MERMIN-WAGNER THEOREM and DLR EQUATIONS in QUANTUM STATISTICS and QUANTUM GRAVITY

Mark Kelbert, Swansea/Sao Paulo

Joint work with Yurii Suhov, Cambridge/Sao Paulo

David Mermin's foot. In his book 'About Time' (2005):
"Henceforth, by 1 foot we shall mean the distance light travels in a nanosecond. A foot, if you will, is a light-nanosecond (and a nanosecond, even more nicely, can be viewed as a light foot). If it offends you to redefine the foot then you may define 0.299792458 meters to be 1 phoot, and think "phoot" whenever you read "foot"."

A nanosecond= 10^{-9} **second** $= 1000$ **picosecond**
1 foot=1 light-nanosecond ≈ 0.3 **m**
The cycle time for radio frequency 1 GHz $(1 \times 10^9$ **hertz)=1 light-nanosecond**

The Mermin-Peres magic square. The product of 9 numbers ± 1 should be either 1 or -1. This means that it is not possible to construct a 3×3 table with entries $+1$ and -1 such that the product of the elements in each row equals $+1$ and the product of elements in each column equals -1. But it is nearly possible to do so with the richer algebraic structure based on Pauli matrices:

$\sigma_x \times \mathbf{I}$	$\sigma_x \times \sigma_x$	$\mathbf{I} \times \sigma_x$
$-\sigma_x \times \sigma_z$	$\sigma_y \times \sigma_y$	$-\sigma_z \times \sigma_x$
$\mathbf{I} \times \sigma_z$	$\sigma_z \times \sigma_z$	$\sigma_z \times \mathbf{I}$

with Pauli matrices $\sigma_x, \sigma_y, \sigma_z$

0	1
1	0

0	-i
i	0

1	0
0	-1

1. Gibbs measures and Dobrushin-Lanford-Ruelle (DLR) equation

The definition of a Gibbs random field on a lattice requires some terminology:
The lattice: A countable set $\mathbf{L} = \mathbf{Z}^d$.
The single-spin space: A probability space $(S, \mathcal{S}, \lambda)$.
The configuration space: $(\Omega, \mathcal{F})$, where $\Omega = S^{\mathbf{L}}$ and $\mathcal{F} = \mathcal{S}^{\mathbf{L}}$.
The set $\mathcal{L}$ of all finite subsets of $\mathbf{L}$.
Given a configuration $\omega \in \Omega$ and a subset $\Lambda \subset \mathbf{L}$, the restriction of ω to Λ is $\omega_\Lambda = (\omega(t))_{t \in \Lambda}$. If $\Lambda_1 \cap \Lambda_2 = \emptyset$ and $\Lambda_1 \cup \Lambda_2 = \mathbf{L}$, then the configuration $\omega_{\Lambda_1}\omega_{\Lambda_2}$ is the configuration whose restrictions to Λ_1 and Λ_2 are ω_{Λ_1} and ω_{Λ_2},

respectively.

For each subset $\Lambda \subset \mathbf{L}, \mathcal{F}_\Lambda$ is the $\sigma-$algebra generated by the family of functions $(\sigma(t))_{t\in\Lambda}$, where $\sigma(t)(\omega) = \omega(t)$. This $\sigma-$algebra is just the algebra of cylinder sets on the lattice.

The potential: A family $\Phi = (\Phi_A)_{A\in\mathcal{L}}$ of functions $\Phi_A : \Omega \to \mathbf{R}$ such that for each $A \in \mathcal{L}$, Φ_A is $\mathcal{F}_A-$measurable. For all $\Lambda \in \mathcal{L}$ and $\omega \in \Omega$, the series $H^\Phi_\Lambda(\omega) = \sum_{A\in\mathcal{L}, A\cap\Lambda\neq\emptyset} \Phi_A(\omega)$ exists.

The Hamiltonian in $\Lambda \in \mathcal{L}$ with boundary conditions $\bar{\omega}$, for the potential Φ, is defined by

$$H^\Phi_\Lambda(\omega|\bar{\omega}) = H^\Phi_\Lambda(\omega_\Lambda \bar{\omega}_{\Lambda^c}), \tag{1.1}$$

where $\Lambda^c = \mathbf{L} \setminus \Lambda$.

The partition function in $\Lambda \in \mathcal{L}$ with boundary conditions $\bar{\omega}$ and inverse temperature $\beta \in \mathbf{R}_+$ (for the potential Φ and λ) is defined by

$$Z^\Phi_\Lambda(\bar{\omega}) = \int \lambda^\Lambda(\mathrm{d}\omega) \exp(-\beta H^\Phi_\Lambda(\omega|\bar{\omega})). \tag{1.2}$$

Here $\lambda^\Lambda(\mathrm{d}\omega)$ is the product measure $\prod_{t\in\Lambda} \lambda(\mathrm{d}\omega(t))$.

A potential Φ is $\lambda-$admissible if $Z^\Phi_\Lambda(\bar{\omega}) < \infty$ for all $\Lambda \in \mathcal{L}, \bar{\omega} \in \Omega$ and $\beta > 0$.

A probability measure μ on $(\Omega, \mathcal{F})$ is a **Gibbs measure** for a $\lambda-$admissible potential Φ if it satisfies the **Dobrushin-Lanford-Ruelle (DLR) equations**

$$\int \mu(\mathrm{d}\bar{\omega}) Z^\Phi_\Lambda(\bar{\omega})^{-1} \int \lambda^\Lambda(\mathrm{d}\omega) \exp(-\beta H^\Phi_\Lambda(\omega|\bar{\omega})) 1_A(\omega_\Lambda \bar{\omega}_{\Lambda^c}) = \mu(A), \tag{1.3}$$

for all $A \in \mathcal{F}$ and $\Lambda \in \mathcal{L}$. We say that a **phase transition** is observed if the solution of (1.3) is not unique.

2. Classical case: Heisenberg model

Werner Heisenberg studied a model of classical statistical mechanics on a d-dimensional lattice $\mathbf{Z}^d$ with spins of the unit length $\mathbf{s}_i \in \mathbf{R}^3, |\mathbf{s}_i| = 1$, each one placed on a lattice node.

This was a prelude to Heisenberg quantum model with formal Hamiltonian

$$\hat{H} = \sum_{|j-j'|=1} J^x \sigma^x_j \sigma^x_{j'} + J^y \sigma^y_j \sigma^y_{j'} + J^z \sigma^z_j \sigma^z_{j'} \tag{2.1}$$

Formal Hamiltonian of classical model:

$$H = \sum_{i,j} \phi_{ij}(\mathbf{s}_i, \mathbf{s}_j) = -\sum_{|i-j|=1} J_{ij} \langle \mathbf{s}_i, \mathbf{s}_j \rangle \tag{2.2}$$

with a coupling J_{ij} between neighboring spins.
Invariant with respect of rotation group $O(3)$.

Mermin-Wagner principle: if $d = 1, 2$ an external magnetic field cannot destroy this symmetry

Original proof is quite involved. A short proof is possible based on Gibbs-Bogolyubov inequality: if $H = H^0 + \lambda H$ and a free energy $F(\lambda) = -\beta^{-1} \ln \mathrm{Tr}\big[e^{-\beta H}\big]$ then $\frac{\mathrm{d}^2 F}{\mathrm{d}\lambda^2} \le 0$. $\forall \lambda$.

3. Schlosman's rotators on $\mathbf{Z}^2$:

A phase transition with spontaneous breaking of discrete symmetry and preservation of continuous symmetry.

$$\phi_{ij}(x, x') = -\beta \cos(x - x') \;\; if \;\; |i - j| = \sqrt{2},$$
$$\phi_{ij}(x, x') = \beta \cos^2(x - x') \;\; if \;\; |i - j| = 1. \tag{3.1}$$

0 in all other cases.
Discrete symmetry is a rotation of all spins on even sub-lattice by π.

When $\beta \to \infty$ the model exhibits a phase transition destroying the discrete symmetry but preserving the continuous symmetry.

4. Quantum Hamiltonian on a graph

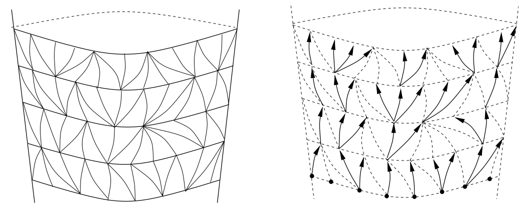
Formal Hamiltonian on frustrated 2D lattice Γ:

$$H = \frac{1}{2}\Big[-\sum_{j \in \Gamma} \Delta_j + \sum_{j \neq j'} J(d(j, j'))V(x_j, x_{j'})\Big] \tag{4.1}$$

Assume $V(gx, gx') = V(x, x')$, $g \in \mathtt{G}-$compact Lie group, and

$$|V(x', x'')|, |\nabla_x V(x', x'')|, |\nabla_{x'} \nabla_{x''} V(x', x'')| \le \bar{V} \tag{4.2}$$

and a reasonable decay of $J(d)$, say $J(d) \sim d^{-3}$.



5. Quantum gravity: Lorentzian triangulations

Random Lorentzian triangulations are parametrized by critical size-biased Galton-Watson (GW) trees

with the offspring numbers $\{k_t, t = 1, 2, \dots\}$ conditional upon non-extinction. Select one particle from k_{t-1} and generate its offspring family with MGF $f'(x)$, i.e. $\tilde{p}_k = kp_k$. All other particles have the same offspring law as in the classical GW process.

The sized-biased critical branching processes have been studied by **Russell Lions** (Indiana University), **Robin Pemantle** (UPenn) and **Yurval Peres** (Microsoft Research).

Lemma 1. *Let the offspring distribution has a finite second moment. Then a.s.*

$$k_t \leq Ct \ln^{\frac{1}{2}+\epsilon} t \tag{5.1}$$

6. Quantum Gibbs states

in a finite volume $\Lambda \subset \Gamma$ are linear positive normalized functionals on the C*-algebra $\mathcal{B}_\Lambda$ of bounded operators in space $\mathcal{H}_\Lambda = \prod\limits_{i\in\Lambda} \mathrm{L}_2(M_i,\nu)$:

$$\varphi_\Lambda(A) = \mathrm{tr}_{\mathcal{H}_\Lambda}\big(R_\Lambda A\big) \tag{6.1}$$

where

$$R_\Lambda = \frac{\exp\big[-\beta H_\Lambda\big]}{\Xi_{\beta,\Lambda}} \quad \text{with} \quad \Xi_{\beta,\Lambda} = \mathrm{tr}_{\mathcal{H}_\Lambda}\big(\exp\big[-\beta H_\Lambda\big]\big). \tag{6.2}$$

Restriction to a finite volume $\Lambda^0 \subset \Lambda$:

$$\varphi_\Lambda^{\Lambda^0}(A_0) = \mathrm{tr}_{\mathcal{H}_{\Lambda^0}}\big(R_\Lambda^{\Lambda^0} A_0\big), A \in \mathcal{B}(\Lambda_0) \tag{6.3}$$

where

$$R_\Lambda^{\Lambda^0} = \mathrm{tr}_{\mathcal{H}_{\Lambda\setminus\Lambda^0}} R_\Lambda. \tag{6.4}$$

In a similar way define operators $R^{\Lambda^0}_{\Lambda|\overline{\mathbf{x}}_{\Gamma'\setminus\Lambda}}$ with boundary conditions $\overline{\mathbf{x}}_{\Gamma'\setminus\Lambda}$.

Theorem 1. *For all given $\beta \in (0,\infty)$ and a finite $\Lambda^0 \subset \Gamma$, operators $R_\Lambda^{\Lambda^0}$ form a compact sequence in the trace-norm topology in $\mathcal{H}_{\Lambda^0}$ as $\Lambda \nearrow \Gamma$. Furthermore, given any family of (finite or infinite) sets $\Gamma' = \Gamma'(\Lambda) \subseteq \Gamma$ and boundary conditions (i.e. particle configurations) $\overline{\mathbf{x}}_{\Gamma'\setminus\Lambda}$, operators $R^{\Lambda^0}_{\Lambda|\overline{\mathbf{x}}_{\Gamma'\setminus\Lambda}}$ also form a compact sequence in the trace-norm topology.*

Moreover, any limiting point, R^{Λ^0}, for $\left\{R^{\Lambda^0}_{\Lambda|\overline{\mathbf{x}}_{\Gamma'\setminus\Lambda}}\right\}$ is a positive definite operator of trace one which possesses the following invariance property: $\forall\ \mathtt{g} \in \mathtt{G}$,

$$U_{\Lambda^0}(\mathtt{g})^{-1} R^{\Lambda^0} U_{\Lambda^0}(\mathtt{g}) = R^{\Lambda^0}. \tag{6.5}$$

The proof is based on the following

Lemma 2. *Let $\rho_n(x,y)$ be a sequence of kernels defining positive-definite operators R_n of trace class and with trace 1 in a Hilbert space $\mathrm{L}_2(M,\nu)$ where $\nu(M) < \infty$. Suppose there exists the following limit, uniform in $x, y \in M$:*

$$\lim_{n\to\infty} \rho_n(x,y) = \rho(x,y),$$

which defines a positive-definite trace-class operator R of trace 1. Then

$$\lim_{n\to\infty} \|R_n - R\|_{\mathrm{tr}} = 0$$

where $\|A\|_{\mathrm{tr}} = \mathrm{tr}\big(AA^*\big)^{1/2}$.

7. Symmetries in the Hubbard model

Brian Flowers, Baron Flowers FRS (1924-2010), was a British physicist, he was educated in Swansea at the Bishop Gore School. Rector of Imperial College (1973-1985), VC of University of London (1985-1990). While VC of the University of London, he became known for making extensive notes during committee meetings. People thought that maybe he didnt trust the minutes. Later, when his textbook "An introduction to numerical methods in C++" came out, it all became clear.

Walter Marshall FRS (1932-1996) from Cardiff gained a PhD under Rudolf Peierls. He succeeded Brian Flowers as Head of Theoretical Physics Division at AERE (Atomic Energy Research Establishment) Harwell.

John Hubbard (1931-1980) from London was the Head of the Theoretical Physics Group at AERE. He left the UK for the US in 1976, following Marshall's promotion to director of the AERE and a subsequent major reform of its facilities in Harwell. He joined Brown University and the IBM Laboratories in San Jose, California, where his research focused on the study of critical phenomena: phase transitions near which universal behaviour, independent of material specific properties, is observed.

When asked what the book "The Many-Body Problem" was about, declared that it was a murder mystery.

With vertex $i \in \Gamma$ associate a bosonic Fock-Hilbert space $\mathcal{H} = \bigoplus \mathcal{H}_k, \mathcal{H}_k = \mathrm{L}_2^{sym}(M^k), M = \mathbf{R}^d/\mathbf{Z}^d$- d-dimensional torus. The action of G is lifted to unitary operator $U_\Lambda(g)$:

$$U_\Lambda(g)\phi(\mathbf{x}^*_\Lambda) = \phi(g^{-1}\mathbf{x}^*_\Lambda), (g,\mathbf{x}) \in G \times M \to g\mathbf{x} \in M.$$

Formal Hamiltonian (with $\emptyset$ boundary conditions)

$$\begin{aligned}
&(H_\Lambda\phi)(\mathbf{x}^*_\Lambda) = \Big[-\tfrac{1}{2}\textstyle\sum_{j\in\Lambda}\sum_{x\in\mathbf{x}^*(j)} \Delta_j^{(x)} + \sum_{j\in\Lambda}\sum_{x\in\mathbf{x}^*(j)} U^{(1)}(x)\\
&+\tfrac{1}{2}\textstyle\sum_{j\in\Lambda}\sum_{x,x'\in\mathbf{x}^*(j)} \mathbf{1}(x\neq x')U^{(2)}(x,x') + \tfrac{1}{2}\sum_{j,j'\in\Lambda}\mathbf{1}(j\neq j')J(d(j,j'))\times\\
&\textstyle\sum_{x\in\mathbf{x}^*(j),j'\in\mathbf{x}^*(j')} V(x,x')\Big]\phi(\mathbf{x}^*_\Lambda) + \sum_{j,j'}\mathbf{1}(\sharp\mathbf{x}^*(j)\geq 1, \sharp\mathbf{x}^*(j') < \kappa)\times\\
&\textstyle\sum_{x\in\mathbf{x}^*(j)} \int \nu(\mathrm{d}y)[\phi(\mathbf{x}^{*(j,x)\to(j',y)}(j)) - \phi(\mathbf{x}^*(j))].
\end{aligned} \tag{7.1}$$

Assume that

$$ze^{\Theta} < 1 \tag{7.2}$$

where $\Theta = \kappa\beta(\bar{U}^{(1)} + \kappa\bar{U}^{(2)} + \kappa\bar{J}(1)\hat{V})$ and

$$\bar{J}(l) = \sup_{j'\in\Gamma}\Big[\sum_{j\in\Gamma} J(d(j',j))\mathbf{1}(d(j',j)\geq l)\Big].$$

***Theorem 2.** Assume that all potentials are invariant under continuous group G and satisfy some addtional conditions. Then for all β, z satisfying (7.2) all* *states corresponding the Hamiltonian (7.1) are $G-$invariant : $\forall A \in \mathcal{B}_{\Lambda_0}$ and $g \in G$*

$$\varphi(A) = \varphi(U^{-1}_{\Lambda_0}(g) A U_{\Lambda_0}(g))$$

8. Bose gas in $\mathbf{R}^2$

The simplest example of breaking translational symmetry is the **wetting transition** *in 2D Ising model for $\beta \to \infty$. Let $\underline{n}$ be a unit vector, and $\mathcal{D}_{\mathbf{n}}$ be the straight line line through the origin with normal $\underline{n}$. Denote by $D_{\underline{n}}$ the length of the segment $\mathcal{D}_{\mathbf{n}} \cap [-1,1]^2$, and define the following boundary condition*

$$\eta_{\underline{n}} = \mathrm{sign}(\underline{x}, \underline{n}).$$

The **surface tension** *in the direction $\underline{n}$ is defined by*

$$\tau(\underline{n}, \beta) = -\lim_{l\to\infty} \frac{1}{l D_{\underline{n}}} \ln \frac{Z^{\eta_{\underline{n}}}_{\Lambda_l}}{Z^{+}_{\Lambda_l}}. \tag{8.1}$$

We prove that the translational invariance is preserved if $\beta < \beta_0$.

The local Hamiltonian

$$(H_{n,\Lambda}\phi_n)(\underline{x}^n_1) = -\frac{1}{2}\sum_{j=1}^{n} (\Delta_j \phi_n)(\underline{x}^n_1) + \sum_{1\le j<j'\le n} V(x(j) - x(j'))(\underline{x}^n_1) \tag{8.2}$$

which acts on functions $\phi_n \in \mathrm{L}^{sym}_2(\Lambda^n_r)$ where Λ^n_r stands for the set of $n-$points configurations in Λ with a hard core of radius r. Define a partition function

$$\Xi_{\beta,n}(\Lambda) = \mathrm{tr}_{\mathrm{L}^{sym}_2(\Lambda^n_r)} G_{\beta,n,\Lambda}, G_{\beta,n,\Lambda} = \exp\left[-\beta H_{n,\Lambda}\right] \tag{8.3}$$

positive-definite trace-class operator in $\mathrm{L}^{sym}_2(\Lambda^n_r)$. Similar define $G_{\beta,n,\Lambda|\mathbf{x}(\Lambda)^c}$ for a boundary condition $\mathbf{x}(\Lambda)^c$. Next, for a given fugacity $z > 0$ define

$$G_{\Lambda|\mathbf{x}(\Lambda)^c} = \sum_{n\ge 0} z^n G_{\beta,n,\Lambda|\mathbf{x}(\Lambda)^c},$$

$$\Xi(\Lambda|\mathbf{x}(\Lambda)^c) = \sum_{n\ge 0} z^n \Xi_{\beta,n}(\Lambda|\mathbf{x}(\Lambda)^c) = \mathrm{tr}_{H(\Lambda)} G_{\Lambda|\mathbf{x}(\Lambda)^c}.$$

Now the **density matrix** *(for simplicity select $\mathbf{x}(\Lambda)^c = \emptyset$)*

$$R_{\beta,\Lambda} = \frac{1}{\Xi_\beta(\Lambda)} G_{\beta,\Lambda} \tag{8.4}$$

defines the **Gibbs state***, i.e., a linear positive normalized functional $\varphi_{z,\beta,\Lambda}$ on the C^*-algebra of bounded operators $A \in \mathcal{B}(\Lambda)$:*

$$\varphi_{z,\beta,\Lambda} = \mathrm{tr}_{\mathcal{H}(\Lambda)}\big(A R_{\beta.\Lambda}\big), A \in \mathcal{B}(\Lambda). \tag{8.5}$$

Reduced density matrices (RDMs) $R^{\Lambda_0}_{\Lambda}, R^{\Lambda_0}_{\Lambda|\mathbf{x}(\Lambda^c)}$ and R^{Λ_0} are integral operators, say

$$\left(R^{\Lambda_0}_{\Lambda}\phi_\Lambda\right)(\mathbf{x}_0) = \int_{C_r(\Lambda)} F^{\Lambda_0}_{\Lambda}(\mathbf{x}_0, \mathbf{y}_0)\phi_\Lambda(\mathbf{y}_0)\mathrm{d}\mathbf{y}_0. \tag{8.6}$$

Next, we define the limiting density matrices as $\Lambda \nearrow \mathbf{R}^2$. Here and below we assume that z, β satisfy the condition

$$\bar{\rho} = z \exp\left(4\beta\bar{V}R^d/r_0^d\right) < 1. \tag{8.7}$$

Here $\bar{V} = -\max[0, -V(r), r_0 \le r \le R]$, R stands for the radius of potential, r_0 stands for the radius of the hard core. However, it is valid $\forall z \in (0,1)$ if the two-body potential $V \ge 0$. We also assume that $\bar{V}^{(2)} = \max[|V''(r)|, r_0 \le r \le R] < \infty$.

Theorem 3. *The family $F^{\Lambda_0}_{\Lambda|\mathbf{x}(\Lambda^c)}(\mathbf{x}_0, \mathbf{y}_0)$ is compact in the space of continuous functions $\mathcal{C}^0(C_r(\Lambda_0) \times C_r(\Lambda_0))$. Any limiting point $F^{\Lambda_0}(\mathbf{x}_0, \mathbf{y}_0)$ determines a positive-definite operator R^{Λ_0} of trace 1. Consider a pair of limit-points along a sequence $\Lambda(l) \nearrow \mathbf{R}^2$*

$$F^{\Lambda_1} = \lim_{l\to\infty} F^{\Lambda_0}_{\Lambda(l)|\mathbf{x}(\Lambda(l)^c)}, F^{\Lambda_0} = \lim_{l\to\infty} F^{\Lambda_0}_{\Lambda(l)|\mathbf{x}(\Lambda(l)^c)}. \tag{8.8}$$

Then for any $\Lambda_1 \subset \Lambda_0$ the compatibility property holds:

$$R_{\beta,\Lambda_1} = \mathrm{tr}_{\mathcal{H}(\Lambda_0\setminus\Lambda_1)} R_{\beta,\Lambda_0}. \tag{8.9}$$

We establish the translation invariance of bose-gas:

Theorem 4. *Under condiitons formulated below*

$$\varphi(A) = \varphi(S(s)A), \quad \forall A \in \mathcal{B}(\Lambda_0) \tag{8.10}$$

or

$$R_{\beta,S(s)\Lambda_0} = U^{\Lambda_0}(s)R_{\beta,\Lambda_0}U^{S(s)\Lambda_0}(-s) \tag{8.11}$$

Here

$$S(s)A = U^{S(s)\Lambda_0}(-s)AU^{\Lambda_0}(s) \in \mathcal{B}(S(s)\Lambda_0). \tag{8.12}$$

Feynman-Kac representation in a cube Λ

$$F^{\Lambda_0}_{\Lambda|\mathbf{x}(\Lambda^c)}(\mathbf{x}_0, \mathbf{y}_0) = \int_{W^*(\mathbf{x}_0,\mathbf{y}_0)} \mathbf{P}_{\mathbf{x}_0,\mathbf{y}_0}(\mathrm{d}\Omega_0^*)z^{K(\Omega_0^*)}\alpha_\Lambda(\Omega_0^*)\chi^{\Lambda_0}(\Omega^*_{\Lambda_0})\frac{\Xi^{\Lambda_0,\Omega_0^*}[\Lambda\setminus\Lambda_0|\mathbf{x}(\Lambda^c)]}{\Xi[\Lambda|\mathbf{x}(\Lambda^c)]}.$$

where $K(\Omega_0^*) = \sum_{\omega^*\in\Omega_0^*} k(\omega^*)$, $\alpha_\Lambda(\Omega_0^*)$ and $\chi^{\Lambda_0}(\Omega^*_{\Lambda_0})$ indicates that the paths are always inside Λ but outside Λ_0 at moments $\sim\beta$.

proves that a.a. quenched Gibbs measures generated by U are $\mathtt{G}-$ *invariant. The basis is the following property of specifications (conditional probabilities) of Gibbs measures*

$$\gamma = \{\gamma_\Lambda(\omega|\bar{\omega}) = \frac{1}{Z_\Lambda^\Phi(\bar{\omega})} e^{-\beta H_\Lambda^\Phi(\omega|\bar{\omega})}\}.$$

Lemma 3. *(***H.-O. Georgii***: Gibbs Measures and Phase Transitions)*
Let for any cylindrical set A $\exists$ $a, b > 0$ and $\Lambda \in \mathbf{Z}^d$ with

$$a\gamma_\Lambda(g^{-1}A|.) + b\gamma_\Lambda(gA|.) \geq \gamma_\Lambda(A|.). \tag{9.1}$$

Then g preserve any measure $\mu \in$ Gibbs(γ).

We illustrate the method for a classical model on Lorenzian triangulation. Let T_r be the union of the first r layers of the Lorentzian tree T. Let $\mathcal{G}$ be a $d-$dimensional torus. Identify g with the vector $\underline{\theta}$ and define a gauged action on the layer $j, r+1 < j < n$:

$$g_n = \underline{\theta} \frac{1}{Q(n-r)} \sum_{t=j+1-r}^{n-j} \frac{1}{t \ln t}$$

where

$$Q(n-r) = \sum_{t=2}^{n-r} \frac{1}{t \ln t}.$$

Let $E_{t,t+1} \leq k_t + k_{t+1}$ be the number of edges between levels j and $j+1$.Then

$$\phi = \sum_{\langle v,v' \rangle} (g_n(v) - g_n(v'))^2 \leq \frac{|\underline{\theta}|^2}{\ln\ln(n-r)} \sum_{t=2}^{n-r} \frac{E_{t,t+1}}{t^2(\ln t)^2} \to 0 \tag{9.2}$$

as $n \to \infty$. The series in (9.2) converges due to Lemma 1.

A tuned-shift argument:

Lemma 4. *Let μ be a FK-DLR measure, and an event $\mathcal{D}$ is localized in Λ_0. Then measure μ is $S(s)$ invariant if and only if*

$$\mu(S(s)\mathcal{D}) + \mu(S(-s)\mathcal{D}) - 2\mu(\mathcal{D}) \geq 0. \qquad (9.2)$$

Proof *Let $\tau = S(s)$. Then*

$$\mu(\tau^{k+1}D) + \mu(\tau^{k-1}D) \geq 2\mu(\tau^k D).$$

The sequence $\{\mu(\tau^k D)\}$ is convex and bounded. Hence, it has to be constant, in particular $\mu(\tau^{-1}D) = \mu(D)$.

10. Feynman-Kac formula for density matrix kernel

The idea goes back to **Jean Ginibre**, *Paris-Sud 11 University.*

$$\Big(\exp\,\big[-\beta H_\Lambda\big]\phi\Big)(\mathbf{x}_\Lambda) = \int_{M^\Lambda} \prod_{j\in\Lambda} v(\mathrm{d}y(j)) K_{\beta,\Lambda}(\mathbf{x}_\Lambda, \mathbf{y}_\Lambda)\phi(\mathbf{y}_\Lambda). \tag{10.1}$$

The integral kernel $K_{\beta,\Lambda}(\mathbf{x}_\Lambda, \mathbf{y}_\Lambda)$ admits a Feynman–Kac (FK) integral representation

$$K_{\beta,\Lambda}(\mathbf{x}_\Lambda, \mathbf{y}_\Lambda) = \int\limits_{\overline{W}^\beta_{\mathbf{x}_\Lambda,\mathbf{y}_\Lambda}} \mathbf{P}^\beta_{\mathbf{x}_\Lambda,\mathbf{y}_\Lambda}(\mathrm{d}\overline{\boldsymbol{\omega}}_\Lambda)\exp\,\big[-h^\Lambda(\overline{\boldsymbol{\omega}}_\Lambda)\big]. \tag{10.2}$$

Define an energy for a path configuration $\overline{\boldsymbol{\omega}}_\Lambda = \{\omega_j,\ j\in\Lambda\}$ over Λ,

$$h^\Lambda(\overline{\boldsymbol{\omega}}_\Lambda) = \sum_{(j,j')\in\Lambda\times\Lambda} h^{j,j'}(\overline{\omega}_j, \overline{\omega}_{j'})$$

where $h^{j,j'}(\overline{\omega}_j, \overline{\omega}_{j'})$ represents an integral along trajectories $\overline{\omega}_j$ and $\overline{\omega}_{j'}$:

$$h^{j,j'}(\overline{\omega}_j, \overline{\omega}_{j'}) = J(\mathtt{d}(j,j'))\int\limits_0^\beta \mathrm{d}\tau\, V\big(\overline{\omega}_j(\tau), \overline{\omega}_{j'}(\tau)\big).$$

$h^{j,j'}(\overline{\omega}_j, \overline{\omega}_{j'})$ yields the 'energy of interaction' between trajectories $\overline{\omega}_j$ and $\overline{\omega}_{j'}$, and $h^\Lambda(\overline{\boldsymbol{\omega}}_\Lambda)$ equals the 'full potential energy' of the path configuration $\overline{\boldsymbol{\omega}}_\Lambda$.

11. Breaking of continuous symmetry

Theorem 5. *Take $\Gamma = \mathbf{Z}^2$ with distance $\mathtt{d}(j,j') = \max\,\big[|j_1 - j_1'|, |j_2 - j_2'|\big]$. Take $M = S^1 = \mathtt{G}$ where $S^1 = \mathbf{R}/\mathbf{Z}$ is a unit circle, with a standard metric $\rho(x,x') = \min\,\big[|x-x'|, 1-|x-x'|\big]$ and the group operation of addition mod 1. Assume that the two-body potentials $J(\mathtt{d}(j,j'))$ and $V(x,x')$, $j,j'\in\mathbf{Z}^2$, $x,x'\in S^1$, are of the form*

$$\begin{aligned} J(\mathtt{d}(j,j')) &= 1,\ |j-j'| = 1,\\ &= 0,\ |j-j'| \neq 1,\end{aligned}$$

$$\begin{aligned} V(x,x') &= -\cos 2\pi(x-x'),\ \rho(x,x') \leq \theta,\\ &= +\infty,\ \rho(x,x') > \theta,\end{aligned}$$

where $\theta \in (0, 1/4)$. Then, $\forall\ \beta \in (0, \infty)$, $\exists$ a measure $\widetilde{\mu} = \widetilde{\mu}_\beta$ which is not S^1-invariant. Consequently, the corresponding state $\widetilde{\varphi} = \widetilde{\varphi}_{\widetilde{\mu}}$ is not S^1-invariant.

Theorem 5: a sketch of the proof:

Consider a sequence of 'cooled' boundary conditions x^* and a measure μ^* which is a limiting point of Gibbs measures in Λ_n. If μ^* is not rotation invariant, we are done. Otherwise, select an arc $\alpha = (x^* - 1/200, x^* + 1/200)$. So, the weight of $\alpha < 1/99$ for n large enough. Next, given $\eta \in (0, 1]$ consider a sequence of boundary conditions

$$\tilde{x}_{j,\eta} = x^* + j_1 \eta \theta, j = (j_1, j_2).$$

For $\eta = 1$ $\exists$ unique compatible configuration inside the box. Hence, $\forall n$ $\exists$ $\eta^* \in (0, 1)$ such that the weight of $\alpha = 2/3$. Any limiting point of this sequence is not rotation invariant.

12. Dobrushin-Lanford-Ruelle (DLR) equations in quantum statistics

The standard approach to phase transitions in quantum statistics are KMS (Kubo-Martin-Schwinger) states. In Heisenberg picture $\alpha_\tau(A) := e^{iH\tau}Ae^{-iH\tau}$

$$\begin{aligned}&\langle \alpha_\tau(A)B\rangle_\beta = \mathrm{Tr}[R\alpha_\tau(A)B]\\ &= \mathrm{Tr}[RB\alpha_{\tau+i\beta}] = \langle B\alpha_{\tau+i\beta}(A)\rangle_\beta.\end{aligned} \tag{12.1}$$

RHS and LHS of (12.1) are the boundary values of an analytic function of z. *If there is a phase transition or spontaneous symmetry breaking, the KMS state is not unique. In the case of the density matrix* R *with positive elements we develop a simpler approach based on the DLR equations. Define*

$$p_\Lambda(\mathbf{x}_\Lambda, \boldsymbol{\omega}_\Lambda) = \frac{1}{\Xi_\Lambda}\exp\big[-h^\Lambda(\boldsymbol{\omega}_\Lambda)\big] \tag{12.2}$$

Given $\Lambda^0 \subset \Lambda$ *consider the partially integrated probability density*

$$p_\Lambda^{\Lambda^0}(\boldsymbol{\omega}^0) := \int_{W_{\Lambda\setminus\Lambda^0}} \mathrm{d}\nu_{\Lambda\setminus\Lambda^0}(\boldsymbol{\omega}_{\Lambda\setminus\Lambda^0}) p_\Lambda(\boldsymbol{\omega}^0 \vee \boldsymbol{\omega}_{\Lambda\setminus\Lambda^0})$$

where $\boldsymbol{\omega}^0 \vee \boldsymbol{\omega}_{\Lambda\setminus\Lambda^0}$ *stands for concatenation of two loop configurations.*

DLR equations: $\forall$ *set* Λ' *such that* $\Lambda^0 \subset \Lambda' \subset \Lambda$, *the density* $p_\Lambda^{\Lambda^0}(\boldsymbol{\omega}^0)$ *obeys*

$$\begin{aligned}p_\Lambda^{\Lambda^0}(\boldsymbol{\omega}^0) = &\int_{W_{\Lambda\setminus\Lambda'}} \mathrm{d}\nu_{\Lambda\setminus\Lambda'}(\boldsymbol{\omega}_{\Lambda\setminus\Lambda'})\\ &\times p_\Lambda^{\Lambda\setminus\Lambda'}(\boldsymbol{\omega}_{\Lambda\setminus\Lambda'})\frac{\Xi_{\Lambda'\setminus\Lambda^0}(\boldsymbol{\omega}^0, \boldsymbol{\omega}_{\Lambda\setminus\Lambda'})}{\Xi_{\Lambda'}(\boldsymbol{\omega}_{\Lambda\setminus\Lambda'})}.\end{aligned}$$

Define the class of infinite-volume Gibbs states $\mathcal{G}(\beta)$ *satisfying DLR equations.*

Theorem 6. *For all* $\beta \in (0,\infty)$, *the sequence of Gibbs states* $\varphi_{\Lambda(n)}$ *contains a subsequence* $\varphi_{\Lambda(n_k)}$ *such that* $\forall$ *finite* $\Lambda^0 \subset \Gamma$ *and* $A_0 \in \mathcal{B}_{\Lambda^0}$, *we have:*

$$\lim_{k\to\infty} \varphi_{\Lambda(n_k)}(A_0) = \varphi(A_0)$$

where state $\varphi \in \mathcal{G}(\beta)$. *Consequently, class* $\mathcal{G}(\beta)$ *is non-empty.*

Theorem 7. *For all* $\beta \in (0,\infty)$ *and finite* $\Lambda^0 \subset \Gamma$, *any Gibbs state* $\varphi \in \mathcal{G}(\beta)$ *is* $\mathtt{G}-$ *invariant.*

References

[1] M. Kelbert, Y. Suhov. *A quantum MerminWagner theorem for quantum rotators on two-dimensional graphs.* Journ. Math. Phys., **54**, No 3, 2013 http://dx.doi.org/10.1063/1.4790885

[2] M. Kelbert, Y. Suhov. *A quantum Mermin–Wagner theorem for a generalized Hubbard model on a 2D graph.* Advances Math. Phys., V.2013, ID 637375, 2013
arXiv; 1210.8344 [math-ph]

[3] M. Kelbert, Y. Suhov, A. Yambartsev. *A Mermin-Wagner theorem for Gibbs states on Lorentzian triangulations.* Journ. Stat. Phys., **150** (2013), 671–677

[4] M. Kelbert, Y. Suhov, A. Yambartsev. *A Mermin-Wagner theorem on Lorentzian triangulations with quantum spins,* Brazilian Journ. Prob., 2013 arXiv:1211.5446v1 [math-ph]

[5] Y. Suhov, M. Kelbert. *FK-DLR states of a quantum bose-gas.* arXiv:1304.0782 [math-ph]

[6] Y. Suhov, M. Kelbert, I. Stuhl. *Shift-invariance of FK-DLR states of a 2D quantum bose-gas*
ArXiv:1304.4177 [math-ph]