Coupling and geometry

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Couplings

- Let $(\Omega_1, \mathcal{F}_1, \mu_1)$ and $(\Omega_2, \mathcal{F}_2, \mu_2)$ be two probability spaces. A coupling of μ_1 and μ_2 is a measure μ on $(\Omega_1 \times \Omega_2, \mathcal{F}_1 \times \mathcal{F}_2)$ with marginals μ_1 and μ_2.
- We will be dealing with coupling of (the laws of) Markov processes X and Y.
- **Coupling Time:** $\tau = \inf\{s > 0 : X_t = Y_t \text{ for all } t > s\}$.

Coupling and Total Variation Distance

- The total variation distance between probability measures μ and ν on a Polish Borel space $(E, \mathcal{E})$ is defined as

$$||\mu - \nu||_{TV} = \sup_{A \subset \mathcal{E}} |\mu(A) - \nu(A)|.$$

- Aldous' Inequality: For any coupling (X, Y) of (μ, ν),

$$||\mu - \nu||_{TV} \leq \mathbb{P}(X \neq Y).$$

Maximal Couplings

- By Aldous' inequality, for any $t > 0$,

$$P(\tau > t) \geq ||\mu_{1,t} - \mu_{2,t}||_{TV},$$

  where

  - $\mu_{1,t}$ and $\mu_{2,t}$ are distributions of X_t and Y_t respectively.
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  - $||\cdot||_{TV}$ is the total variation distance between measures.

- A coupling of Markov processes X and Y with laws μ_1 and μ_2, with coupling time τ, is called a **Maximal Coupling** if $P(\tau > t) = ||\mu_{1,t} - \mu_{2,t}||_{TV}$ for all $t > 0$.

Existence

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- Pitman's construction simulates the *meeting point* first and then constructs the *forward* and *backward* chains.
- The coupling *cheats* by looking into the future.

Markovian couplings

- A coupling of Markov processes X and Y starting from x_0 and y_0 is called **Markovian** if

$$(X_{t+s}, Y_{t+s})_{t \geq 0} \mid \mathcal{F}_s$$

is again a coupling of the laws of X and Y starting from (X_s, Y_s). Here $\mathcal{F}_s = \sigma\{(X_{s'}, Y_{s'}) : s' \leq s\}$.

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- Usually easy to describe explicitly in forward time.
- Enable efficient application of stochastic calculus to derive near-optimal estimates for gradients, spectral gaps, etc., for diffusions.

Question

There is much work on how quickly coupling can happen (Rogers, 1999; Burdzy-Kendall, 2000; ...). Here we focus on a very specific question.

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- What is the class of Markov processes which admit a Markovian maximal coupling (MMC) for two copies started from distinct points?

- **Popular belief:** Rather limited class! MMC exhibits rigidity.

Known Examples of MMC

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- Hsu and Sturm (2013) showed that this is the unique Markovian maximal coupling of these Brownian motions. NOT unique maximal coupling.
- Similar conclusions hold for Ornstein-Uhlenbeck process by Doob's representation. (Connor, 2007)

Theorem (Kuwada, 2009)

Brownian motion on a homogeneous Riemannian manifold M can be coupled by MMC, starting from x_0 and y_0, if and only if the manifold admits a reflection structure, i.e. a continuous map $R : M \mapsto M$ with $R^2 = \mathrm{Id}$ such that

(i) $Rx_0 = y_0$

(ii) $\exists$ *open* M_0 *such that* $M = M_0 \sqcup H \sqcup R(M_0)$ *where* H *is the set of fixed points of* R.

and the coupling is a reflection coupling determined by R.

Consequences of Kuwada's result

Assume M is an irreducible global symmetric space.

(i) If M is of non-constant curvature, no reflection structure, hence no MMC.

(ii) If M is a sphere, Euclidean space or Hyperbolic space, then a MMC of Brownian motions exists from every pair of starting points.

(iii) If M is a Real Projective space, no MMC from any pair of starting points.

(iv) If M is a torus, then MMC exists from starting points $(x_1, \ldots, x_d)$ and $(y_1, \ldots, y_d)$ if and only if there exists $j \in \{1, \ldots, d\}$ such that $x_i = y_i$ for all $i \neq j$.

Stable MMC

Bearing in mind the torus example, we ask what happens when existence of MMC remains stable under a *slight perturbation of starting points*.

Local Perturbation Condition (LPC): There exist arbitrary open sets $U, V \subseteq M$ such that a MMC of the diffusion processes X and Y starting from x and y exists for every $x \in U$ and $y \in V$.

We say that an MMC is stable if LPC holds.

Goal

Our goal is to investigate *when a stable MMC exists* for elliptic diffusions given by generator of the form

$$\mathcal{L} = \sum_{i,j=1}^{d} a_{ij}(x)\partial_{ij} + \sum_{i=1}^{d} b_i(x)\partial_i$$

on $\mathbb{R}^d$ and later, more generally, on a complete smooth manifold M.

To ease exposition, we deal only with the case of smooth coefficients.

[A construction of successful Markovian couplings for elliptic diffusions achieved in some cases by Lindvall and Rogers ('86).]

Euclidean case

We first investigate this question for Euclidean diffusions with constant diffusion matrix:

$$dX_t = \mathbf{b}(X_t)dt + dB_t$$

started from distinct points x_0 and y_0.

A brief outline: Structure of the Euclidean MMC

If a MMC exists, then we can show that it should satisfy the following:

- There is a *deterministic* system of hyperplanes $\{M(t)\}_{t\geq 0}$ which can *evolve in time* such that, for each t, Y_t is obtained by reflecting X_t in $M(t)$.
- Under mild regularity assumptions, the moving mirror can be parametrized in a *smooth way*.
- These lead to (implicit) functional equations on the drift, via stochastic calculus.

Characterisation of Euclidean MMC

Theorem (B.-Kendall, 2014)

A stable MMC exists for time-homogeneous Euclidean diffusions X and Y if and only if there exist a real scalar λ, a skew-symmetric matrix $\mathcal{T}$ and a vector $c \in \mathbb{R}^d$ such that

$$\mathbf{b}(x) = \lambda x + \mathcal{T}x + c$$

for all $x \in \mathbb{R}^d$.

(Ornstein-Uhlenbeck + rotation)

Theorem (B.-Kendall, 2014)

When the drift is of the above form, the MMC is described by

$$Y_t = F_t(X_t)$$

where F_t denotes reflection in the hyperplane parametrized by its normal vector

$$\mathbf{n}(t) = \exp(\mathcal{T} t)\frac{x_0 - y_0}{|x_0 - y_0|}$$

and distance from the origin

$$l(t) = e^{\lambda t}\frac{|x_0|^2 - |y_0|^2}{2|x_0 - y_0|} + e^{\lambda t}\int_0^t \frac{(x_0 - y_0)^T}{|x_0 - y_0|}\exp\{-(\mathcal{T} + \lambda I)s\}c \; ds.$$

Stronger version for one dimension

Theorem (B.-Kendall, 2014)

There exists a MMC of a one-dimensional diffusion X starting from x_0 and y_0 if and only if, when X is transformed so that the martingale part is Brownian, then the drift $\mathbf{b}$ is either linear or $\mathbf{b}(x) = -\mathbf{b}(x_0 + y_0 - x)$ for all $x \in \mathbb{R}$.

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Remark: This determines *all one-dimensional diffusions* (with general diffusion coefficient) for which MMC holds. Essentially they must be (transformations of) either Brownian motion with constant drift or Ornstein-Uhlenbeck processes, or the drift obeys a symmetry condition with respect to the starting points.

Manifold case

If the generator of the diffusion on a connected smooth manifold M of dimension d is given (in local coordinates) by

$$\mathcal{L} = \sum_{i,j=1}^{d} a_{ij}(x)\partial_{ij} + \sum_{i=1}^{d} b_i(x)\partial_i,$$

then we can give M a metric $g_{ij} = a^{ij}$ under which, the generator becomes

$$\mathcal{L} = \frac{1}{2}\Delta_M + \mathbf{b}$$

where Δ_M is the Laplace-Beltrami operator and $\mathbf{b}$ is a 'drift' vector field.

The diffusion thus becomes Brownian motion plus drift under this metric. It can now be represented as the solution to a Stratonovich SDE. (Stochastic Parallel Transport.)

The Isometry Group of M

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3. Killing vector fields are vector fields corresponding to *generators of these one parameter subgroups*, given by

$$\kappa(x) = \frac{d}{dt}\Big|_{t=0} F_t(x).$$

These form the Lie Algebra corresponding to the Lie group of isometries.

Rigidity Theorem I: Classifying the space M

Theorem (B.-Kendall, 2014)

If a stable MMC exists on M, then M has to be maximally symmetric (i.e. the dimension of $\mathrm{Iso}(M)$ is $d(d+1)/2$).

The only complete, connected Riemannian manifolds which are maximally symmetric are the sphere($\mathbb{S}^d$), Euclidean space ($\mathbb{R}^d$), hyperbolic space ($\mathbb{H}^d$) and Real Projective space ($\mathbb{RP}^d$).

But (Kuwada, 2009) $\mathbb{RP}^d$ does not support any MMC.

Classifying the space M: continued

Thus, we have the following classification of M:

Corollary

A stable MMC exists on M if and only if M is $\mathbb{S}^d$, $\mathbb{R}^d$ or $\mathbb{H}^d$.

Let K denote the (sectional) curvature of M.

Theorem (B.-Kendall, 2014)

A stable MMC exists on M if and only if the following hold:

(i) *For $K \neq 0$, the drift $\mathbf{b}$ is a Killing vector field $\mathcal{K}$ on M.*

(ii) *For $K = 0$, $\mathbf{b}$ is described in Euclidean co-ordinates by $\mathbf{b}(x) = \lambda x + \mathcal{T}x + \mathbf{c}$ for any scalar λ, skew-symmetric matrix $\mathcal{T}$ and vector $\mathbf{c}$, where $x \mapsto \lambda x$ is a dilation vector field about the origin and $x \mapsto \mathcal{T}x + \mathbf{c}$ is a Killing vector field.*

This is the general rigidity result for MMC.
It confirms the intuition that MMC are very rare.

Description of the MMC for $K \neq 0$

We have already described the Euclidean case $K = 0$. Now, we describe the stable MMC for $K \neq 0$.

Theorem (B.-Kendall, 2014)

For $K \neq 0$, the stable MMC of X and Y starting from x_0 and y_0 is given by

$$(\mathcal{G}_t(W_t), \mathcal{G}_t(\tilde{W}_t))_{t \geq 0}$$

where $(W, \tilde{W})$ is the MMC of Brownian motions on M starting from (x_0, y_0) and $(\mathcal{G}_t)_{t \geq 0}$ is the one parameter subgroup of isometries generated by $\mathcal{K}$.

Sketch of Proof

Define

- $\alpha(t, z) = p(x_0; t, z) - p(y_0; t, z)$,
- $\mathcal{I}_t = \{z \in M : \alpha(t, z) = 0\}$. We will call this the interface.
- $\mathcal{I}_t^+ = \{z \in M : \alpha(t, z) > 0\}$ and $\mathcal{I}_t^-$ similarly.

The time evolution of the interface plays a pivotal role in our arguments.

A Support Lemma for maximal couplings

Let μ be a maximal coupling of X and Y.

Lemma

$$\mu(X_s = dz, \tau > s) = \alpha^+(s,z)m(dz),$$

$$\mu(Y_s = dz, \tau > s) = \alpha^-(s,z)m(dz).$$

Thus, X_s and Y_s are supported on disjoint regions of the state space $\mathcal{I}_s^+$ and $\mathcal{I}_s^-$ before they couple.

Flow property of MMC

Let μ denote the law of a MMC of X and Y and μ_s denote law of (X_s, Y_s). Let θ denote the time-shift operator.

Lemma

For μ_s-almost every (x, y) with $x \neq y$, $(\theta_s X, \theta_s Y \mid X_s = x, Y_s = y)$ gives a Markovian maximal coupling of (X, Y) starting from (x, y).

This can be interpreted as a flow property of MMC.

Varadhan small-time asymptotics

Lemma (Varadhan 1967, Molchanov 1975)

Let M_1 and M_2 be compact subsets of M. Then the density p of X_t satisfies the following:

$$\lim_{t \downarrow 0} 2t \log p(x; t, y) = -d^2(x, y) \tag{1}$$

uniformly for all $x, y \in M_1 \times M_2$, where d is the Riemannian distance function.

The Mirror

Let $H(x, y)$ denote the set of points in M equidistant from x and y.

The previous lemmas, along with the Strong Maximum principle for parabolic PDEs, yield the following important observation.

Theorem

For any $t > 0$,

$$\mathcal{I}_t = H(X_t, Y_t).$$

almost surely with respect to the coupling law μ.

Thus, the set of equidistant points from X_t and Y_t is a deterministic set $\mathcal{I}_t$ almost surely.

Defining the reflection structure

Following Kuwada (2009), we can define a 'reflection' on $\mathcal{I}_t$ given by

$$\begin{aligned} F_t(x) &= \text{the unique } y \in M \text{ such that } y \neq x \\ &\quad \text{and } d(x,z) = d(y,z) \text{ for all } z \in \mathcal{I}_t; \text{ if } x \in M/\mathcal{I}_t \end{aligned}$$

$$F_t(x) = x; \text{ if } x \in \mathcal{I}_t$$

This turns out to be a well defined global involutive isometry on M such that

$$Y_t = F_t(X_t)$$

on $\{0 \leq t < \tau\}$ almost surely. Further, we can prove by using this relation that $t \mapsto F_t$ is a C^1 curve in $\mathrm{Iso}(M)$.

Space classification: sketch

The LPC can be used now to get sufficiently many isometries by the above recipe to show that M is homogeneous (the Isometry group acts transitively on M) and isotropic (the isometries fixing a point generate all the rotations about it), which yields maximal symmetry.

The space classification follows from this.

Drift characterisation: sketch

Applying stochastic calculus on the equation $Y_t = F_t(X_t)$ on $\{0 \leq t < \tau\}$ and using the fact that X and Y have the same generator, we arrive at the following functional equation for the drift.

$$b(x) = F_{t*}b(x) + \kappa_t(x)$$

where $F_{t*}b$ denotes the pushforward of the vector field b by F_t given by $F_{t*}b(x) = dF_t\Big|_{F_t(x)} b(F_t(x))$ and κ_t represents the Killing vector field

$$\kappa_t(x) = \frac{d}{ds}\Big|_{s=t} F_s(F_t(x)).$$

We then carefully analyse the equation for the drift under LPC. This reduces the drift to a 'dilation + Killing field' form.

For nonzero curvature, the Toponogov triangle comparison theorem can be used to show that the dilation part must be zero.

This gives the drift characterisation.

Implications of Rigidity

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- This turned out to be a pivotal observation in devising coupling strategies for Kolmogorov diffusions (simplest example of a hypoelliptic diffusion).

Coupling Kolmogorov diffusions

- Kolmogorov diffusion of order n given by

$$X_t = \left(B_t, \int_0^t B_s ds, \ldots, \int_0^t \int_0^{s_{n-1}} \cdots \int_0^{s_2} B_{s_1} ds_1 \ldots ds_{n-1} \right).$$

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- There exist Markovian couplings of two copies of the Kolmogorov diffusion (Ben Arous, Cranston and Kendall 1995, Kendall and Price 2004).
- We show that there are no MMC from any pair of starting points. (B.-Kendall 2015).
- In fact there are pairs of starting points from which no MC is efficient (coupling rates and total variation distance have same order of decay in time). (B.-Kendall 2015).

An efficient non-Markovian coupling (B.-Kendall 2015)

- For $i = 1, 2$, we consider the Brownian paths $\{B^{(i)}(t) : 0 \leq t \leq T\}$ as $B^{(i)}(t) = \mathcal{B}^{(i)}(T, t)$ where $\mathcal{B}^{(i)}$ is the infinite-dimensional Brownian motion on $L^2(0, T)$ given by

$$\mathcal{B}^{(i)}(\zeta, t) = \sum_{k=1}^{\infty} \sqrt{\lambda_k} w_k^{(i)}(\zeta) f_k(t/T)$$

for $0 \leq \zeta, t \leq T$ (Karhunen-Loève expansion).

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- On each block $[T_n, T_{n+1}]$ ($T_n = 1 + \cdots + 2^n$), apply a 'reflection coupling' of the Brownian vectors $\mathbb{W}^{(i)} = (w_1^{(i)}, w_2^{(i)}, \ldots)$ in a hyperplane determined by $X^{(2)}(T_n) - X^{(1)}(T_n)$.

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- This projects down to an efficient non-Markovian coupling of the Kolmogorov diffusions $X^{(1)}$ and $X^{(2)}$.

Conclusion, Remarks and Future Work

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- For hypoelliptic diffusions of the form

$$dX_t = V_0(X_t)dt + \sum_{i=1}^{n} V_i(X_t) \circ dW_i(t)$$

where the vector fields $V_0, V_1, \ldots, V_n$ satisfy the Hörmander condition (Lie brackets span the whole tangent space), the existence of density implies existence of a maximal coupling. But does this imply existence of a Markovian coupling?

If so, when is it maximal/efficient?

We are investigating this question for Levy stochastic areas.

Conclusion, Remarks and Future Work (contd.)

- For general cost functions, couplings have deep connections with Optimal Transport. We can ask similar questions for general cost functions.

Conclusion, Remarks and Future Work (contd.)

- For general cost functions, couplings have deep connections with Optimal Transport. We can ask similar questions for general cost functions.
- Couplings (especially Markovian ones) are an important toolbox for studying mixing times of discrete Markov chains. The present question becomes harder in this context when the chains are not skip-free. It would be interesting to obtain comparable results about discrete state-space Markov chains.

Thank You!