Scaling limit of dynamical percolation on critical Erdös-Rényi random graphs

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Erdös-Rényi random graphs

- n vertices
- E_n: set of edges of the complete graph
- $(U_{e \in E_n})$ i.i.d uniform on $[0, 1]$
- For $p \in [0, 1]$, state of edge e in $\mathcal{G}(n, p)$: present iff $U_e \leq p$

The birth of a giant connected component (giant $=$ of size $\Theta(n)$):

- $p = c/n$, $c < 1 \Longrightarrow$ no giant component
- $p = c/n$, $c > 1 \Longrightarrow \exists$ a unique giant component
- Critical window: $p = p(n, \lambda) = \frac{1}{n} + \frac{\lambda}{n^{4/3}}$, $\lambda \in \mathbb{R}$

Background: convergence of rescaled components, λ fixed

$X^{n,\lambda}$:= sequence of rescaled connected components listed in decreasing order of size, viewed as measured metric spaces:

measure = counting measure on vertices $\times n^{-2/3}$
distance = graph distance $\times n^{-1/3}$

Theorem (Aldous'97 + Addario-Berry, Broutin & Goldschmidt'10)

$(X^{n,\lambda})$ converges in distribution ($n \to \infty$) to a random sequence of measured metric spaces (which are real graphs) $X^{\infty,\lambda}$.

Topology: product of Gromov-Hausdorff-Prokhorov

$$d_{GHP}((X,\mu,d),(X',\mu',d')) := \inf_{\delta \text{ on } X \sqcup X'} \{\delta_H(X,X') \vee \delta_P(\mu,\mu')\}$$

Background: multiplicative coalescent, λ varying

$x^{n,\lambda}$:= sequence of masses in $X^{n,\lambda}$

When λ increases, connected components i and j, of masses x_i and x_j merge at rate

$$\frac{1}{n^{4/3}} \cdot x_i n^{2/3} \cdot x_j n^{2/3} = x_i x_j$$

$\rightarrow$ multiplicative coalescent

Theorem (Aldous '97 + Aldous & Limic'98)

Multiplicative coalescent has the Feller property on $\ell^2(\mathbb{N})$, and the process $(x^{n,\lambda})_{\lambda \in \mathbb{R}}$ converges as $(n \rightarrow +\infty)$ to some extremal eternal version of the multiplicative coalescent (in the sense of fidi).

Dynamical percolation, coalescence and fragmentation on $\mathcal{G}(n,p)$

Fix some intensity γ_n

- Coalescence: create the edge at rate $\gamma_n p$
- Fragmentation: kill the edge at rate $\gamma_n(1-p)$
- Dynamical Percolation: perform coalescence and fragmentation independently

Dynamical percolation, coalescence and fragmentation on measured real graphs

- Coalescence on a measured metric space (X, μ, d): identify points of a Poisson process of intensity $\frac{1}{2}\mu^2$ on X^2.
- Fragmentation on a real graph (X, d): cut points of a Poisson process of intensity the length measure.
- Dynamical percolation on a measured real graph (X, μ, d): perform coalescence and fragmentation, simultaneously and independently.

Main result

Choose intensity $\gamma_n = n^{-1/3}$

Theorem

The dynamical percolation (resp. coalescence, resp. fragmentation) process on $\mathcal{G}(n, n^{-1} + \lambda n^{-4/3})$ in the critical window converges in distribution ($n \to \infty$) to the dynamical percolation (resp. coalescence, resp. fragmentation) process on the limit $X^{\infty,\lambda}$

Work in progress: Show that dynamical percolation on the limit $X^{\infty,\lambda}$ is mixing. Interpretation in terms of noise-sensitivity.

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Main ideas

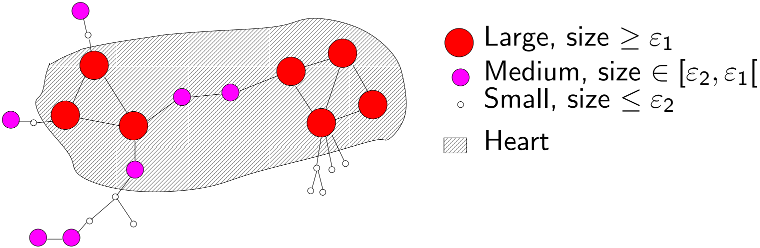
- Main difficulty for coalescence: total mass is infinite (sizes are in ℓ^2)
  Structure lemma for the multiplicative coalescent gives:

  almost-Feller property for coalescence:
  $Coal(X,t) \to Coal(Y,t)$ when $d_{GHP}(X,Y) \to 0$ + extra condition

- Feller property for fragmentation

Structure for the multiplicative coalescent

For every $\varepsilon > 0$, there exists $\varepsilon_2 \leq \varepsilon_1 \leq \varepsilon$ such that with probability larger than $1 - \varepsilon$, every component $\mathcal{C}$ of $Coal(x, t)$ of size at least ε has the following structure



Crown:=$\mathcal{C} \setminus Heart$, $mass(Crown) < \varepsilon_1$ is a forest of components

Almost-Feller property

Suppose:

- $d_{GHP}(X^{(n)}, X^{(\infty)}) + \|sizes(X^{(n)}) - sizes(X^{(\infty)})\|_2 \xrightarrow[n\infty]{} 0$
- (extra-condition) $\forall\varepsilon > 0$,

$$\limsup_{n\infty} \mathbb{P}(diam(Coal(X^{(n)}_{<\varepsilon_1}, t)) > \varepsilon) \xrightarrow[\varepsilon_1 \to 0]{} 0$$

Then $Coal(X^{(n)}, t)$ converges to $Coal(X^{(\infty)}, t)$ in distribution.

Main steps of AdBrGo'10, λ fixed

- Define a brownian motion with drift $W_t^\lambda := B_t + \lambda t - \frac{t^2}{2}$,
- reflect it above past minima: $h_t^{\infty,\lambda} := 2(W_t^\lambda - \min_{0\leq s\leq t} W_s^\lambda)$,
- then, the rescaled height $h^{n,\lambda}$ of the exploration process converges in distribution to $h^{\infty,\lambda}$ (Marckert &Mokkadem '03 + AdBrGo'10).
- Additional edges correspond to a Poisson point process of intensity $1/2$ below $h^{\infty,\lambda}$

Control of the width of the crown for $\mathcal{G}(n, p)$

- In depth-first order, a component of the crown is explored in at most two intervals
- Thus, the width of the crown is w.h.p less than $w_n(\varepsilon_1)$ with w_n the modulus of continuity of $4h^{n,\lambda+t}$, which goes to zero when ε_1 goes to zero, uniformly in n.

Consequence of mixing

$N_\varepsilon(X^n)$: refresh each edge with probability ε.
A graph property A_n is ε_n-noise sensitive if

$$\mathrm{Cor}(\mathbb{1}_{A_n}(X^n), \mathbb{1}_{A_n}(N_\varepsilon(X^n))) \xrightarrow[n\infty]{} 0$$

and ε_n-noise stable if

$$\mathrm{Cor}(\mathbb{1}_{A_n}(X^n), \mathbb{1}_{A_n}(N_\varepsilon(X^n))) \xrightarrow[n\infty]{} 1$$

Suppose A_n is a sequence of properties which can be "seen" in the scaling limit

- Main result $\Rightarrow$ ε_n-noise stability of A_n when $\varepsilon_n \ll n^{-1/3}$
- Mixing $\Rightarrow$ ε_n-noise sensitivity of A_n when $\varepsilon_n \gg n^{-1/3}$

Example: having a complex component, being planar

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The End

Thanks !!!