = \mathbf{W}_{a,a}(\,\cdot\,|\,K).$$

Then

$$\nu^a(\mathrm{d}\zeta) \;:=\; \frac{1}{\hat{Z}^a}\,\exp\left(2\langle F(\zeta),1\rangle\right)\,\mathbf{P}^3_{a,a}(\mathrm{d}\zeta),$$

and as $a \downarrow 0$

$$\nu^a(\mathrm{d}\zeta) \Longrightarrow \nu^0(\mathrm{d}\zeta) \;:=\; \frac{1}{\hat{Z}^a}\,\exp\left(2\langle F(\zeta),1\rangle\right)\,\mathbf{P}^3_{0,0}(\mathrm{d}\zeta).$$

In particular if $f \equiv 0$ then the invariant measure of the SPDE with reflection is simply $\mathbf{P}^3_{a,a}$.

Integration by parts

Consider a regular bounded open set $O \subset \mathbb{R}^d$. Then the classical Gauss-Green formula states that for all $h \in \mathbb{R}^d$

$$\int_O (\partial_h \varphi)\, \rho \, \mathrm{d}x \;=\; -\int_O \varphi \, \frac{\partial_h \rho}{\rho}\, \rho \, \mathrm{d}x \;-\; \int_{\partial O} \varphi \, \langle \hat{n}, h \rangle \, \rho \, \mathrm{d}\sigma$$

- $\varphi, \rho \in C^1_b(O)$ with $\lambda \leq \rho \leq \lambda^{-1}$, $\lambda \in]0,1]$ is a constant,
- $\hat{n}$ is the inward-pointing normal vector to the boundary ∂O
- σ is the surface measure on ∂O
- $\partial_h \varphi$ is the directional derivative of φ along h
- $\partial_h \log \rho = (\partial_h \rho)/\rho$.

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For us, $\mathbf{W}_{a,a} = \rho \, \mathrm{d}x, \quad K = O.$

What is the analog of $\rho \, \mathrm{d}\sigma$? and of $\hat{n}$? and of ∂O?

The boundary measure

$$\mathbf{P}^3_{a,a}\left[\partial_h\varphi\right] = -\mathbf{P}^3_{a,a}\left[\varphi(X)\,\langle X, h''\rangle\right] - \int_0^1 \mathrm{d}r\, h(r)\,\gamma(r,a)\,\mathbf{P}^3_{a,a}\left[\varphi(X)\,|\,X_r = 0\right].$$

where $\gamma(r,a) \geq 0$ is an explicit function of $r \in]0,1[$, $a \geq 0$.

$$\int_O (\partial_h\varphi)\,\rho\,\mathrm{d}x \;=\; -\int_O \varphi\,\frac{\partial_h\rho}{\rho}\,\rho\,\mathrm{d}x \;-\; \int_{\partial O} \varphi\,\langle \hat{n}, h\rangle\,\rho\,\mathrm{d}\sigma.$$

The boundary measure

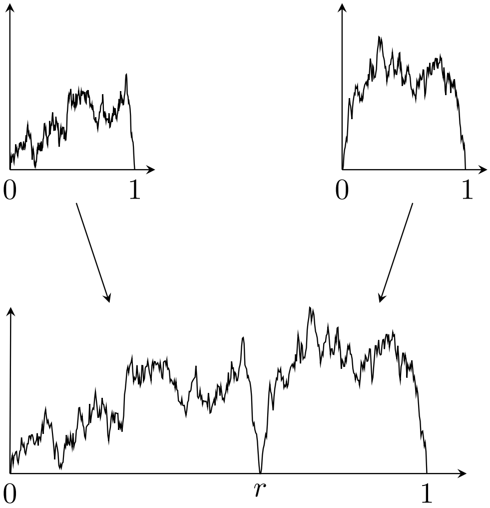


Figure: A typical path under the boundary measure.

The Revuz measure of η

Theorem

For all bounded Borel $\varphi : H \mapsto \mathbb{R}$ *and* $h \in C_c(0,1)$

$$\int \nu^a(\mathrm{d}u_0)\, \mathbb{E}\left[\int_0^t \int_0^1 h(x)\, \varphi(u_s)\, \eta(\mathrm{d}s, \mathrm{d}x)\right] =$$

$$= \frac{t}{2Z^a} \int_0^1 \mathrm{d}r\, h(r)\, \gamma(r,a) \int \varphi(\zeta)\, e^{2F(\zeta)}\, \Sigma_a(r, \mathrm{d}\zeta).$$

where $\Sigma_a(r,\cdot) := \mathbf{P}^3_{a,a}[\,\cdot \mid X_r = 0]$.

The contact set

We denote by $\pi : [0,+\infty[\times[0,1] \mapsto [0,+\infty[$ the projection $(t,x) \mapsto t$, and for a set $S \subset [0,+\infty[\times[0,1]$ we write

$$S_t := \{x \in [0,1] : (t,x) \in S\}, \qquad t \geq 0.$$

Theorem

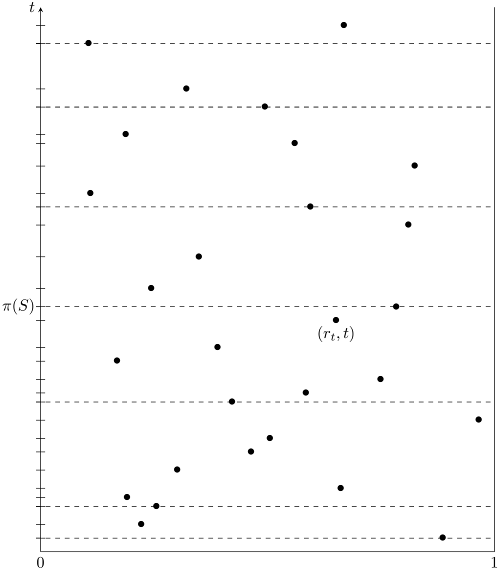
Let (u,η) be the stationary solution to equation (6). Let us denote by

$$\mathscr{C} := \{(t,x) : u(t,x) = 0,\ t > 0,\ x \in]0,1[\,\}$$

the contact set and let us recall that the support of η is contained in $\mathscr{C}$. Then a.s. the set $\pi(\mathscr{C})$ has zero Lebesgue measure and there exists a measurable set $S \subset \mathscr{C}$ such that

1. $\eta(\mathscr{C} \setminus S) = 0$
2. *for all $t > 0$, either $S_t = \emptyset$ or $S_t = \{r_t\}$, with $r_t \in]0,1[$.*
3. *if $S_t = \{r_t\}$, then $u(t,x) > 0$ for all $x \in]0,1[\setminus\{r_t\}$ and $u(t,r_t) = 0$.*

The contact set



SPDEs with repulsion from 0

We study now the SPDE

$$\begin{cases} \dfrac{\partial u}{\partial t} = \dfrac{1}{2}\dfrac{\partial^2 u}{\partial x^2} + \dfrac{c}{u^3} + W \\ u(t,0) = u(t,1) = a, \quad t \geq 0 \\ u(0,x) = u_0(x), \quad x \in [0,1] \end{cases}$$

where $a \geq 0$ and $c > 0$ are fixed and we search for solutions $u \geq 0$.

This SPDE has the same invariance scaling as the linear and the reflected SPDE.

This SPDE is an analogue of Bessel processes for $\delta > 1$ (see the slide no. 28).

Theorem

Let $a \geq 0$, $c > 0$ *and* $u_0 \in C([0,1]) \cap K$. *Then there exists a unique continuous* $u : [0,+\infty[\times[0,1] \mapsto [0,+\infty[$ *such that*

1. $u^{-3} \in L^1_{loc}([0,+\infty[\times]0,1[)$
2. *A.s. for all* $t \geq 0$ *and* $h \in C^\infty_c(0,1)$

$$\begin{aligned} \langle u_t, h\rangle = \langle u_0, h\rangle + \frac{1}{2}\int_0^t \langle u_s, h''\rangle \, \mathrm{d}s + \int_0^1 h(x)\, W(\mathrm{d}s, \mathrm{d}x) \\ + c\int_0^t \int_0^1 h(x)\, u^{-3}(s,x)\, \mathrm{d}s\, \mathrm{d}x. \end{aligned} \tag{10}$$

If $\delta > 3$ *is such that* $c = \frac{(\delta-3)(\delta-1)}{8}$, *then the only invariant probability measure of* (10) *is* $\mathbf{P}^\delta_{a,a}$, *law of the* δ*-Bessel bridge.*

Hitting of zero

We have now functions $u = u^\delta$ for $\delta \geq 3$, stationary solutions to equations with reflection ($\delta = 3$) or repulsion from 0 ($\delta > 3$).

One of the main results of this course is the following

Theorem (Dalang, Mueller, Z. 2006)

Let $\delta \geq 3$. If $k \in \mathbb{N}$ satisfies

$$k > \frac{4}{\delta - 2},$$

the probability that there exist $t > 0$ and $x_1, \ldots, x_k \in [0, 1]$ such that $0 < x_1 < \cdots < x_k < 1$ and $u(t, x_i) = 0$ for all $i = 1, \ldots, k$, is zero.

Hitting of zero

In particular, setting for $\delta \geq 3$

$$\zeta(\delta) := \sup\{k : \exists\, (t, x_1, \dots, x_k) \in]0,1] \times]0,1[, u(t, x_i) = 0\}$$

then

- for $\delta = 3$, a.s. $\zeta(\delta) \leq 4$
- for $\delta \in]3, 3+1/3]$, a.s. $\zeta(\delta) \leq 3$
- for $\delta \in]3+1/3, 4]$, a.s. $\zeta(\delta) \leq 2$
- for $\delta \in]4, 6]$, a.s. $\zeta(\delta) \leq 1$
- for $\delta > 6$, a.s. $\zeta(\delta) = 0$.

In any case $\zeta(\delta) \leq 4$ a.s. for all $\delta \geq 3$. The behavior at the transition points $\delta \in \{3, 3+1/3, 4, 6\}$ might be non-optimal. Indeed, we conjecture that a.s.

$$\zeta(3) \leq 3, \qquad \zeta(3+1/3) \leq 2, \qquad \zeta(4) \leq 1, \qquad \zeta(6) = 0.$$

Hitting of zero

Theorem (Dalang, Mueller, Z.)

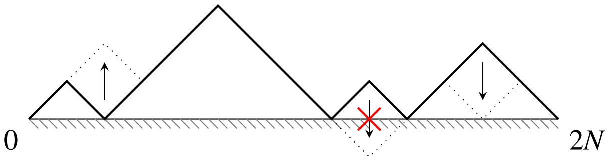
(a) *For all* $\delta \in [3, 5]$, *with positive probability, there exist* $t > 0$ *and* $x \in]0, 1[$ *such that* $u_t(x) = 0$.

(b) *For* $\delta = 3$, *with positive probability there exist* $t > 0$ *and* $\{x_1, x_2, x_3\} \subset]0, 1[$, $x_1 < x_2 < x_3$, *such that* $u_t(x_i) = 0$, $i = 1, 2, 3$.

We conjecture that for all $\delta \geq 3$ a.s.

$$\zeta(\delta) = \left\lceil \frac{4}{\delta - 2} \right\rceil - 1.$$

Back to the discrete interface



- What can be said on the contact set of the dynamical discrete interface as $N \to +\infty$?

Other open problems

- Construct SPDEs whose invariant measure is the δ-Bessel bridge for $\delta < 3$ (log-concavity is lost).
- One can conjecture that a.s.

$$\zeta(\delta) = \left\lceil \frac{4}{\delta - 2} \right\rceil - 1, \qquad \delta \geq 2.$$

- For $\delta < 2$ the situation is even more complicated since 0 is hit by the stationary profile.
- The case $\delta = 1$ (reflecting BM) is the most intriguing since it is the limit of homogeneous pinning models.
- There is an IbPF for $\delta = 1$ but the form of the dynamics is hard even to conjecture.
- Dynamics of random trees (Aldous' CRT)