Condensation in stochastic particle systems - recent results on statics and dynamics

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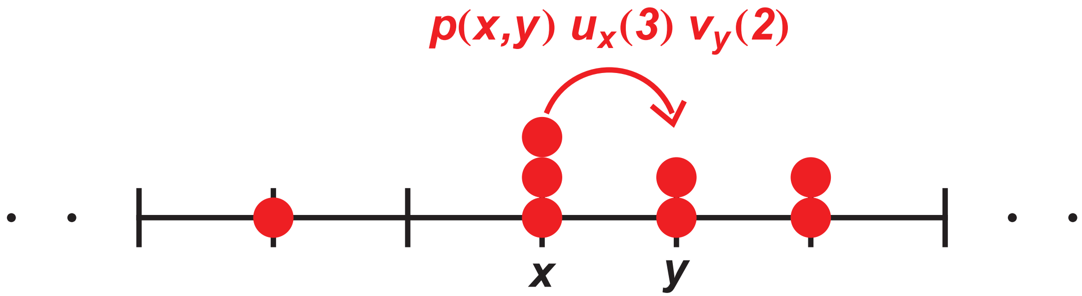
March 7, 2014

IPS with condensation

Lattice: Λ of size L

State space: $X = \{0, 1, ..\}^{\Lambda}$

$\boldsymbol{\eta} = (\eta_x)_{x\in\Lambda}$



Jump rates: $p(x,y)\, u_x(\eta_x)\, v_y(\eta_y)$, $\quad d > 0$

$p(x,y) \geq 0$ irreducible on Λ

Generator: $\mathcal{L}f(\boldsymbol{\eta}) = \sum_{x,y\in\Lambda_L} p(x,y)\, u_x(\eta_x)\, v_y(\eta_y) \big(f(\boldsymbol{\eta}^{x,y}) - f(\boldsymbol{\eta}) \big)$

Inclusion process: $u_x(n) = n,\ v_y(m) = d + m$, $\quad d > 0$

[Giardina, Kurchan, Redig, Vafayi (2009); G., Redig, Vafayi (2011)]

Misanthrope models: $u_x(n) v_y(m) = c(n,m)$, $\quad p(x,y) = q(y-x)$

[Cocozza-Thivent (1985)]

IPS in this class

- inclusion process (IP) $u_x(n) = n, \; v_y(m) = d + m \;, \quad d > 0$
- zero-range processes (ZRP) $v_y(m) = 1 \;, \quad u_x(n)$ arbitrary
- target process (TP) $u_x(n) = \mathbb{1}_{[1,\infty)}(n) \;, \quad v_y(m)$ arbitrary

  [Luck, Godrèche (2007)]
- explosive condensation model (ECM) [Waclaw, Evans (2012)]

  $v_y(m) = (d+m)^\gamma \;, \quad u_x(n) = (d+m)^\gamma - d^\gamma \;, \quad \gamma \geq 1$

Applications of IP

- 2 sites, N particles: rates $d\,k + k(N-k)$

  $\rightarrow$ multi-species Moran model (related to Wright-Fisher)
- duality with Brownian energy/momentum process

  [Giardina, Kurchan, Redig, Vafayi (2009); Giardina, Redig, Vafayi (2010)]

Condensation

- spatial heterogeneity $p(x,y)$ or u_x, v_y

  $\Rightarrow$ condensation on 'slow sites'

  ZRP with $u_x(n) = u_x$ or $u_x(n) \nearrow$

  [Evans (1996); Krug, Ferrari (1996); Landim (1996); Benjamini, Ferrari, Landim (1996); Andjel, Ferrari, Guiol, Landim (2000); Ferrari, Sisko (2007); G., Redig, Vafayi (2011); Chleboun, G. (2013)]

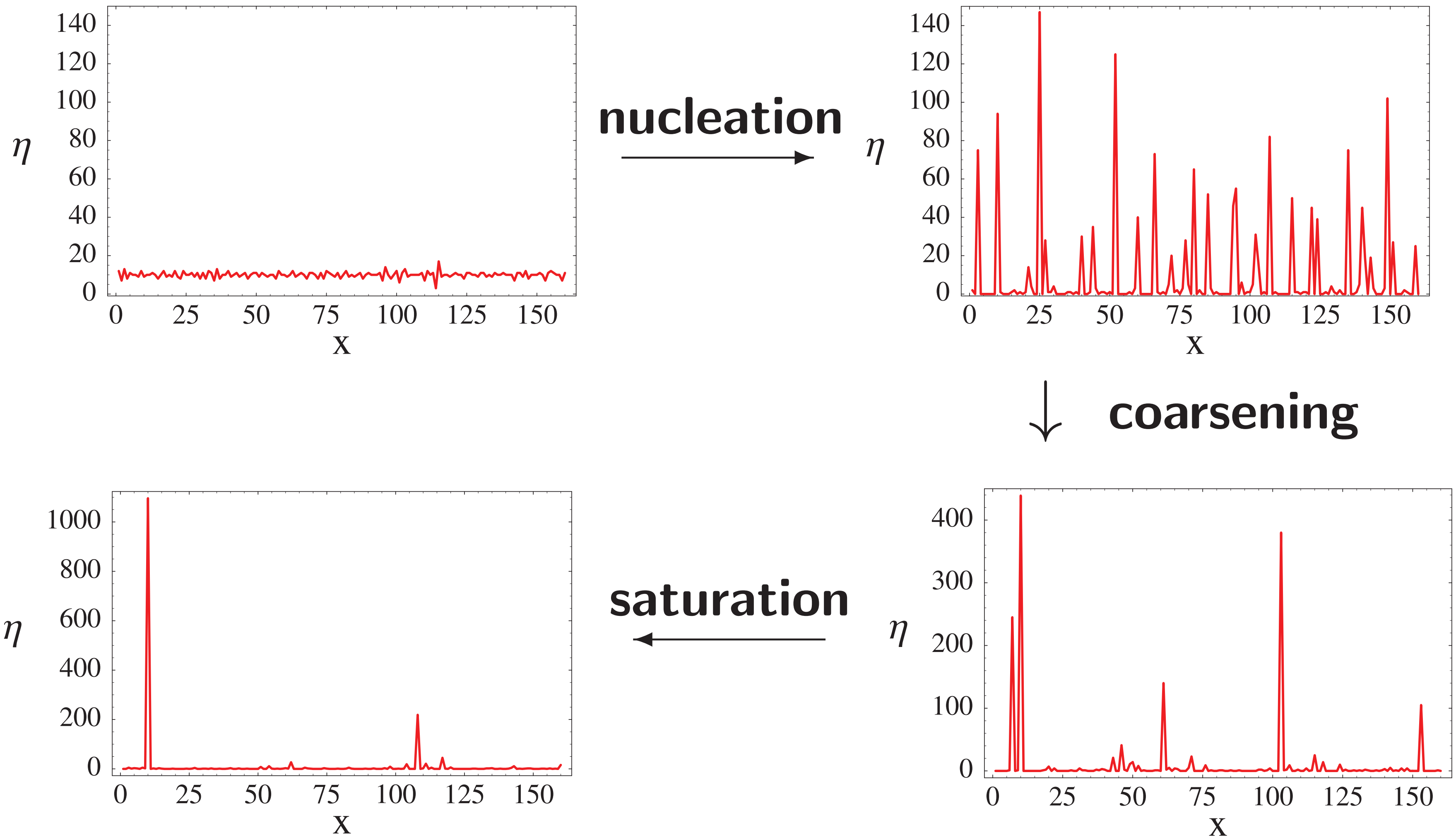
- effective attraction of particles

  $\Rightarrow$ condensation on a random site

  ZRP, IP, TP, ECM with $u(n) \searrow$ and/or $v(m) \nearrow$

  [Evans (2000); Jeon, March, Pittel (2000); G., Schütz, Spohn (2003); Ferrari, Landim, Sisko (2007); Armendáriz, Loulakis (2009); G., Chleboun (2010); Armendáriz, G., Loulakis (2012); Beltran, Landim (2010-12)]

Condensation



I Stationary Results

S. G., F. Redig, K. Vafayi, J. Stat. Phys. 142, 952-974 (2011)

P. Chleboun, S. G., J. Stat. Phys. 154, 432465 (2014)

Stationary product measures

Generator $\mathcal{L}f(\boldsymbol{\eta}) = \sum_{x,y\in\Lambda_L} p(x,y)\, u_x(\eta_x)\, v_y(\eta_y)\big(f(\boldsymbol{\eta}^{x,y}) - f(\boldsymbol{\eta})\big)$

harmonic function $\lambda_x > 0$ $\quad \sum_{x\in\Lambda}\big(\lambda_x p(x,y) - \lambda_y p(y,x)\big) = 0$

product measure $\quad \nu_\phi^\Lambda(d\boldsymbol{\eta}) = \prod_{x\in\Lambda} \nu_\phi^x(\eta_x)\, d\boldsymbol{\eta}$ $\quad$ with

$$\nu_\phi^x(n) = \frac{1}{z_x(\phi)}(\lambda_x\phi)^n\, w_x(n) \quad \text{with} \quad w_x(n) = \prod_{k=1}^{n} \frac{v_x(k-1)}{u_x(k)}$$

with $z_x(\phi) = \sum_{n\geq 0} w_x(n)(\lambda_x\phi)^n$ $\quad$ and $\quad \phi < \phi_c = \inf_{x\in\Lambda}\phi_c^x$

For IP: $\quad w_x(n) = w(n) = \frac{\Gamma(d+n)}{n!\,\Gamma(d)} \simeq n^{d-1}\,, \qquad \phi_c = 1$

Stationary product measures

Grand-canonical measures

The IPS with generator $\mathcal{L}$ has SPM ν_ϕ^Λ provided that

$$v_y(m) \equiv 1 \qquad \text{(ZRP)}$$

OR

$$\lambda_x p(x,y) = \lambda_y p(y,x) \quad \text{for all } x,y \in \Lambda \qquad (\Rightarrow \;\; \nu_\phi \text{ reversible})$$

OR

$$\sum_{y\in\Lambda} \big(p(x,y) - p(y,x)\big) = 0 \quad \text{for all } x \in \Lambda \quad \text{AND}$$

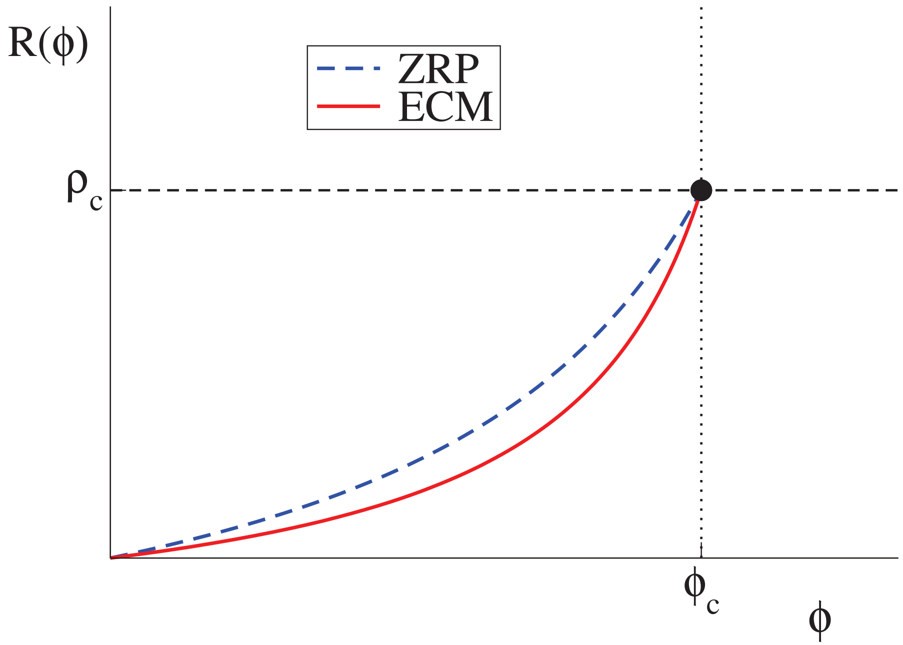
$$u_x = u; v_x = v; u(n)v(m) - u(m)v(n) = u(n) - u(m)\ , \quad n,m \geq 0\ ,$$

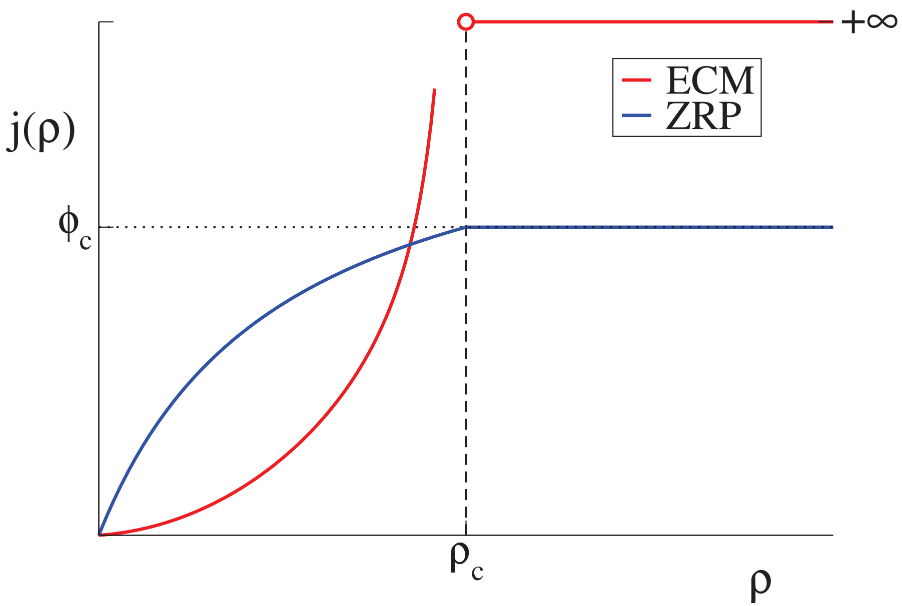
which implies that $\lambda_x \equiv 1$ (homogeneous) .

Homogeneous condensation

density $\quad R(\phi) = \mathbb{E}_{\nu_\phi}[\eta_x] \nearrow \rho_c < \infty$

- rates $\quad \eta_x^{\gamma}(d + \eta_y^{\gamma})\ , \quad w(n) \sim n^{-\gamma}$
- ECM $\quad \big((d+\eta_x)^{\gamma} - d^{\gamma}\big)(d+\eta_y)^{\gamma}$ [Waclaw, Evans (2012)]
- ZRP $\quad u(\eta_x) = 1 + b/\eta_x\ , \quad w(n) \sim n^{-b}$





Equivalence of ensembles

finite lattices $|\Lambda| = L$ (e.g. $\Lambda = \mathbb{T}_L$)

grand-canonical measures ν_ϕ^Λ , $\phi \in [0, \phi_c)$

Conservation law $S_L\big(\boldsymbol{\eta}(t)\big) := \sum_{x \in \Lambda_L} \eta_x(t) = const.$

Canonical measures fix $S_L(\boldsymbol{\eta}) = N$

$$\pi_{L,N}(d\boldsymbol{\eta}) = \nu_\phi^\Lambda\big(d\boldsymbol{\eta} | S_L = N\big) = \frac{1}{Z_{L,N}} \mathbb{1}_{S_L = N} \prod_{x \in \Lambda} w(\eta_x)\, d\boldsymbol{\eta}$$

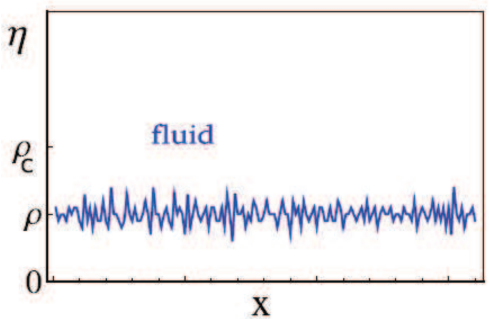
$\rightarrow$ Equivalence in the limit of large systems?

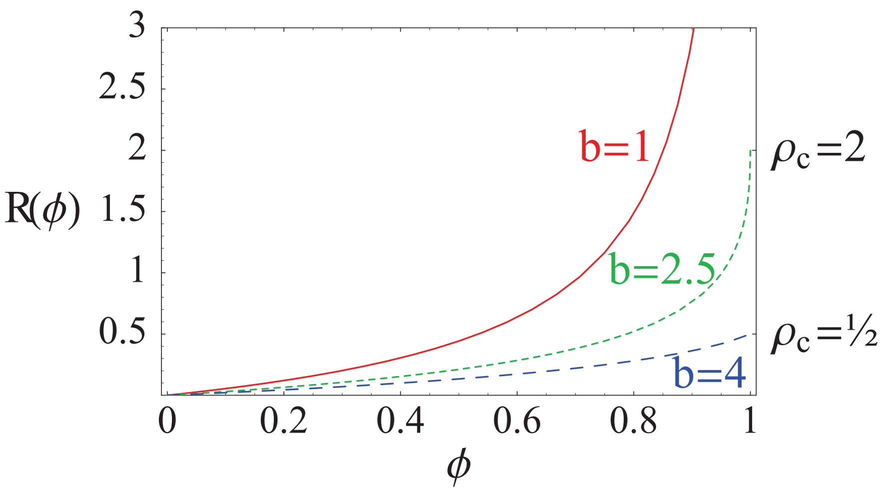
Equivalence of ensembles

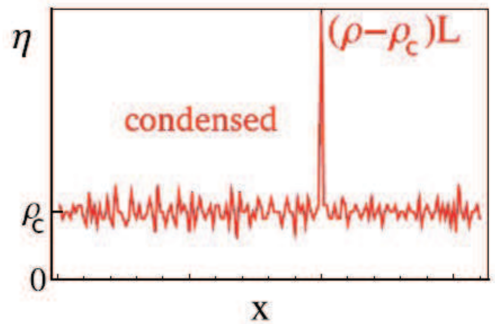
Theorem [G., Schütz, Spohn (2003)]

Assume that $\lim_{n\to\infty} \frac{1}{n} \log w(n)$ exists.
Then, in the thermodynamic limit $\quad L, N \to \infty\ , \quad N/L \to \rho$

$$\frac{1}{L} H\big(\pi_{L,N}; \nu_\phi\big) \to 0 \quad \text{if} \quad \begin{cases} R(\phi) = \rho\ , & \rho < \rho_c \\ \phi = \phi_c\ , & \rho \geq \rho_c \end{cases} .$$







Equivalence of ensembles

Theorem

Assume that $\lim_{n\to\infty} \frac{1}{n} \log w(n)$ exists.
Then, in the thermodynamic limit $L, N \to \infty \ , \quad N/L \to \rho$

$$\pi_{L,N}(f) \to \nu_\phi(f) \quad \text{if} \quad \begin{cases} R(\phi) = \rho \ , & \rho < \rho_c \\ \phi = \phi_c \ , & \rho \geq \rho_c \end{cases} .$$

- $f \in C_0^b(X)$ [G., Schütz, Spohn (2003), G. (2008)]
- $\rho \leq \rho_c$ also $f \in C_0(X) \cap L^{1+\epsilon}(\nu_\phi)$ [Chleboun, G. (2014)]
- $\rho > \rho_c$, for $w(n) \sim n^{-b}$ $\Rightarrow$ $f \in C^b(\hat{X})$

  [Armendáriz, Loulakis (2009)]

  implies $M_L/L \to \rho - \rho_c$, where $M_L := \max_{x\in\Lambda_L} \eta_x$

  cf. also [Jeon, March, Pittel (2000), Ferrari, Landim, Sisko (2007)]

Relative entropy

$$h_{L,N}(\phi) = \frac{1}{L} H\big(\pi_{L,N} | \nu_\phi^L\big) := \frac{1}{L} \sum_{\boldsymbol{\eta}\in X_{L,N}} \pi_{L,N}(\boldsymbol{\eta}) \log \frac{\pi_{L,N}(\boldsymbol{\eta})}{\nu_\phi^L(\boldsymbol{\eta})}$$

$$= -\frac{1}{L} \log \nu_\phi^L\big(\{S_L = N\}\big) \to 0$$

$\rho \leq \rho_c \quad \Rightarrow \quad R(\phi) = \rho$ local limit theorem

$\rho > \rho_c \quad \Rightarrow \quad \phi = \phi_c$ large deviation for subexponential rvs

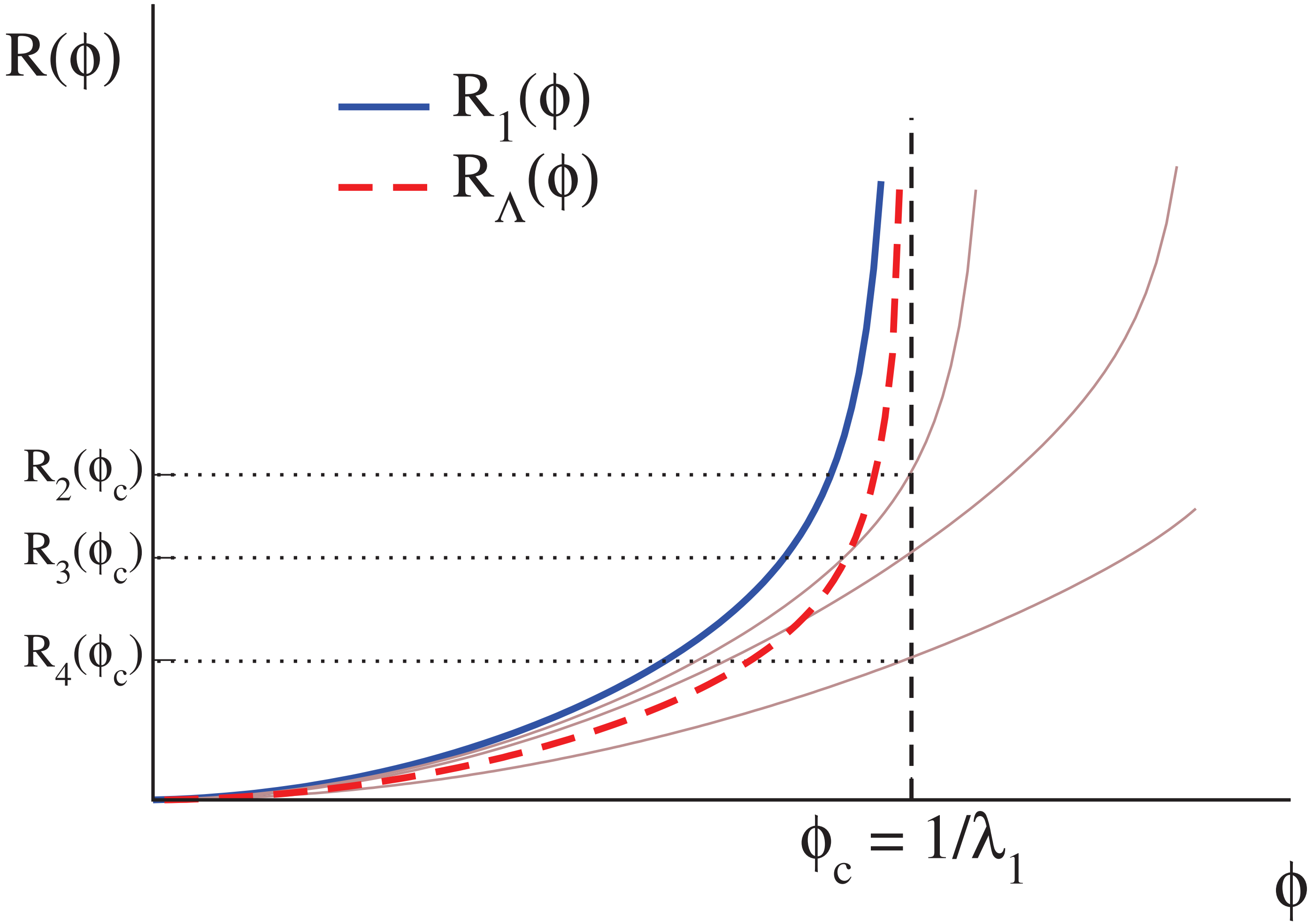
$$\nu_{\phi_c}^L\big(\{S_L = N\}\big) \geq \nu_{\phi_c}^1\big(N - [\rho_c L]\big)\, \nu_{\phi_c}^{L-1}\big(\{S_{L-1} = [\rho_c L]\}\big)$$

$$h_{L,N}(\phi) = \frac{1}{L} \log Z_{L,N} - \sup_{\phi\in[0,\phi_c]} \Big(\frac{N}{L} \log\phi - \log z(\phi)\Big)$$

$$\to s_{can}(\rho) \qquad - \qquad s_{gcan}(\rho)$$

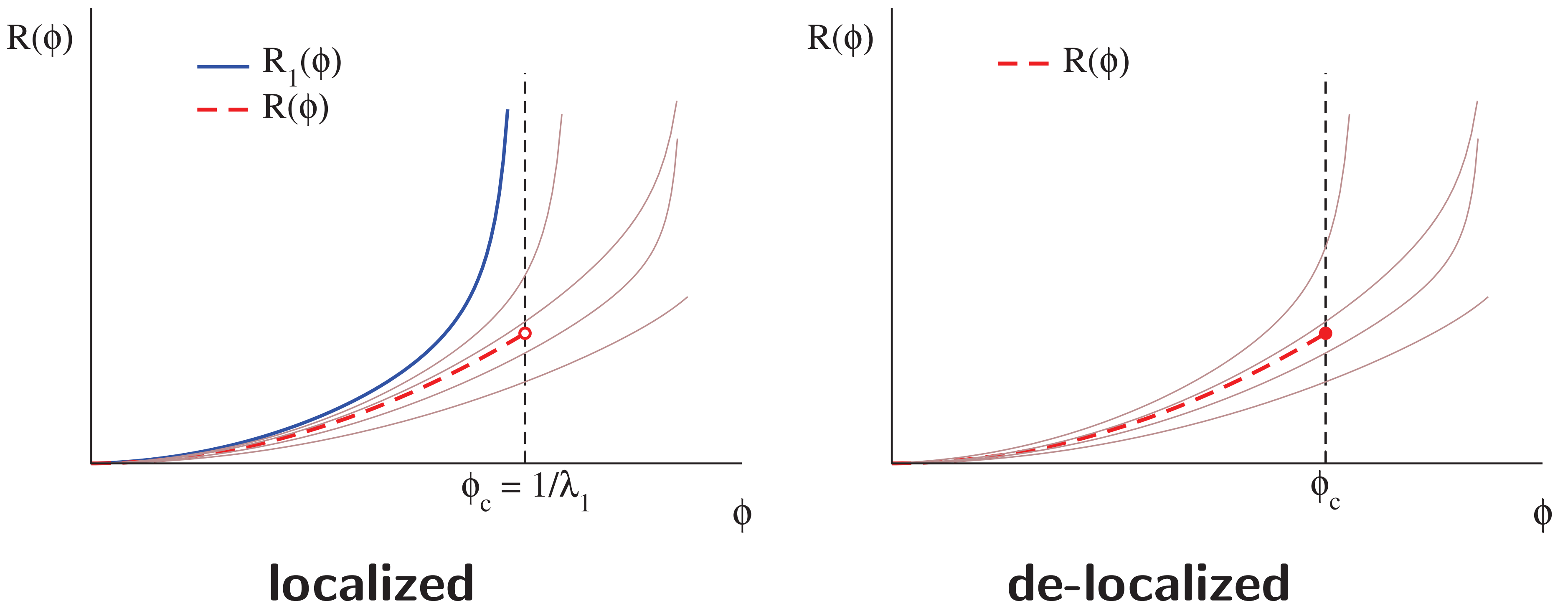
Condensation in inhomogeneous systems

Finite systems



Condensation in inhomogeneous systems

Thermodynamic limit



$$R(\phi) = \lim_{L\to\infty} \frac{1}{L} \sum_{x\in\Lambda} R_x(\phi) \ , \qquad \rho_c := \lim_{\phi\nearrow\phi_c} R(\phi)$$

Condensation in inhomogeneous systems

product measure $\quad \nu_\phi^\Lambda(d\boldsymbol{\eta}) = \prod_{x\in\Lambda} \nu_\phi^x(\eta_x)\, d\boldsymbol{\eta} \quad$ with

$$\nu_\phi^x(n) = \frac{1}{z_x(\phi)}\,(\lambda_x\phi)^n\, w_x(n) \quad \text{with} \quad w_x(n) = \prod_{k=1}^{n} \frac{v_x(k-1)}{u_x(k)}$$

Condensation in inhomogeneous systems

Theorem [Chleboun, G. (2014)]

Assume that $\lim_{n\to\infty} \frac{1}{n} \log w_x(n) = 0$ and $\rho_c < \infty$, and that $\lambda_1, \lambda_2, \ldots$ are uniformly bounded. Then

1. **Delocalized case.** If $\lambda_x < 1/\phi_c$ for all $x \in \mathbb{N}$, and the critical measure ν_{ϕ_c} has finite second moments we have for all $\rho \geq \rho_c$

$$\frac{1}{L} H\left(\pi_{\Lambda,N}; \nu_{\phi_c}^{\Lambda}\right) \to 0\,, \quad \text{as } L \to \infty \text{ and } N/L \to \rho\,.$$

2. **Localized case.** If $\Delta = \{x : \lambda_x = 1/\phi_c\} \neq \emptyset$ and for all $y \notin \Delta$, $1/\lambda_y > \phi_c + \delta$ for some $\delta > 0$, we have for all $\rho \geq \rho_c$

$$\frac{1}{L} H\left(\pi_{\Lambda,N}^{\Lambda\setminus\Delta}; \nu_{\phi_c}^{\Lambda\setminus\Delta}\right) \to 0, \quad \text{as } L \to \infty \text{ and } N/L \to \rho\,.$$

Furthermore, the volume fraction of the condensed phase vanishes, $|\Delta \cap \Lambda|/L \to 0$ as $L \to \infty$.

II Dynamics of condensation

I. Armendáriz, S. G., M. Loulakis (in preparation)

S. G., F. Redig, K. Vafayi, EJP 18, no. 66, 123 (2013)

Metastability for Markov processes

$(\boldsymbol{\eta}^L(t) : t \geq 0)$ sequence of MPs with state space X_L

$(\boldsymbol{\eta}^L(t) : t \geq 0)$ exhibits metastability as $L \to \infty$

- w.r.t. the observable $f_L : X_L \to E$ (e.g. $E = \Lambda, \subset \mathbb{R}$)
- on the timescale $\theta_L(t)$ ($\theta_L(t) \nearrow t$, e.g. $\theta_L t$)
- with initial distribution μ_0^L (stationary or non-st.)

if

$$\Big(f_L\big(\boldsymbol{\eta}^L(\theta_L(t))\big) : t \geq 0 \Big) \stackrel{L\to\infty}{\longrightarrow} \big(Y(t) : t \geq 0\big)$$

where

$$\big(Y(t) : t \geq 0\big) \quad \text{is a MP on } E$$

with $Y(0) \sim \mu$ and $\mu = \lim_{L\to\infty} \mu_0^L \circ f_L^{-1}$.

Metastability

Stationary dynamics of the condensate

$\Lambda = \mathbb{T}_L$, $g(k) = \mathbb{1}_{k>0}(1 + \frac{b}{k})$, $b > 5$, $p(x, y)$ NN

$M_L(\boldsymbol{\eta}) = \max_{x\in\Lambda} \eta_x$, $\psi_L(\boldsymbol{\eta}) = \inf \{x \in \Lambda : \eta_x = M_L(\boldsymbol{\eta})\}$

In preparation

[Armendáriz, G., Loulakis]

Let $\boldsymbol{\eta}_0 \sim \pi_{L,N}$, thermodynamic limit $L, N \to \infty$ with $L/N \to \rho > \rho_c$. Then on scale $\theta_L = L^{1+b}$, $\big(\frac{1}{L}\psi_L(\boldsymbol{\eta}_{\theta_L t}) : t \geq 0\big)$ converges weakly on path space $D\big([0, \infty), \mathbb{T}\big)$ to a Lévy-type process $(Y_t : t \geq 0)$ with generator

$$\mathcal{L}f(y) = \int_{\mathbb{T}\setminus\{0\}} \big(f(x + y) - f(x)\big) \frac{C_{b,\rho}}{|y|(1 - |y|)}\, dy$$

for all $f \in C^1(\mathbb{T})$.

$C_{b,\rho} = \big(\frac{b-1}{b}\big)(\rho - \rho_c)^b \Big(\Gamma(1 + b) \int_0^{\rho-\rho_c} u^b(\rho - \rho_c - u)^b du\Big)^{-1}$

Method of proof

Potential theory. [Bovier, Eckhoff, Gayrard, Klein (2001,2002)]

- **valleys** $\mathcal{E}_x \subset \{\boldsymbol{\eta} : \psi_L(\boldsymbol{\eta}) = x\}$, $\pi_{L,N}(\cup_{x\in\Lambda}\mathcal{E}_x) \to 1$
  time spend out of valleys can be ignored
- **effective rates** $R_L(x,y) = \mathbb{E}_{\pi_{L,N}|\mathcal{E}^x} \sum_{\boldsymbol{\zeta}} r(.,\boldsymbol{\zeta})\, \mathbb{P}_{\boldsymbol{\zeta}}(\boldsymbol{\eta}_\tau \in \mathcal{E}_y)$
  sharp bounds via capacities $\simeq C_{b,\rho}\mathrm{cap}_\Lambda(x,y)L/\theta_L$

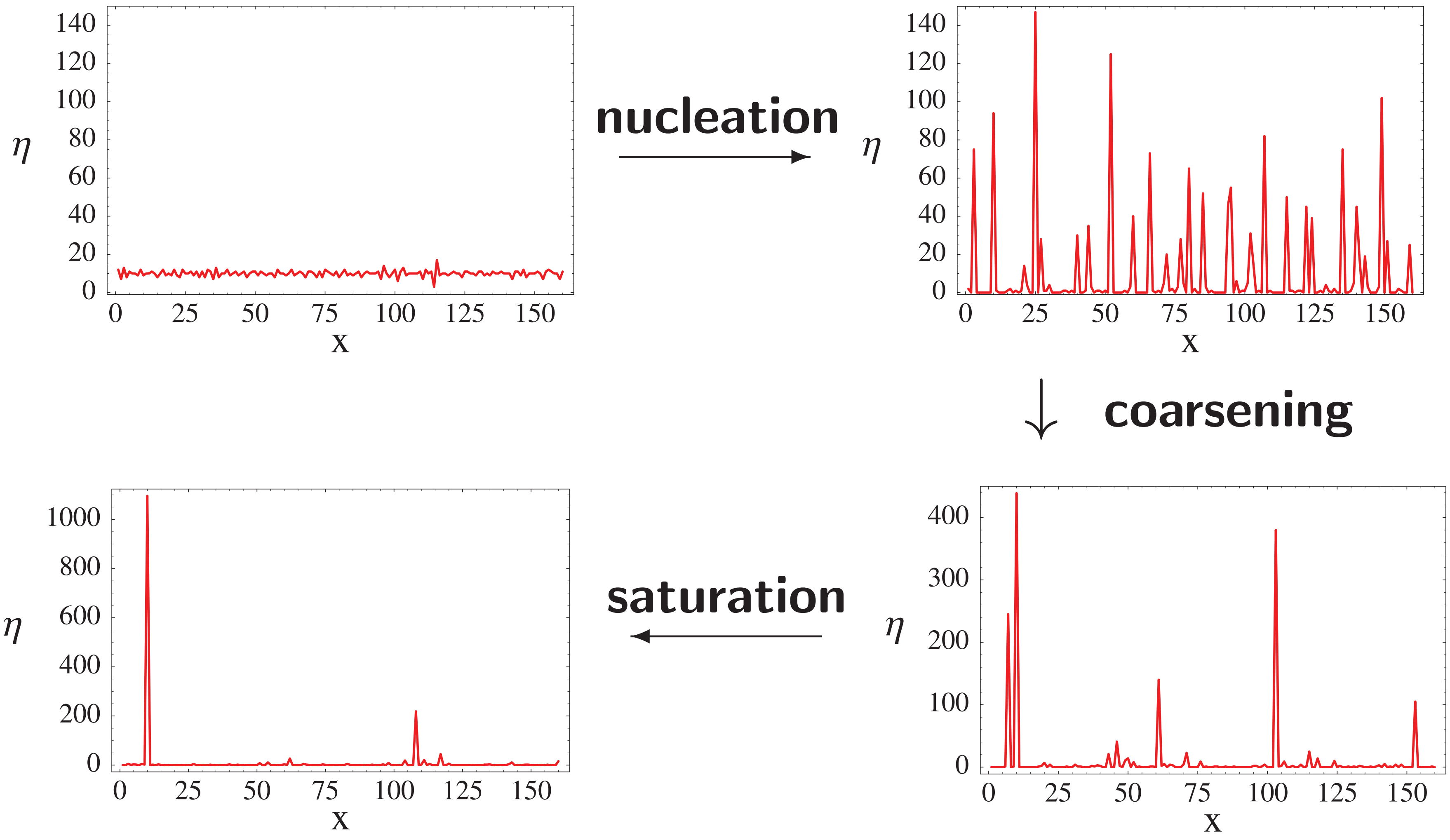
Martingale approach. [Landim, Beltran (2011,2012)]

- **tightness** of $\psi_L(\boldsymbol{\eta}_{\theta_L t})$ as $L\to\infty$
  involves pointwise upper bounds on rates (coupling)
- **martingale problem** for all $f \in C^1(\mathbb{T})$

$$f(Y_t) - f(Y_0) - \int_0^t \mathcal{L}f(Y_s)\,ds \quad \text{is a martingale}$$

- **equilibration** replace $\psi_L(\boldsymbol{\eta}_t)$ by process on Λ with rates R_L
  T_{rel} for $\boldsymbol{\eta}_t$ on the valley , $L^2 T_{rel} \ll \theta_L$

Dynamics of condensation



Coarsening dynamics

Λ fixed ; $N \to \infty$, $d_N \to 0$ such that $d_N \gg 1/N$

$$\mathcal{L}_N f(\boldsymbol{\eta}) = \sum_{x,y\in\Lambda_L} p(x,y)\, \eta_x (d_N + \eta_y) \big(f(\boldsymbol{\eta}^{x,y}) - f(\boldsymbol{\eta}) \big)$$

time scale $\theta_N := 1/d_N$

$$\mathbf{u}^N(t) := \big(\eta_x(\theta_N t)/N : x \in \Lambda \big)$$

process on the simplex $E = \big\{ \mathbf{u} \in [0,1]^\Lambda : \sum_{x\in\Lambda} u_x = 1 \big\}$

Taylor expansion ($p(x,y)$ symmetric)

$$\theta_N \mathcal{L}_N f(\mathbf{u}) = \frac{1}{2} \sum_{x,y\in\Lambda} p(x,y)(u_x - u_y)(\partial_{u_y} - \partial_{u_x}) f(\mathbf{u})$$

$$+ \frac{1}{2} \sum_{x,y\in\Lambda} p(x,y) u_x u_y \theta_N (\partial_{u_x} - \partial_{u_y})^2 f(\mathbf{u}) + O(\theta_N/N) =$$

$$= L\, f(\mathbf{u}) + \theta_N\, L'\, f(\mathbf{u}) + O(\theta_N/N)$$

two-scale system with drift and fast Wright-Fisher diffusion

Coarsening dynamics

WF-diffusion has absorbing set

$$\mathcal{A} := \big\{\mathbf{u} \in E \,:\, p(x,y)\, u_x u_y = 0 \text{ for all } x,y \in \Lambda\big\}\ .$$

corner points $\mathcal{C} := \big\{\mathbf{e}_x : x \in \Lambda\big\} \subset \mathcal{A}$

Theorem 1

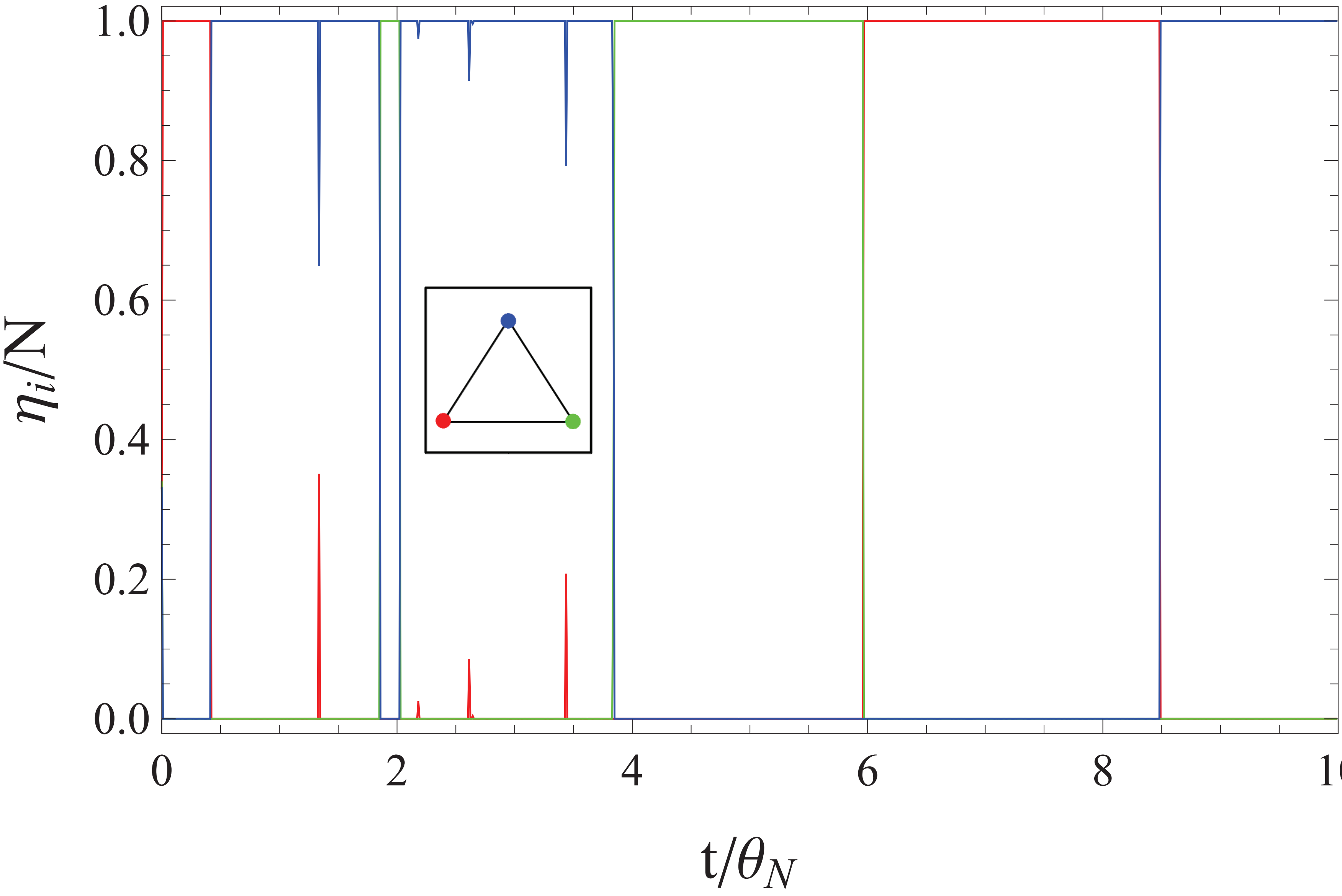
Assume $\mathbf{u}^N(0) \xrightarrow{d} \mathbf{u}^0 \in \mathcal{C}$. Then $\big(\mathbf{u}^N(t) : t \geq 0\big)$ converges weakly on path space to $\big(\mathbf{u}(t) : t \geq 0\big)$ on $\mathcal{C}$ with $\mathbf{u}(0) = \mathbf{u}^0$ and generator

$$Af(\mathbf{e}_x) = \sum_{y\in\Lambda} p(x,y)\big(f(\mathbf{e}_y) - f(\mathbf{e}_x)\big)\ .$$

If $p(x,y) > 0$ for all $x,y \in \Lambda$ the same holds (with $t > 0$) for general initial conditions $\mathbf{u}^N(0) \xrightarrow{d} \mathbf{u}^0 \in E$ with $\mathbb{P}(\mathbf{u}(0) = e_x) = u_x^0$.

Illustration

3-site ring $N = 10000,\ d_N = 0.001$



Coarsening dynamics

Theorem 2

Let $p(x,y)\in\{0,1\}$, $\mathbf{u}^N(0)\xrightarrow{d}\mathbf{u}^0\in E$ and write

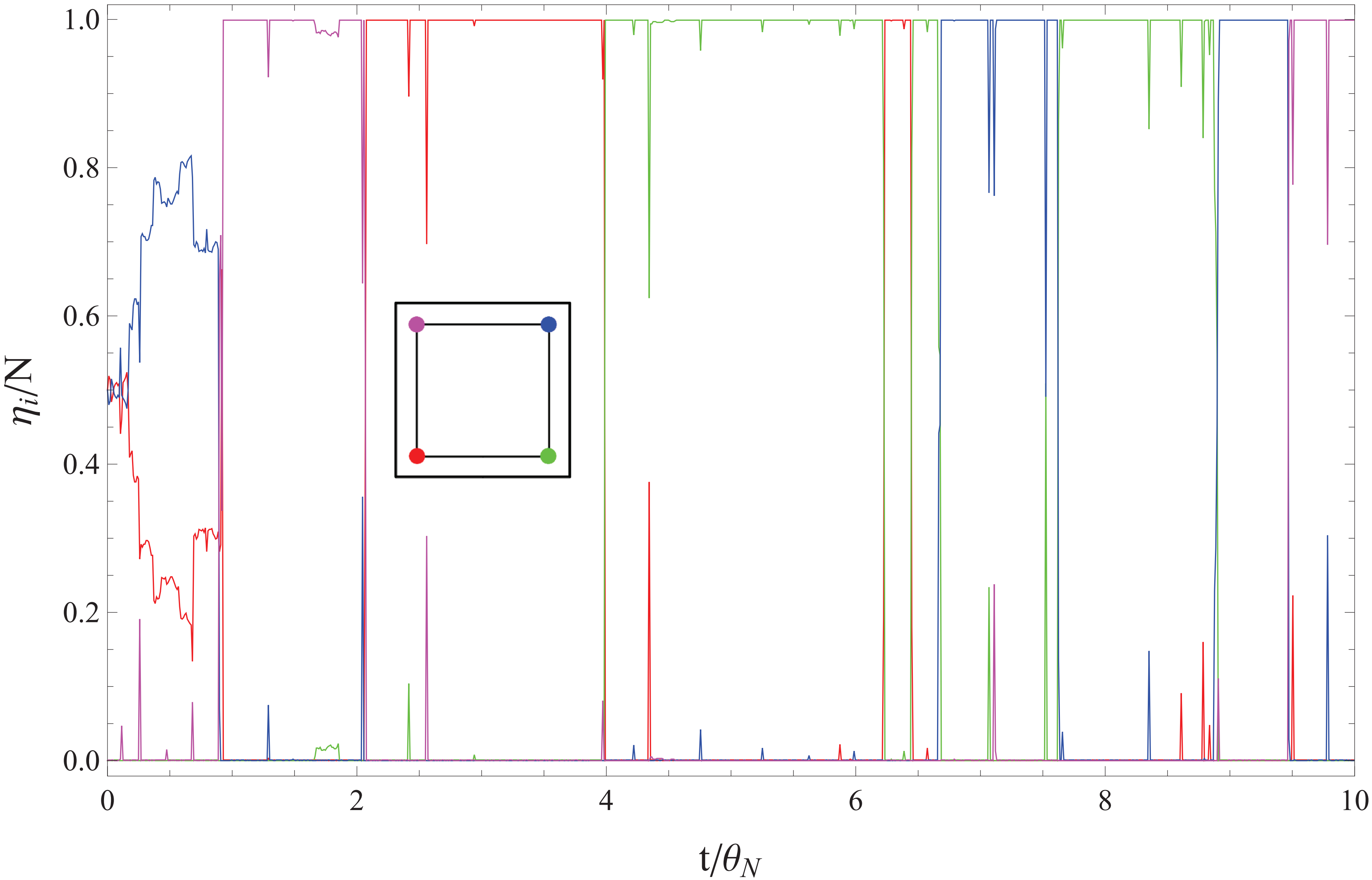
$$\hat{p}(x,y)=\big(1-p(x,y)\big)\sum_{z\in\Lambda}p(x,z)p(z,y)\geq 0\ .$$

Then $\big(\mathbf{u}^N(t):t>0\big)$ converges weakly on path space to $\big(\mathbf{u}(t):t>0\big)$ on $\mathcal{A}$ with initial condition $\mathbf{u}(0)\sim\nu_{\mathbf{u}^0}$ and generator

$$\begin{aligned}Af(\mathbf{u})=&\sum_{x,y\in\Lambda}\frac{1}{2}\hat{p}(x,y)\,u_x u_y\left(\partial_{u_x}-\partial_{u_y}\right)^2 f(\mathbf{u})\\&+\sum_{y\in\Lambda}\delta_{u_y,0}\Big(\sum_{x\in\Lambda}p(x,y)u_x\Big)\Big(f\Big(\mathbf{u}+\sum_{x\in\Lambda}p(x,y)u_x(\mathbf{e}_y-\mathbf{e}_x)\Big)-f(\mathbf{u})\Big)\end{aligned}$$

Illustration

4-site ring $N = 1000,\ d_N = 0.01$



Method of proof

convergence of the semigroups $e^{t\theta_N \mathcal{L}_N}$ and $e^{(L+\theta_N L')t}$

Central lemma. For all $t > 0$

$$p(x,y) \sup_{\mathbf{u}\in E} \mathbb{E}_{\mathbf{u}}\big[u_x^N(t) u_y^N(t)\big] \to 0 \quad \text{as } N \to \infty \ ,$$
$$p(x,y) \limsup_{N\to\infty} \theta_N \sup_{\mathbf{u}\in E} \mathbb{E}_{\mathbf{u}}\big[u_x^N(t) u_y^N(t)\big] \leq C \ .$$

from **Gronwall**-type estimate due to two-scale structure

- **tightness** of $(\mathbf{u}^N(t) : t > 0)$ with Lemma
  for $t = 0$ use right-continuity of paths
- for Theorem 1, characterize through **martingale problem** on $\mathcal{C}$

$$M_x(t) := u_x(t) - u_x(0) - \sum_{y\in\Lambda} \int_0^t p(x,y)\big(u_y(s) - u_x(s)\big)\, ds$$

Method of proof

for Theorem 2 (general initial condition)

- **harmonic projection** $Pf(x) := \int_{\mathcal{A}} f(a)\nu_x(da)$ [Kurtz (1973)]

  $P : C(E,\mathbb{R}) \to \mathcal{H}(E,\mathbb{R})$, $L'(Pf) = 0$ with BC $f(a)$

- convergence $e^{(L+\theta_N L')t} \to S(t)$ with $S(0) = P$

  semigroup on $\mathcal{H}(E,\mathbb{R})$ with **generator** $Af := (PL)f$

  process on $C(\mathcal{A},\mathbb{R})$ by uniqueness of harmonic functions

- computation $PL = \lim_{h\searrow 0} P\left(\frac{e^{hL}-I}{h}\right)$

  use **martingales** $u_x(t)$, $u_x(t)u_y(t)$ if $p(x,y) = 0$

Conclusion

- stationary results, relative entropy
- de-/localization in inhomogeneous systems
- stationary dynamics in the thermodynamic limit
- dynamics of condensation on finite lattices

Work in progress on coarsening.

generalize condition on θ_N, include asymmetry,

dynamics of correlation functions in thermodynamic limit,

more general rates (such as $\gamma > 2$, ECM)