The survival probability in high dimensions

Remco van der Hofstad

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Based on

▷ vdH and Mark Holmes. The survival probability and r-point functions in high dimensions. Ann. Math. **178**(2): 665-685, (2013).

Branching processes

Branching process is simple model for

evolution of number of individuals

in population. Individuals each have random number of

identically and independently distributed (i.i.d.) offspring.

Formally, let N_n be number of individuals in generation n in branching process started from single individual. Then

$$N_{n+1} = \sum_{i=1}^{N_n} X_{n,i},$$

where offsprings $(X_{n,i})_{n,i\geq 0}$ form array of i.i.d. random variables.

Questions:

▷ When does population survive with positive probability?

▷ How many individuals are there at time n?

BP phase transition

Theorem 0. $\theta = \mathbb{P}(N_n \geq 1 \forall n \geq 0) = 0$ precisely when $\mathbb{E}[X] \leq 1$.
[Except for boring case $X = 1$ a.s.]

Proof for simple example:

$$\mathbb{P}(X = 2) = p = 1 - \mathbb{P}(X = 0),$$

for which $\mathbb{E}[X] = 2p$. Let $\theta_n(p) = \mathbb{P}(N_n \geq 1)$. Then, $\theta_0(p) = 1$ and

$$\theta_n(p) = p\mathbb{P}(N_n \geq 1 \forall n \geq 0 \mid X_{0,1} = 2) = p[2\theta_{n-1}(p) - \theta_{n-1}(p)^2].$$

Further, $\theta(p) = \mathbb{P}(N_n \geq 1 \forall n \geq 0) = \lim_{n\to\infty} \theta_n(p)$ is largest solution

$$\theta(p) = p[2\theta(p) - \theta(p)^2].$$

Solution:

$\theta(p) = 0$ for $p \leq 1/2$;
$\theta(p) = (2p - 1)/p$ for $p > 1/2$.

BP phase transition

We can compute $\mathbb{E}[N_n] = \mathbb{E}[X]^n$.

$\triangleright$ When $\mathbb{E}[X] < 1$, Markov's inequality shows that

$$\mathbb{P}(N_n \geq 1) \leq \mathbb{E}[N_n] = \mathbb{E}[X]^n,$$

which is exponentially small. Thus, total population $\sum_{n \geq 0} N_n$ has finite mean: **subcritical branching process.**

$\triangleright$ When $\mathbb{E}[X] > 1$,

$$M_n = N_n / \mathbb{E}[N_n]$$

is non-negative martingale, and assuming that $\mathbb{E}[X \log X] < \infty$,

$$M_n \xrightarrow{a.s.} M,$$

where $\theta = \mathbb{P}(M = 0)$. Thus, conditionally on survival,

N_n grows exponentially: **supercritical branching process.**

Critical BPs

Branching processes with offspring X are called **critical** when

$$\mathbb{E}[X] = 1.$$

$\triangleright$ Simplest example of phase transition. Many statistical physics models have phase transition. For branching processes explicit computations are possible.

$\triangleright$ Most interesting behavior occurs close to phase transition, i.e., for critical branching processes. For example,

$$\theta_n = \mathbb{P}(N_n \geq 1) \to 0, \qquad \text{but} \qquad \mathbb{E}[N_n] = 1.$$

$\triangleright$ Implies that $N_n = 0$ most of the times, but when $N_n \geq 1$, in fact N_n is very large.

Critical BPs

Let N_n be number of individuals in generation n in critical branching process with offspring distribution having variance γ.

Kolmogorov (1938):

$$n\theta_n = n\mathbb{P}(N_n \geq 1) \to 2/\gamma.$$

Yaglom (1947): Conditionally on $N_n \geq 1$,

$$N_n/n \xrightarrow{d} \mathrm{Exp}(2/\gamma).$$

How?

Kolmogorov: induction on n, Yaglom: moment method on N_n.

Goal:

Prove Kolmogorov and Yaglom's Theorems for spatial statistical physics models in high dimensions, where interaction between far-away pieces is small.

Oriented percolation

Oriented bonds join (x,n) to $(y,n+1)$ for $n \geq 0$ and $x,y \in \mathbb{Z}^d$.
Make bond $((x,n),(y,n+1))$ independently

occupied	with probability $pD(y-x)$,
vacant	with probability $1-pD(y-x)$.

Here, $p \in [0, 1/\|D\|_\infty]$ is percolation parameter, and $x \mapsto D(x)$ is some random walk transition probability.

Spread-out models: range of D grows proportionally with L and

$$\sup_x D(x) \leq CL^{-d}, \qquad \sum_x |x|^2 D(x) \approx cL^2.$$

Simplest example: $D(x) = (2L+1)^{-d} \mathbb{1}_{\{\|x\|_\infty \leq L\}}$.

OP phase transition

Survival probability: N_n is number of particles alive at time n and

$$\theta_n(p) = \mathbb{P}_p(N_n \geq 1).$$

Oriented percolation has a phase transition, i.e, there is a critical probability $p_c = p_c(d, L) \in (0, \infty)$, such that

- For $p < p_c$, a.s. no infinite cluster, $\theta_n(p)$ exponentially small.
- For $p > p_c$, a.s. unique infinite cluster, $\theta_n(p) \downarrow \theta(p) > 0$.
- For $p = p_c$, $\theta_n(p_c) \downarrow 0$ (Bezuidenhout and Grimmett (1990)), $\theta_n = \theta_n(p_c)$ not understood and dimension dependent.

Goal: Prove that $n\theta_n$ converges in high dimensions.

Related models

$\triangleright$ **Contact process.** Continuous-time version OP.
Bezuidenhout-Grimmett (90): Exists critical infection rate λ_c above which disease survives with positive prob., below it dies out a.s.
$\triangleright$ Survival probability: N_t is number of infected individuals at time t when started from single infected individual, and $\theta_t = \theta_t(\lambda_c) = \mathbb{P}(N_t \geq 1)$.

$\triangleright$ **Lattice trees.** T is finite connected set of bonds containing no cycles. Fix $z > 0$ and define

$$\rho_z(x) = \sum_{T \ni 0,x} z^{|T|} \prod_{(x,y)\in T} D(y-x), \qquad \mathbb{P}(T) = \frac{z_c^{|T|}}{\rho_{z_c}(0)} \prod_{(x,y)\in T} D(y-x)$$

where z_c is radius of convergence of $\rho_{z_c}(0)$.
$\triangleright$ Survival probability: N_n is number of vertices at tree distance n from origin, and $\theta_n = \theta_n(z_c) = \mathbb{P}(N_n \geq 1)$.

Main result

Theorem 1 (Kolmogorov's and Yaglom's Theorem)
Let $L \gg 1$, and $d > 4$ for oriented percolation and contact process, and $d > 8$ for lattice trees. Then, there exist $A, V > 0$ s.t.

$$\lim_{n\to\infty} n\theta_n = 2/(AV),$$

and, conditionally on $N_n > 0$,

$$N_n/n \xrightarrow{d} \mathrm{Exp}(2/(AV)).$$

Interpretation (vdH-Slade03, vdH-Sakai10, Holmes08):

$$A = \lim_{n\to\infty} \mathbb{E}[N_n], \qquad VA^3 = \lim_{n\to\infty} \mathbb{E}[N_n^2]/n.$$

Oriented percolation: reproves result vdH-den Hollander-Slade (07a,07b: $\pm$ 100 pages), at expense of weaker error estimates.

Proof: three conditions

Condition 1 (Cluster tail bound) There exists $C_{\mathcal{C}}$ s.t.

$$\mathbb{P}\Big(\sum_{n\geq 1} N_n \geq k\Big) \leq C_{\mathcal{C}}/\sqrt{k}.$$

Condition 2 (Self-repellence survival property) Let $\mathcal{F}_m$ be σ-field generated by vertices at distance $\leq m$ from 0 and N_m their number. Then there exists C_θ s.t. for every stopping time $M \leq n$,

$$\mathbb{P}(0 \longrightarrow n \mid \mathcal{F}_M) \leq C_\theta N_M \theta_{n-M}.$$

Condition 3 (Convergence r-point functions) There exist $A, V > 0$ s.t. for each $r \geq 2$ and $\vec{t} \in \mathbb{R}_+^{(r-1)}$,

$$n^{-(r-2)}\mathbb{E}\Big[\prod_{i=1}^{r-1} N_{t_i n}\Big] \rightarrow A(VA^2)^{r-2}\widehat{M}_{\vec{t}}^{(r-1)}(0), \quad \text{as } n \rightarrow \infty,$$

where $\widehat{M}_{\vec{t}}^{(r-1)}(0)$ are moments total mass super-Brownian Motion.

General result

Theorem 2 (Kolmogorov's and Yaglom's Theorem)
When Conditions 1-3 hold,

$$\lim_{n\to\infty} n\theta_n = 2/(AV),$$

and, conditionally on $N_n > 0$,

$$N_n/n \xrightarrow{d} \mathrm{Exp}(2/(AV)).$$

▷ Proof relies on lace expansion results formulated in Conditions 1 and 3, but does not use lace expansion itself.

Proof structure

Conditions 1-3 follow from lace expansion results:

▷ Condition 1 is $\delta = 2$ which follows from triangle condition OP, CP (Aizenman-Newman 84), Derbez-Slade (97,98) for lattice trees.

▷ Condition 2 is Markov property for OP/CP, self-repellence for LT.

▷ Condition 3 is convergence r-point functions to SBM moment measures: vdH-Slade (03), vdH-Sakai (10), Holmes (08).

Proof structure:
(a) Upper bound using Conditions 1 and 2, similar to Kozma-Nachmias (09);
(b) Weak convergence arguments using Condition 3, extending ideas from Holmes-Perkins (07).

Upper bound

Investigate θ_{4n}. Split according to whether there exists $j \in [n, 3n]$ s.t. $1 \le N_j \le \varepsilon n$, where $\varepsilon > 0$ is chosen later.

If such j does not exist, then $\sum_{j\ge1} N_j \ge 2\varepsilon n^2$. Otherwise, let stopping time J be first. Leads to

$$\theta_{4n} \le \mathbb{P}\big(\sum_{j\ge1} N_j \ge 2\varepsilon n^2\big) + \mathbb{P}\big(0 \longrightarrow 4n, J \in [n, 3n]\big).$$

Use Condition 1 for first term. For second term, by Condition 2,

$$\theta_{4n} \le C_{\mathcal{C}}/\sqrt{2\varepsilon n^2} + C_\theta \mathbb{E}\big[\theta_{4n-J} N_J \mathbb{1}_{\{J\in[n,3n]\}}\big].$$

By monotonicity of $n \mapsto \theta_n$ and bound $N_J \le \varepsilon n$,

$$\theta_{4n} \le C_{\mathcal{C}}/\sqrt{2\varepsilon n^2} + C_\theta \varepsilon n \theta_n \mathbb{P}(J \in [n, 3n]) \le C_{\mathcal{C}}/\sqrt{2\varepsilon n^2} + C_\theta \varepsilon n \theta_n^2.$$

Claim follows from induction in n.

Lower bound convergence

Rescale time by n and space by $\sqrt{n}$:

$$X_t^{(n)}(f) = \frac{1}{VA^2n} \sum_{x \in A_{nt}} f(x/\sqrt{vn}), \quad \text{and} \quad \mu_n(\cdot) = nVA\mathbb{P}(\cdot).$$

Let $X_s^{(n)}(1) = N_{sn}/n$. Then Condition 3 implies [Holmes+Perkins 07]

$$\mathbb{E}_{\mu_n}\left[\mathbb{1}_{\{X_s^{(n)}(1)>\eta\}} H(X_t^{(n)}(1))\right] \to \mathbb{E}_{\mathbb{N}_0}\left[\mathbb{1}_{\{X_s(1)>\eta\}} H(X_t(1))\right],$$

where $(X_s(1))_{s\geq 0}$ is total mass canonical measure of SBM.

In particular,

$$\mathbb{N}_0(X_t(1) > 0) = 2/t,$$

so that, as $\eta \searrow 0$,

$$\liminf_{n\to\infty} n\theta_n \geq (AV)^{-1}\mathbb{E}_{\mu_n}\left[\mathbb{1}_{\{X_1^{(n)}(1)>\eta\}}\right] \to (AV)^{-1}\mathbb{N}_0(X_1(1) > \eta) \to 2/(AV).$$

Upper bound convergence

By upper bound on $n\theta_n$, exists subsequence $(n_k)_{k\geq 1}$ s.t.

$$n_k\theta_{n_k} \to \limsup_n n\theta_n \equiv b, \qquad (1-\delta)n_k\theta_{(1-\delta)n_k} \to b_\delta,$$

where, by lower bound, $b, b_\delta \geq 2/AV$. Key split:

$$\begin{aligned} n_k\theta_{n_k} = {} & n_k\mathbb{P}(N_{(1-\delta)n_k} > \varepsilon n_k, N_{n_k} > \varepsilon' n_k) \\ & + n_k\mathbb{P}(0 < N_{(1-\delta)n_k} \leq \varepsilon n_k, N_{n_k} > 0) \\ & + n_k\mathbb{P}(N_{(1-\delta)n_k} > \varepsilon n_k, 0 < N_{n_k} \leq \varepsilon' n_k). \end{aligned}$$

Weak convergence: first term $\to 2/(AV)$ when $k \to \infty, \delta, \varepsilon, \varepsilon' \searrow 0$.
Condition 2: second term $\to 0$ when $k \to \infty, \delta, \varepsilon, \varepsilon' \searrow 0$.
Condition 3: third term $\to 0$ where limits are taken in order $k \to \infty$, $\varepsilon' \searrow 0$, $\varepsilon \searrow 0$, $\delta \searrow 0$.

$\triangleright$ Relies on fact that $\mathbb{N}_0(X_1(1) = 0 \mid X_{1-\delta}(1) > 0) = \delta$ together with weak convergence arguments for $(N_{(1-\delta)n}/n, N_n/n)$ on event $N_{(1-\delta)n}/n > \varepsilon$.

Conclusions & extensions

▷ Proof relies on simple weak convergence estimates.
Rather general. For example, also applies to voter model in $d > 2$.

▷ Holmes-Perkins 07: Convergence in finite-dimensional distributions to canonical measure super-Brownian motion (CSBM) follows.

CSBM is scaling limit critical branching random walk started from single individual, where

▷ particles split or die as in branching process;
▷ particles move according to random walk;
▷ probability measure is multiplied by a factor n.

▷ Percolation. Would extend Kozma-Nachmias to right constant.
Problem: Scaling r-point functions in Condition 3 yet unknown.

Conclusions & extensions

▷ **Tightness:** General lace expansion criterion vdH-Holmes-Perkins (2015). Involves condition on five-point function.
Verified for lattice trees above 8 dimensions.
Tightness for oriented percolation? Incipient infinite structures?

▷ **Extrinsic one-arm probabilities:**
Identified for percolation in high-dim by Kozma-Nachmias (2011):

$$\mathbb{P}_{p_c}(0 \longrightarrow Q_r^c) \asymp r^{-2}.$$

Results imply lower bound with correct constant.

▷ **Long-range percolation:**
Heydenreich-vdH-Hulshof (2014): Identified lower bound in long-range setting, Hulshof (2015) matching upper bound

$$\mathbb{P}_{p_c}(0 \longrightarrow Q_r^c) \asymp r^{-(\alpha \wedge 4)/2},$$

when $\mathbb{P}_{p_c}(\{x, y\} \text{ occ.}) \sim |x - y|^{-(d+\alpha)}$.

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