Einstein relation and monotonicity of the speed for random walk among random conductances

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Outline

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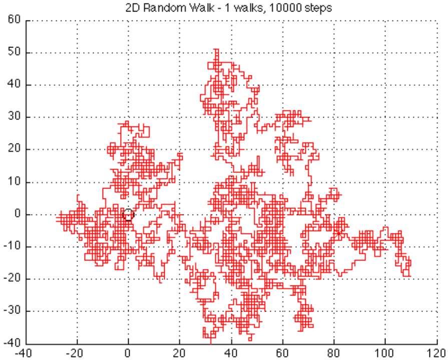
1. The Random Conductance Model
2. Random walks on supercritical percolation clusters
3. Random walks with drift in random environment
4. Einstein-Relation
5. A result about monotonicity

Warm-up: simple random walk

Take a simple random walk on the *d*-dimensional lattice.

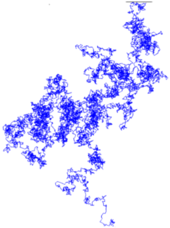


It starts from the origin and moves, with equal probabilities, to the nearest neighbours.

It is well-known that the scaling limit of simple random walk is a Brownian motion, a Gaussian process in continuous time on $\mathbb{R}^d$. More precisely, the law of (the linear interpolation of)

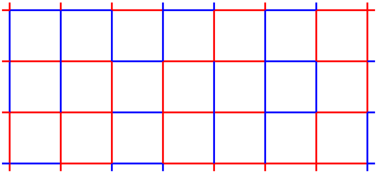
$$(X_m/\sqrt{n})_{m=0,1,\ldots,n}$$

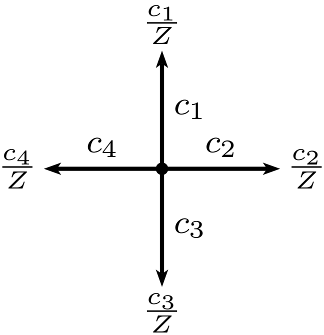
converges to the law of $(\sigma B_t)_{0\leq t\leq 1}$ where σ is a constant depending on the dimension d. This convergence is "universal" and holds as well, for instance, for triangular lattices.



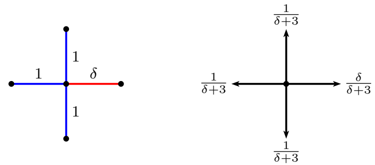
The Random Conductance Model

We define a random medium by giving random weights - often called "conductances" - to the bonds of the lattice.
Consider first the case where the weights are independent, with the same law. Assume that the conductances are bounded above and bounded away from zero.
The configurations of the weights is called "environment". For a fixed environment, define the law of a random walk, where the transition probabilities from a point to its neighbours are proportional to the weights of the bonds.





wobei $Z := c_1 + c_2 + c_3 + c_4$.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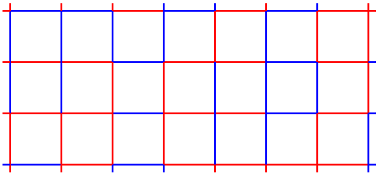

General question:

Question

Can the random medium be replaced by an "averaged" deterministic medium?

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There are two, contradicting paradigms in the theory of random media: *Homogenization* versus *Intermittency*.

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Answer: Yes! (This was proved in in several papers by Martin Barlow, Luis Renato Fontes/Pierre Mathieu, S. M. Kozlov, Vladas Sidoravicius/Alain-Sol Sznitman,... and extended to the case of bounded, strictly positive conductances).

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An active direction of research is the extension of this theorem to the case when the conductances form a stationary, ergodic random field. It is not true in general, but true under boundedness conditions on the conductances. In this case, σ has to be replaced with a deterministic, positive definite covariance matrix Σ.

Method of proof: decompose the walk in a martingale part and a "corrector". Show that the corrector can be neglected and apply the CLT for martingales.

The corrector is in itself an interesting process, see Jean-Christophe Mourrat/Felix Otto for recent results.

Question

How does Σ depend on the law of the conductances?

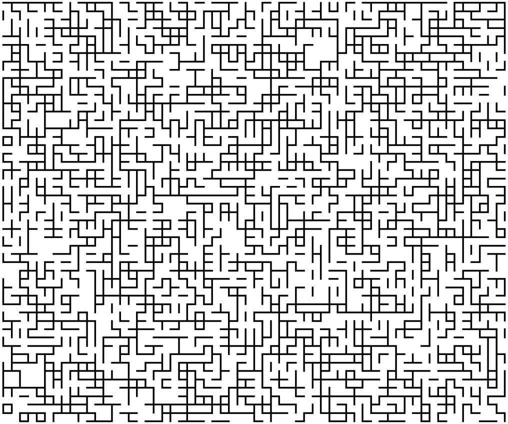
Note that this is important from the viewpoint of "material sciences"! Analytical counterpart of this question, many papers but still open questions. Recent results by Antoine Gloria, Jean-Christophe Mourrat, Stefan Neukamm, Felix Otto.

Random walks on supercritical percolation clusters

To be more radical, consider bond percolation with parameter p on the d-dimensional lattice: all bonds are *open* with probability p and *closed* with probability $1 - p$, independently of each other. This corresponds to conductances with values either 1 or 0.

Take bond percolation on $\mathbb{Z}^d$, $d \geq 2$. Choose p close enough to 1 such that there is a (unique) infinite cluster.
Condition on the event that the origin is in the infinite cluster.
Start a random walk in the infinite cluster which can only walk on open bonds, and which goes with equal probabilities to all neighbours. (In particular, this random walk never leaves the infinite cluster.)

Bond percolation p= 0.51



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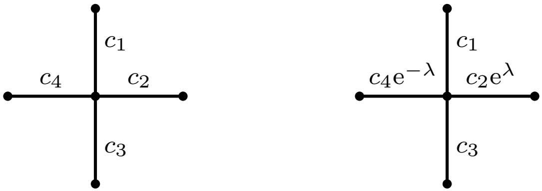
Question

How does σ depend on p?

Einstein-Relation

The Einstein relation gives a different interpretation of the variance as the derivative of the speed of the random walk, when one has a drift in a "favourite" direction ℓ. This leads us to **random walks with drift in random environments**.

We add a drift in direction $\ell = e_1$: choose a parameter $\lambda > 0$ for the strength of the drift and multiply the conductances with powers of e^{λ}.
Example: $d = 2, e_1 = (1, 0)$.



In the same way, define the random walk with drift on the supercitical percolation cluster.

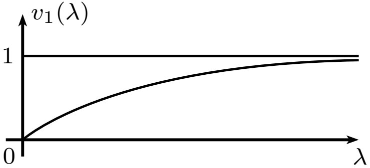
Questions:

–does the random walk move with a constant linear speed, i.e.

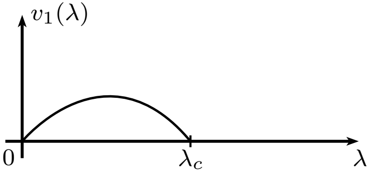
$$\text{does } v(\lambda, p) := \lim_{n\to\infty} \frac{X_n}{n} \text{ exist, and is it deterministic?}$$

–If yes, is the component of $v_1(\lambda, p) = v(\lambda, p) \cdot e_1$ in the favourite direction strictly positive?
–How does $v_1(\lambda, p)$ depend on λ and on p?

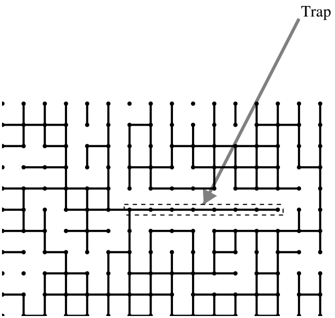
Back to the homogeneous medium: in this case, $v(\lambda)$ can be computed and $v_1(\lambda)$ looks as follows:



For the speed of the random walk on an infinite percolation cluster, the following picture is conjectured:
for each $p \in (p_c, 1)$ we have, with $v_1(\lambda) = v_1(\lambda, p)$:



Reason for the zero speed regime:



Alexander Fribergh and Alan Hammond showed recently that there is, for each $p \in (p_c, 1)$, a critical value λ_c such that $v_1(\lambda) > 0$ for $\lambda < \lambda_c$ and $v_1(\lambda) = 0$ for $\lambda > \lambda_c$. Quoting from their paper:

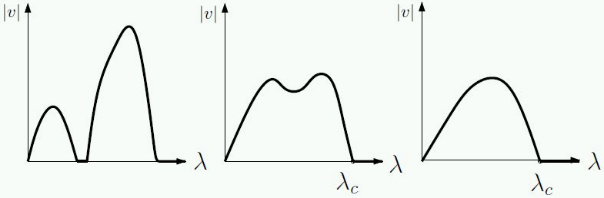


FIGURE 3. The speed as a function of the bias. Sznitman and Berger, Gantert and Peres established positive speed at low λ, but their works left open the possibility depicted in the first sketch. Our work rules this out, though the behavior of the speed in the ballistic regime depicted in the second sketch remains possible. The third sketch shows the unimodal form predicted physically.

For the random walk among iid, bounded (above and away from 0) random conductances, the following is known.

Theorem

(Lian Shen 2002)
For fixed drift, there is a law of large numbers:
For any $\lambda > 0$,

$$\lim_{n\to\infty} \frac{1}{n} X(n) = v(\lambda), \ \ a.s.$$

where $v(\lambda)$ *is deterministic and* $v(\lambda) \cdot e_1 > 0$.

Theorem

Einstein-Relation (NG, Jan Nagel und Xiaoqin Guo, in progress)
Assume that the conductances are iid, bounded above and bounded away from 0*.*

$$\lim_{\lambda \to 0} \frac{v(\lambda)}{\lambda} = \Sigma e_1 \,.$$

Further, $v(\lambda)$ *is differentiable for all* λ *and we can write a formula for the derivative.*

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It can be proved easily in the one-dimensional case and in the periodic case.

Einstein relation for symmetric diffusions in random environment

Consider diffusion $X(t)$ in $\mathbb{R}^d$ with generator

$$L^\omega f(x) = \frac{1}{2} e^{2V^\omega(x)} \operatorname{div}(e^{-2V^\omega} a^\omega \nabla f)(x), \tag{1}$$

where V^ω is a real function and a^ω is symmetric matrix. V^ω and a^ω are realizations of a random environment, defined on some prob. space $(\Omega, \mathcal{A}, Q)$.

Assumptions:
(1) Translation invariance, ergodicity
(2) Smoothness: $x \to V^\omega(x)$ and $x \to a^\omega(x)$ are smooth (for simplicity)
(3) Uniform ellipticity: V^ω is bounded and a^ω is uniformly elliptic, namely there exists a constant κ such that, for all ω, $x \in \mathbb{R}^d$ and $y \in \mathbb{R}^d$,

$$\kappa|y|^2 \leq |a^\omega(x)y|^2 \leq \kappa^{-1}|y|^2\,.$$

(4) Finite range dependence.

Then, with $\sigma^\omega = \sqrt{a^\omega}$ and $b^\omega = \frac{1}{2}\mathrm{div}\, a^\omega - a^\omega \nabla V^\omega$, X solves the stochastic differential equation

$$dX(t) = b^\omega(X(t))\, dt + \sigma^\omega(X(t)) dW_t \tag{2}$$

where W is a Brownian motion.

Theorem

(George Papanicolaou, Srinivasa Varadhan, Hirofumi Osada, S. M. Kozlov 1980, 1982) The process X satisfies a Central Limit Theorem i.e. $\frac{1}{\sqrt{t}}X(t)$ converges in law towards a Gaussian law. More is known: the rescaled process

$$\left(X^{(n)}(t)\right)_{t\geq 0} := \left(\frac{1}{\sqrt{n}}X(nt)\right)_{t\geq 0} \tag{3}$$

satisfies an invariance principle: there exists a non-negative (deterministic) symmetric matrix Σ such that the law of $(X^{(n)}(t))_{t\geq 0}$ converges to the law of $(\sqrt{\Sigma}\,W(t))_{t\geq 0}$.

The statement holds for almost any realization of the environment. Note that Σ is in general **not** the average of a^{ω}.

Now, add a local drift in the equation satisfied by X: let $\ell \in \mathbb{R}^d$ be a vector, $\ell \neq 0$, and take the equation

$$dX^\lambda(t) = b^\omega(X^\lambda(t))dt + \sigma^\omega(X^\lambda(t))dW_t + a^\omega(X^\lambda(t))\lambda\ell dt. \qquad (4)$$

Theorem

(Lian Shen 2003)
Assume Q has finite range of dependence and V^ω is smooth and bounded. Then the diffusion in random environment X^λ satisfies the law of large numbers: For any $\lambda > 0$,

$$\lim_{t\to\infty} \frac{1}{t} X^\lambda(t) = v(\lambda), \ \ a.s. \tag{5}$$

where $v(\lambda)$ is a deterministic vector and $\ell \cdot v(\lambda) > 0$.

Again, v is called the effective drift.

Strategy of the proof (for fixed λ): Show that the process is transient in direction ℓ,

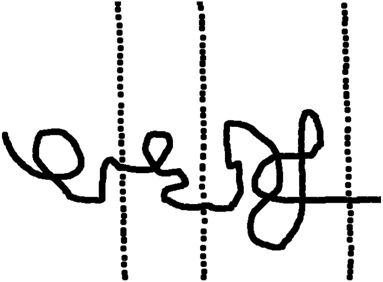
$$\lim_{t} \ell \cdot X^{\lambda}(t) = +\infty\,, \text{ a.s.} \tag{6}$$

Define regeneration times $\tau_1, \tau_2, \ldots$ Show that $\mathbb{E}_0\left[\tau_2 - \tau_1\right] < \infty$. Conclude that

$$\lim_{t\to\infty} \frac{1}{t} X^{\lambda}(t) = \frac{\mathbb{E}_0\left[X^{\lambda}(\tau_2) - X^{\lambda}(\tau_1)\right]}{\mathbb{E}_0\left[\tau_2 - \tau_1\right]} \quad \text{a.s.} \tag{7}$$

Regeneration times

... cut the path AND the environment in independent pieces (or: almost independent pieces, sufficiently independent pieces, . . .).



Theorem

Einstein relation. (NG, Pierre Mathieu, Andrey Piatnitski)
The effective diffusivity can be interpreted with the derivative of the effective drift:

$$\lim_{\lambda\to 0} \frac{v(\lambda)}{\lambda} = \Sigma\ell\,. \tag{8}$$

In other words, the function $\lambda \to v(\lambda)$ *has a derivative at* 0 *and we have for any vector* e

$$\lim_{\lambda\to 0} \frac{1}{\lambda} e\cdot v(\lambda) = e\cdot \Sigma\ell. \tag{9}$$

Why it should be true: heuristics

A key ingredient is **Girsanov transform**. For any t, the law of $(X^\lambda(s))_{0\leq s\leq t}$ is absolutely continuous w. r. t. the law of $(X(s))_{0\leq s\leq t}$ and the Radon-Nikodym density is the exponential martingale

$$e^{\lambda B(t)-\frac{\lambda^2}{2}\langle B\rangle(t)} \tag{10}$$

where

$$B(t)=\int_0^t \ell^T\sigma^\omega(X(s))\cdot dW_s \tag{11}$$

and

$$\langle B\rangle(t)=\int_0^t \left|\ell^T\sigma^\omega(X(s))\right|^2 ds \tag{12}$$

In particular,

$$\mathbb{E}_0\left[X^\lambda(t)\right] = \mathbb{E}_0\left[X(t)e^{\lambda B(t)-\frac{\lambda^2}{2}\langle B\rangle(t)}\right] \tag{13}$$

Hence

$$\frac{d}{d\lambda}\,\mathbb{E}_0\left[X^\lambda(t)\right]\Big|_{\lambda=0} = \mathbb{E}_0\left[X(t)B(t)\right] \tag{14}$$

and

$$\lim_{t\to\infty}\frac{1}{t}\frac{d}{d\lambda}\,\mathbb{E}_0\left[X^\lambda(t)\right]\Big|_{\lambda=0} = \lim_{t\to\infty}\frac{1}{t}\mathbb{E}_0\left[X(t)B(t)\right] \tag{15}$$

Exchanging the order of the limits yields

$$\frac{d}{d\lambda}\, v(\lambda)\Big|_{\lambda=0} = \lim_{t\to\infty} \frac{1}{t}\mathbb{E}_0\left[X(t)B(t)\right] \tag{16}$$

A symmetry argument (using the reversibility) shows that

$$\mathbb{E}_0\left[X(t)B(t)\right] = \mathbb{E}_0\left[X(t)(\ell \cdot X(t))\right] \tag{17}$$

and we conclude that

$$\frac{d}{d\lambda}\, v(\lambda)\Big|_{\lambda=0} = \lim_{t\to\infty} \frac{1}{t}\mathbb{E}_0\left[X(t)(\ell \cdot X(t))\right] = \Sigma\ell \tag{18}$$

Why it is true: strategy of the proof

Joel Lebowitz and Hermann Rost showed, using the invariance principle and Girsanov transform:

Theorem

(Joel Lebowitz, Hermann Rost, 1994)
Let $\alpha > 0$. Then

$$\lim_{\lambda \to 0\,, t \to +\infty\,, \lambda^2 t = \alpha} \mathbb{E}_0 \left[\frac{X^\lambda(t)}{\lambda t} \right] = \Sigma\, \ell\,. \tag{19}$$

Idea: work on the scale $\lambda \to 0$, $t \to \infty$, $\lambda^2 t \to \alpha$ and eventually $\alpha \to \infty$. We show that

Proposition

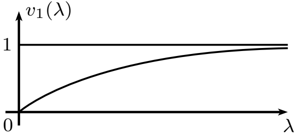
$$\lim_{\alpha \to +\infty} \limsup_{\lambda \to 0, t \to +\infty, \lambda^2 t = \alpha} \left| \mathbb{E}_0 \left[\frac{X^\lambda(t)}{\lambda t} \right] - \frac{v(\lambda)}{\lambda} \right| = 0. \tag{20}$$

In order to show the proposition, follow Lian Shen's construction of regeneration times, but take into account the dependence on λ. To carry this through, need uniform estimates for hitting times (on our scale).

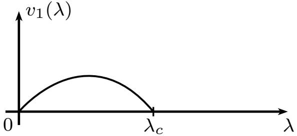
The Einstein relation is conjectured to hold for many models, but it is proved for few. Apart from the results mentioned, examples include:
–Balanced random walks in random environment (Xiaoqin Guo).
–Random walks on Galton-Watson trees (Gérard Ben Arous, Yueyun Hu, Stefano Olla, Ofer Zeitouni).
–Tagged particle in asymmetric exclusion (Michail Loulakis).
The following examples are in progress:
–Random walks on percolation clusters of ladder graphs (NG, Matthias Meiners, Sebastian Müller).
–Mott random walks (Alessandra Faggionato, NG, Michele Salvi).

How does $v_1(\lambda)$ depend on λ for a random walk among random conductances?

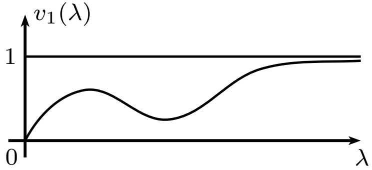
For the homogeneous medium, we have



For the infinite percolation cluster, the conjectured picture is



For the random walk among random conductances, we believe that the picture can be



We show (Noam Berger, NG, Jan Nagel, in progress): the speed in the favourite direction is *not* increasing, provided δ is small enough and the conductances take the values 1 (with probability $> p_c$) and δ with probability $1 - p$.

Thanks for your attention!