Some percolation processes with infinite range dependencies

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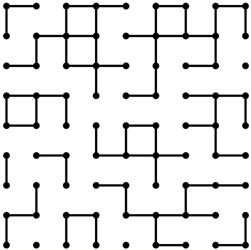
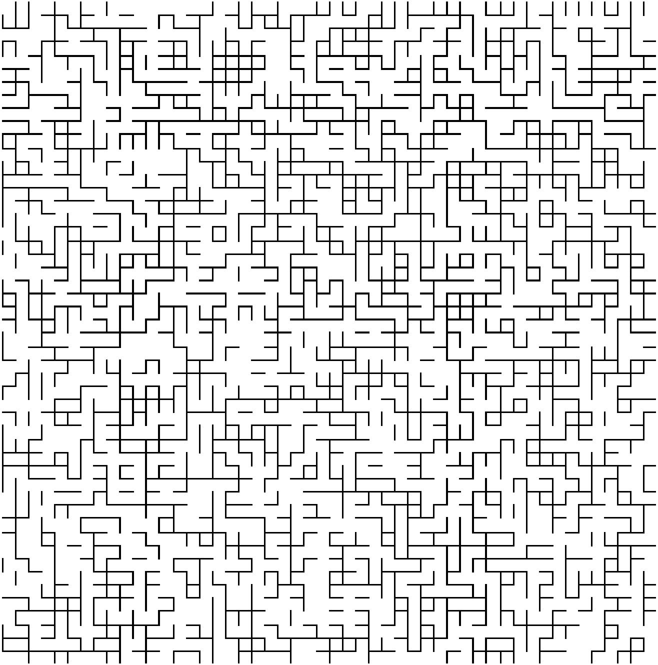
Bernoulli Percolation

- $\mathbb{Z}^d$-lattice $= (V(\mathbb{Z}^d), E(\mathbb{Z}^d))$.
- $p \in [0,1]$.
- Declare each site (or edge) independently
$$\begin{cases} \text{open} & \text{with prob. } p \\ \text{closed} & \text{with prob. } 1-p. \end{cases}$$



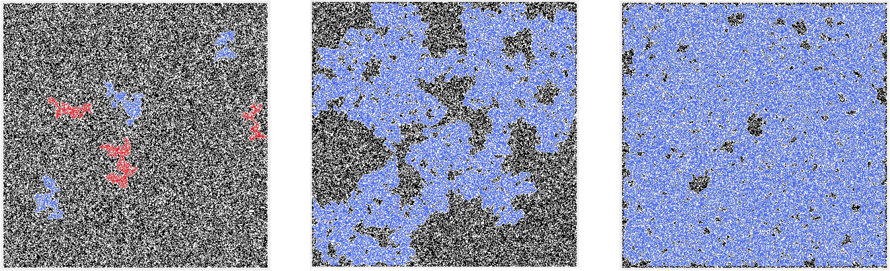
- $\mathbb{P}_p$ corresponding law in $\{0,1\}^{\mathbb{Z}^d}$ (or $\{0,1\}^{E(\mathbb{Z}^d)}$).
- Important events

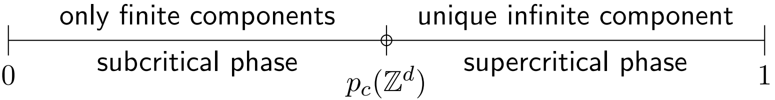
$$\{0 \leftrightarrow \partial B(n)\} \quad \{0 \leftrightarrow \infty\} \quad \mathcal{C}_{RL}(\tau n, n) \quad \mathcal{C}_{TB}(\tau n, n).$$



The phase transition

- $p \mapsto \mathbb{P}_p(0 \leftrightarrow \infty)$ is a non-decreasing function.
- Critical point: $p_c(\mathbb{Z}^d) = \inf\{p \in [0,1];\ \mathbb{P}_p(0 \leftrightarrow \infty)\} > 0$.
- Phase transition: For $d \geq 2$, $0 < p_c(\mathbb{Z}^d) < 1$.





Exponential decay of connectivity

How does the quantity $\{0 \leftrightarrow \partial B(n)\}$ behaves as a function of n?

Subcritical phase:

If $p < p_c(\mathbb{Z}^d)$, then there exists $\alpha = \alpha(p, d) > 0$ such that

$$\mathbb{P}_p(\{0 \leftrightarrow \partial B(n)) \leq e^{-\alpha(p)n}.$$

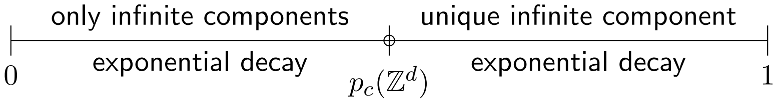
(Menshikov '86, Aizenman & Barsky '87).

Supercritical phase:

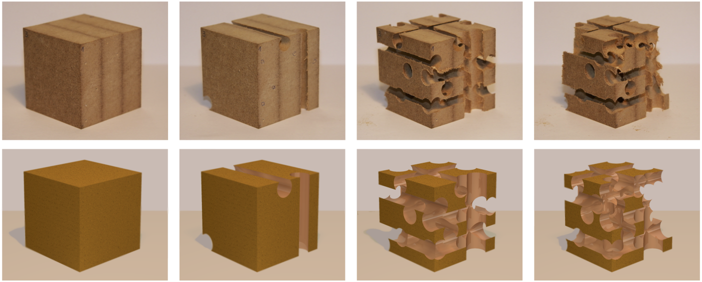
If $p > p_c(\mathbb{Z}^d)$, then there exists $\sigma = \sigma(p, d) > 0$ such that

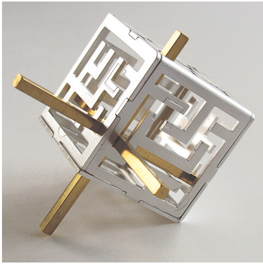
$$\mathbb{P}_p(\{0 \leftrightarrow \partial B(n) \not\leftrightarrow \infty) \leq e^{-\sigma(p)n}.$$

(Chayes, Chayes & Newman '87).



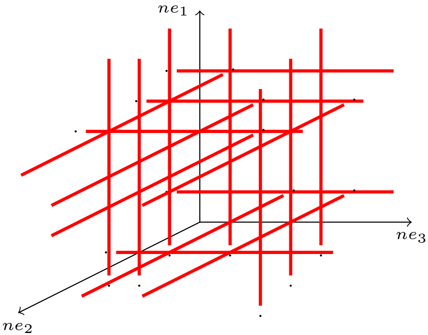
Drilling a wooden cube or playing with the Oskar's puzzle





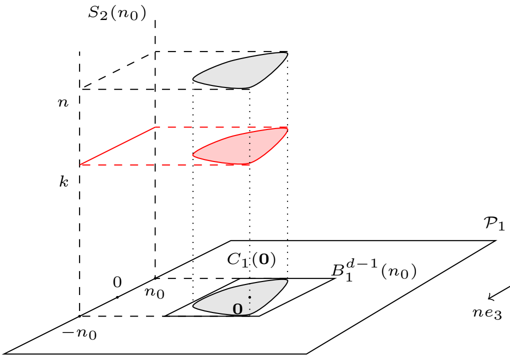
Coordinate Percolation

- $\mathbb{Z}^d$-lattice, $d \geq 3$.
- $\{e_1, \ldots, e_d\}$ standard orthonormal basis.
- $p_1, \ldots, p_d \in [0, 1]$ intensity parameters.
- Remove at random lines parallel to e_i with probability p_i independently.

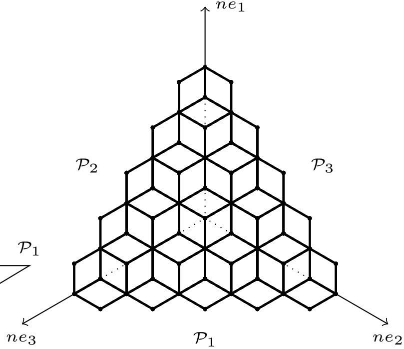


- $\mathcal{L}$ = set of removed sites.
- $\mathcal{V} = \mathbb{Z}^d \backslash \mathcal{L}$ vacant set.
- $\mathbf{p} = (p_1, \ldots, p_d)$.
- $\mathbb{P}_{\mathbf{p}}$ = law of the process.

Phase transition



Existence of the subcritical phase



Existence of the supercritical phase

Phase transition

Theorem (H., Sidoravicius, '11)

Assume that $p_i < p_c(\mathbb{Z}^{d-1})$ *for some* $i \in \{1, \ldots, d\}$, *and that* $p_j \neq 1$ *for some* $j \in \{1, \ldots, d\} \setminus \{i\}$ *then*

$$\mathbb{P}_{\mathbf{p}}(\{\mathbf{0} \leftrightarrow \infty\}) = 0. \tag{1}$$

On the other hand if $p_1, \ldots, p_d$ *are sufficiently close to 1, then*

$$\mathbb{P}_{\mathbf{p}}(\{\mathbf{0} \leftrightarrow \infty\}) > 0. \tag{2}$$

Theorem (H., Sidoravicius, '11)

Let N *be the number of infinite connected components. Almost surely under* $\mathbb{P}_{\mathbf{p}}$, N *is a constant random variable taking values in the set* $\{0, 1, \infty\}$.

Decay of correlations

Theorem (H., Sidoravicius '11)

If $p_i < p_c(\mathbb{Z}^{d-1})$ *and* $p_j < p_c(\mathbb{Z}^{d-1})$ *for some* $i \neq j \in \{1, \dots, d\}$, *then there exists a constant* $\psi = \psi(\mathbf{p}, d) > 0$ *such that, for* $n \geq 0$,

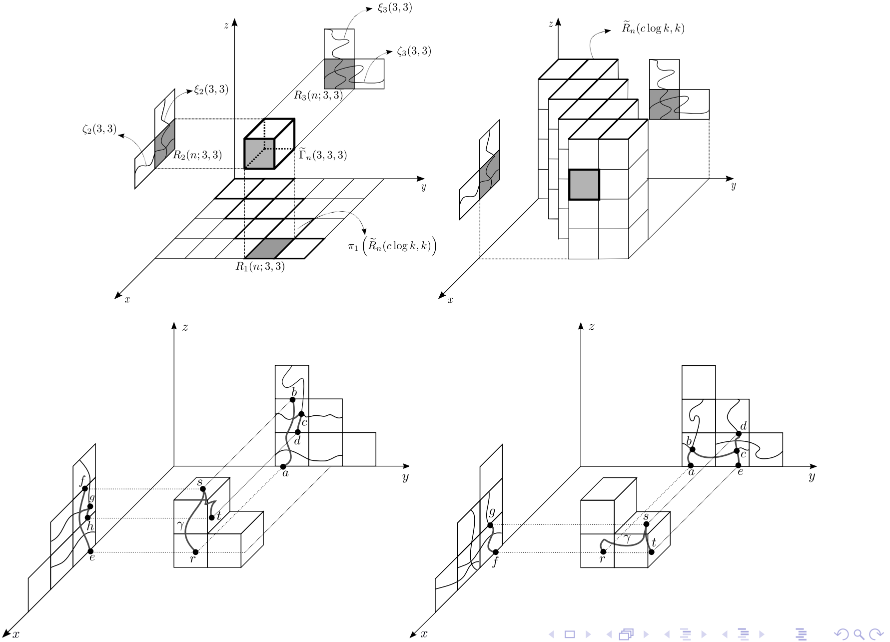
$$\mathbb{P}_{\mathbf{p}}(\{\mathbf{0} \leftrightarrow \partial B(n)\}) \leq e^{-\psi(\mathbf{p},d)n}. \tag{3}$$

Theorem (H., Sidoravicius '11)

Let $d = 3$. *Assume that* $p_2 > p_c(\mathbb{Z}^2)$, $p_3 > p_c(\mathbb{Z}^2)$ *and* $0 < p_1 < 1$. *Then, there exists constants* $\alpha(\mathbf{p}) > 0$ *and* $\alpha'(\mathbf{p}) > 0$ *such that, for all* $n \geq 0$,

$$\mathbb{P}_{\mathbf{p}}(\{\mathbf{0} \leftrightarrow \partial B(n),\ \mathbf{0} \not\leftrightarrow \infty\}) \geq \alpha'(\mathbf{p}) n^{-\alpha(\mathbf{p})}. \tag{4}$$

‘Proof’ of the power-law decay



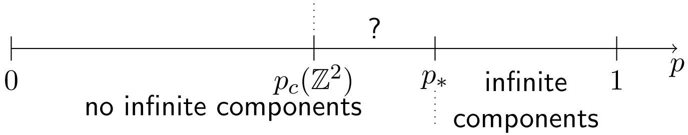
More about the phase transition for $d = 3$.

$d = 3$, $\tilde{p}_c(\mathbb{Z}^2) =$ critical point for site oriented percolation in $\mathbb{Z}^2$.

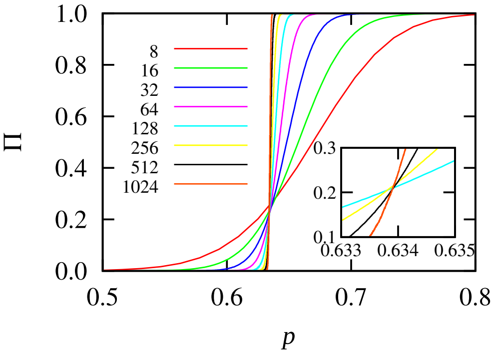
Theorem (H., Sidoravicius '11)

- *If* $p_i > \tilde{p}_c(\mathbb{Z}^2)^{1/3}$ *for all* $i = 1, 2, 3$, *then* $\mathbb{P}_{\mathbf{p}}(\{\mathbf{0} \leftrightarrow \infty\}) > 0$.
- *If* $p_2 > p_c(\mathbb{Z}^2)$ *and* $p_3 > p_c(\mathbb{Z}^2)$, *then there exists* $\epsilon = \epsilon(p_2, p_3) > 0$ *such that for all* $p_1 > 1 - \epsilon$, $\mathbb{P}_{\mathbf{p}}(\{\mathbf{0} \leftrightarrow \infty\}) > 0$.

$p_1 = p_2 = p_3 = p$. Define $p_* =$ critical point. Open question: show that $p_* > p_c(\mathbb{Z}^2)$.

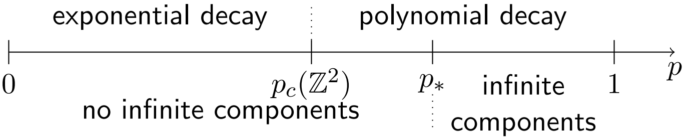


What happens when $p_1 = p_2 = p_3 = p_c(\mathbb{Z}^2) + \delta$ with $\delta \approx 0$?



Probability of having a crossing from bottom to top in a box with indicated side length.
The data suggest that $p_* = 0.6339(5)$.
Compare:
$p_c(\mathbb{Z}^2) \approx 0.5927$ and $\tilde{p}_c(\mathbb{Z}^2)^{1/3} \approx 0.8902$.

Simulation by K.J. Schrenk, N.A.M Araújo, H. J. Herrmann



Cilinders' Percolation

- $\mathbb{R}^d$, $d \geq 3$, $\mathbb{L} =$ space of lines of $\mathbb{R}^d$.

Construct a Poisson point process in $\mathbb{L}$

Parametrization:

- $l =$ line parallel to the canonical vector e_d.
- $\tau_x(y) = y + x$.
- $(x, \theta) \in \mathbb{R}^{d-1} \times \mathsf{SO}_d$.

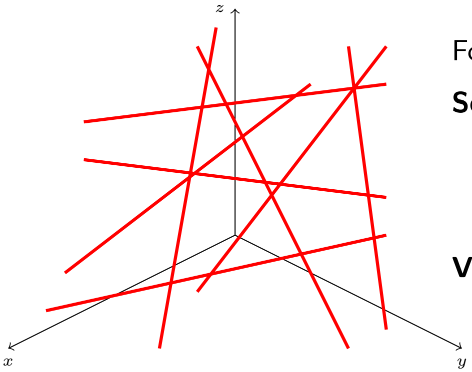
$$\begin{array}{rccc} \alpha : & \mathbb{R}^{d-1} \times \mathsf{SO}_d & \rightarrow & \mathbb{L} \\ & (x, \theta) & \mapsto & \theta(\tau_x(l)). \end{array}$$

Measure:

- λ: Lebesgue on $\mathbb{R}^{d-1}$; ν: Haar on SO_d.
- $\mu(A) = (\lambda \times \nu)(\alpha^{-1}(A))$.

Cilinders' Percolation

- $\Omega =$ set of locally finite point measures on $\mathbb{L}$.
- $u > 0$ parameter.
- $\mathbb{P}_u =$ law of a PPP in $\mathbb{L}$ with intensity $u \cdot \mu$.
- $l \in \mathbb{L} \to C(l) =$ cylinder of radius one and axis l.



For $\omega \in \Omega$

Set of cylinders:

$$\mathcal{L}(\omega) = \bigcup_{l \in \mathrm{supp}(\omega)} C(l).$$

Vacant set:

$$\mathcal{V}(\omega) = \mathbb{R}^d \backslash \mathcal{L}(\omega).$$

Critical point

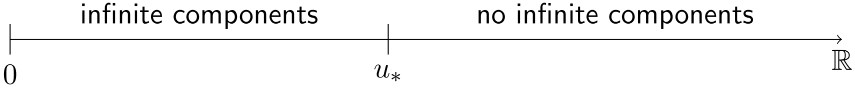
Main goal: To study the connectivity properties of $\mathcal{V}$ under $\mathbb{P}_u$.

$$\begin{array}{ccc} u \text{ small} & \to & \text{few cylinders drilled;} \\ u \text{ large} & \to & \text{many cylinders drilled.} \end{array}$$

$\mathbb{P}_u\left[\mathcal{V} \text{ has an unbounded component}\right]$ is non increasing in u.

$$u_* = \inf\{u > 0;\ \mathbb{P}_u\left[\mathcal{V} \text{ has an unbounded component }\right] = 0\}$$

Question: $0 < u_* < \infty$?



Phase transition

Theorem (Tykesson, Windisch '11)

$$\textit{For } d \geq 3, \quad u_* < \infty;$$
$$\textit{For } d \geq 4, \quad u_* > 0.$$

$d \geq 4, u$ small $\Rightarrow \mathcal{V} \cap \mathbb{R}^2$ has an unbounded component.

Why to look at $\mathcal{V} \cap \mathbb{R}^2$?

Duality

If the component of $\mathcal{V} \cap \mathbb{R}^2$ containing $\mathbf{0}$ is bounded, then there exists a circuit in $\mathcal{L} \cap \mathbb{R}^2$ surrounding the origin.

Multi-scale analysis for ruling out the existence of long circuits

The three dimensional case

Slow decay of correlations:

$$\text{cov}(\mathbf{1}_{x\in\mathcal{V}}, \mathbf{1}_{y\in\mathcal{V}}) \asymp \frac{1}{|x-y|^{d-1}} \qquad d=3 \text{ is slower!.}$$

Theorem (Tykesson, Windisch '11)

$d = 3$, for all $u > 0, \mathcal{V} \cap \mathbb{R}^2$ has only bounded connected components $\mathbb{P}_u$– a.s..

Infinitely many triangles surrounding the origin in $\mathcal{L} \cap \mathbb{R}^2$.

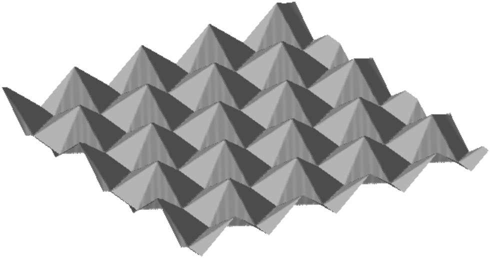
- u small.
- Look for unbounded connected components beyond $\mathcal{V} \cap \mathbb{R}^2$.
- Avoiding being trapped by few cylinders.
- Still being use the duality principle.

The three dimensional case

Idea : Replace $\mathcal{V} \cap \mathbb{R}^2$ by $\mathcal{V} \cap H$.

$\mathcal{H} =$ hexagonal lattice in $\mathbb{R}^2$ with mesh size 2000.

$H =$ graph of the application $x \mapsto \text{dist}(x, \mathcal{H})$.



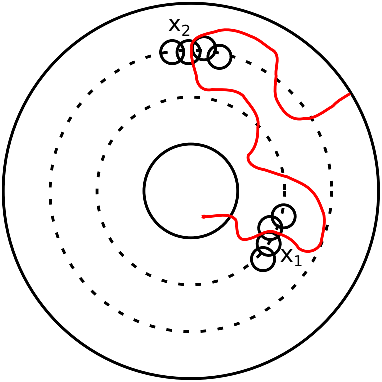
Theorem (H., Sidoravicius, Teixeira '12)

For $d = 3$, for all $u > 0$ small enough

$$\mathbb{P}_u \left[\mathcal{V} \cap H \text{ has an unbounded component } \right] = 1.$$

Show that there are typically no long paths from $\mathbf{0}$ in $\mathcal{L} \cap H$.

The multiscale analysis

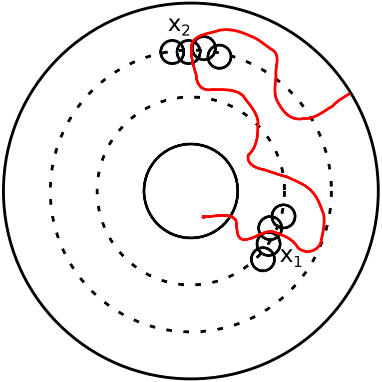


- $\gamma = 7/6$ fixed.
- a_0 large.
- $a_n = a_{n-1}^{\gamma} = a_0^{\gamma^n}$ (super exponential growth of scales).
- $p_n(u) = \sup_{x \in \mathbb{R}^2} \mathbb{E}_u[A_n(x)]$.

$$A_n(x) = \mathbf{1}\{S(x, a_n/10) \leftrightarrow \partial S(x, a_n) \text{ in } \pi(\mathcal{L} \cap H)\}.$$

Show that $p_n(u)$ decays very fast with n.

The multiscale analysis



cover intermediate spheres with at most $c\left(\frac{a_n}{a_{n-1}}\right)$ balls of radius $\frac{a_{n-1}}{10}$

$$p_n(u) = \sup_{x\in\mathbb{R}^2} \mathbb{E}_u[A_n(x)] \leq c\Big(\frac{a_n}{a_{n-1}}\Big)^2 \sup \mathbb{E}_u[A_{n-1}(x_1)A_{n-1}(x_2)]$$

suppremum over x_1 and x_2, centre of balls in the coverings.

The multiscale analysis

$$\begin{aligned} p_n(u) &\le c\left(\frac{a_n}{a_{n-1}}\right)^2 \sup \mathbb{E}_u[A_{n-1}(x_1)A_{n-1}(x_2)] \\ &\le c\left(\frac{a_n}{a_{n-1}}\right)^2 [p_{n-1}(u)^2 + \text{error}]. \end{aligned}$$

Forget about the error: $p_n(u) \le c\left(\frac{a_n}{a_{n-1}}\right)^2 p_{n-1}(u)^2$.

Recursion:

$$p_{n-1}(u) \le a_{n-1}^{5/2(1-\gamma)} \Rightarrow p_n(u) \le a_n^{5/2(1-\gamma)}.$$

$$p_n(u) \le c\left(\frac{a_n}{a_{n-1}}\right)^2 [p_{n-1}(u)^2 + \underbrace{\left(\frac{a_{n-1}}{a_n}\right)^6 + \left(\frac{a_{n-1}}{a_n}\right)^2 q_{n-1}^2(u)]}_{\text{error}}.$$

$$q_n(u) = \sup_{x\in\mathbb{R}^2} \sup_{l_1,l_2\in\mathbb{L}} \mathbb{E}_u[A_n(x,\omega+\delta_{l_1}+\delta_{l_2})] \le R\left(\frac{a_n}{a_{n-1}}, p_{n-1}, q_{n-1}\right).$$

The multiscale analysis

Recursion: a_0 big and u small

$$\begin{cases} p_{n-1} \le a_{n-1}^{5/2(1-\gamma)} \\ q_{n-1} \le a_{n-1}^{3/2(1-\gamma)} \end{cases} \Rightarrow \begin{cases} p_n \le a_n^{5/2(1-\gamma)} \\ q_n \le a_n^{3/2(1-\gamma)} \end{cases}$$

Triggering: As $u \to 0$ both $p_0(u)$ and $q_0(u)$ vanish.

The rough shape of H plays a crucial hole for showing that $q_0(u)$ vanishes (would be false for $\mathbb{R}^2$).

$$\mathbb{P}_u \left\{ \begin{array}{c} \text{there exists a circuit in } \pi(\mathcal{L} \cap H) \\ \text{surrounding the origin of } \mathbb{R}^2 \end{array} \right\} < 1,$$

$$\mathbb{P}_u \left\{ \begin{array}{c} \text{the origin belongs to an unbounded} \\ \text{component of } \pi(\mathcal{L} \cap H) \end{array} \right\} > 0.$$

Brochette percolation

- Bernoulli edge percolation in $\mathbb{Z}^2$.
- Choose a random set of vertical lines.
- Increase the parameter in this set.
- How does it affect the critical point?
- $\Lambda \subset \mathbb{Z}$, deterministic set.
- $E_{\text{vert}}(\Lambda \times \mathbb{Z})$ = set of brochettes.
- $p, q \in [0, 1]$ parameters.
- $\mathbb{P}_{p,q}^{\Lambda}$: open edge e with prob. $= \begin{cases} p, & \text{if } e \in E_{\text{vert}}(\Lambda \times \mathbb{Z}), \\ q, & \text{otherwise.} \end{cases}$

Brochette percolation

Make the set of the brochettes random.

- $\xi = \{\xi_z\}_{z\in\mathbb{Z}}$ i.i.d. Bernoulli(ρ).
- $\Lambda(\xi) = \{j \in \mathbb{Z} : \xi_j = 1\}$.
- $\nu(\rho) =$ law of ξ.
- $\mathbb{P}^{\rho}_{p,q}(\cdot) := \int \mathbb{P}^{\Lambda(\xi)}_{p,q}(\cdot) d\nu_\rho(\xi)$.

Theorem (Duminil-Copin, H., Kozma, Sidoravicius '15)

For every $\varepsilon > 0$ and $\rho > 0$ there exists $\delta > 0$ such that

$$\mathbb{P}^{\rho}_{p_c+\varepsilon,p_c-\delta}(0 \leftrightarrow \infty) > 0.$$

Remark: For the rest of the talk, we fix ε and ρ.

Enhancements induced by K-syndetic sets

$\Lambda \subset \mathbb{Z}$ is k-syndetic if all its gaps have diameter smaller than k.
The Aizenman-Grimmett argument (1991) implies that:

Proposition

If Λ is k-syndetic then for every $\varepsilon > 0$ there exists $\delta > 0$ such that,

$$\mathbb{P}^{\Lambda}_{p_c+\varepsilon, p_c-\delta}(0 \longleftrightarrow \infty) > 0.$$

Russo's Formula:
For A an increasing event depending on the state of finitely many edges only (*e.g.*: $\{0 \leftrightarrow \partial B(n)\}$),

$$\frac{d}{dp}\mathbb{P}_p(A) = \sum_e \mathbb{P}_p(e \text{ is pivotal for } A),$$

where, $\{e \text{ is pivotal for } A\} = \{\omega^e \in A, \omega_e \notin A\}$.

The Aizenman-Grimmett argument

- By Russo's Formula we have:

$$\frac{\partial}{\partial p}\mathbb{P}^{\Lambda}_{p,q}(0 \leftrightarrow \partial B(n)) = \sum_{f \in E_{\text{vert}}(\Lambda \times \mathbb{Z})} \mathbb{P}^{\Lambda}_{p,q}(f \text{ is piv. for } 0 \leftrightarrow \partial B(n)).$$

$$\frac{\partial}{\partial q}\mathbb{P}^{\Lambda}_{p,q}(0 \leftrightarrow \partial B(n)) = \sum_{e \notin E_{\text{vert}}(\Lambda \times \mathbb{Z})} \mathbb{P}^{\Lambda}_{p,q}(e \text{ is piv. for } 0 \leftrightarrow \partial B(n)).$$

- By local modification arguments, using that Λ is k-syndetic:

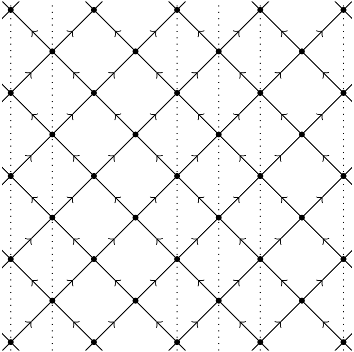
$$\mathbb{P}^{\Lambda}_{p,q}(f(e) \text{ piv. for } 0 \leftrightarrow B(n)) \geq c(k,p,q)\mathbb{P}^{\Lambda}_{p,q}(e \text{ piv. for } 0 \leftrightarrow B(n)).$$

- This ultimately leads to:

$$\frac{\partial}{\partial q}\mathbb{P}^{\Lambda}_{p,q}(0 \leftrightarrow \partial B(n)) \geq c(k,p,q)\frac{\partial}{\partial p}\mathbb{P}^{\Lambda}_{p,q}(0 \leftrightarrow \partial B(n)),$$

with $c(k,p,q)$ bounded in a neighbourhood of (p_c, p_c).

The KSV Theorem



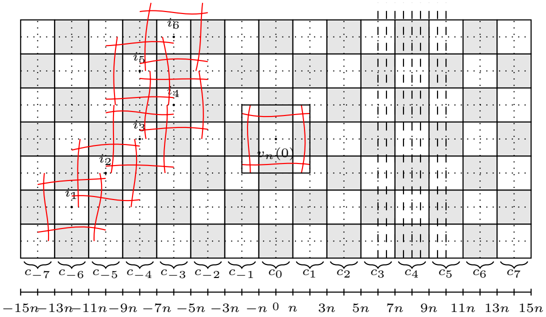
- $\mathbb{Z}^2_\Diamond$
- Edges oriented in the NE and NW sense.
- Declare columns *good* independently with probability ρ'.
- Parameter in good lines: p_G.
- Parameter in bad lines: p_B.
- $\tilde{\mathbb{P}}^{\rho'}_{p_G,p_B}$ = law

Theorem (Kesten, Sidoravicius, Vares, '12)

For all $p_B > 0$ *and* $p_G > \tilde{p}_c(\mathbb{Z}^2)$ *there exists* $\rho' > 0$ *such that*

$$\tilde{\mathbb{P}}^{\rho'}_{p_G,p_B}(\textit{oriented infinite path in } \mathbb{Z}^2_\Diamond) > 0.$$

The renormalisation scheme



Scale n
Blocks:
$v_n(z) = [-n, n]^2 + 2nz$.
Lattice:
$\mathbb{Z}_n^2 = \{v_n(z);\ z \text{ even}\}$.

- Columns: $c_n(i) = \{v_n(i, j);\ i + j \text{ is even}\}$.
- $c_n(i)$ is good if $\Lambda(\xi) \cap [2n(i-1), 2n(i+1)]$ is $\frac{2}{\rho} \log 2n$-syndetic.
- $v_n(z)$ is good if crossed as above.

Crossing probabilities in k-syndetic boxes

Lemma

$\lim_{n\to\infty} \mathbb{P}^{\rho}_{p,q}(c_n(i)$ *is a good column*$) = 1$.

Proposition

There exists $c > 0$ *and* $\alpha > 0$ *such that for all* Λ k*-syndetic,*

$$\mathbb{P}^{\Lambda}_{p_c+\varepsilon,p_c}(\mathcal{C}_{RL}(\tau n, n)) \geq \mathbb{P}_{p_c+ck^{-\alpha}}(\mathcal{C}_{RL}(\tau n, n)).$$

Lemma

$\lim_{n\to\infty} \mathbb{P}_{p_c+[\frac{2c}{\rho}\log(2n)]^{-\alpha}}(\mathcal{C}_{RL}(\tau n, n)) = 1.$

- **Conclusion:** For n large, process in good columns dominates a 0.999 Bernoulli site percolation.
- Also one can show that the process in bad columns dominates a 0.001 Bernoulli site percolation.

Proof of the theorem

- Define $p_B = 0.0001$ and $p_G = 0.99$.
- By KSV, there exists ρ' such that $\tilde{\mathbb{P}}^{\rho'}_{p,q}(0 \leftrightarrow \infty) > 0$.
- Fix n large enough so that:
  – The process of good lines dominates a 1-d i.i.d. Bernoulli(ρ') sequence.
  – The process of occupied blocks in good lines dominates an 0.999 Bernoulli percolation.
- With n fixed, find δ small enough so that, under $\mathbb{P}^{\rho'}_{p_c+\varepsilon, p_c-\delta}$,
  – The process of occupied sites in bad columns still dominates an independent Bernoulli percolation with parameter 0.0001.
  – The process of occupied sites in good columns still dominates an independent Bernoulli percolation with parameter 0.99.
- The result follows from KSV.