THE LONDON SCHOOL OF ECONOMICS AND POLITICAL SCIENCE

# Quantile Cross-Spectral Measures of Dependence


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# Univariate Example: $X_t = Y_t / \mathrm{Var}(Y_t)^{1/2}$

- QAR(1) process, Koenker and Xiao (2006),

$$Y_t = 1.9(U_t - 0.5)Y_{t-1} + 0.1\Phi^{-1}(U_t)$$

  - $(U_t)$ i. i. d. standard uniform random variables,
  - $\Phi$ the distribution function of the standard normal distribution.

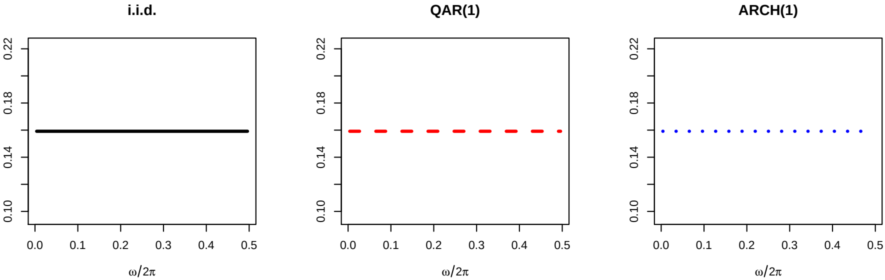
- ARCH(1) process, Engle (1982),

$$Y_t = \left(1/1.9 + 0.9Y_{t-1}^2\right)^{1/2}\varepsilon_t$$

  - $(\varepsilon_t)$ standard normal white noise.

- Independent Gaussian white noise.

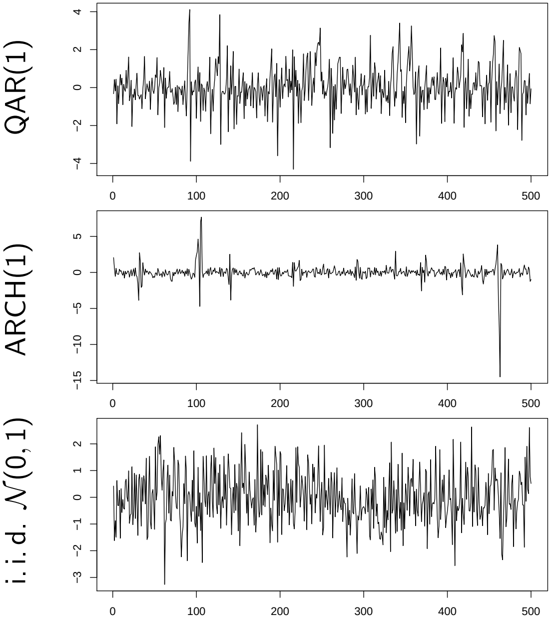
# Spectral densities

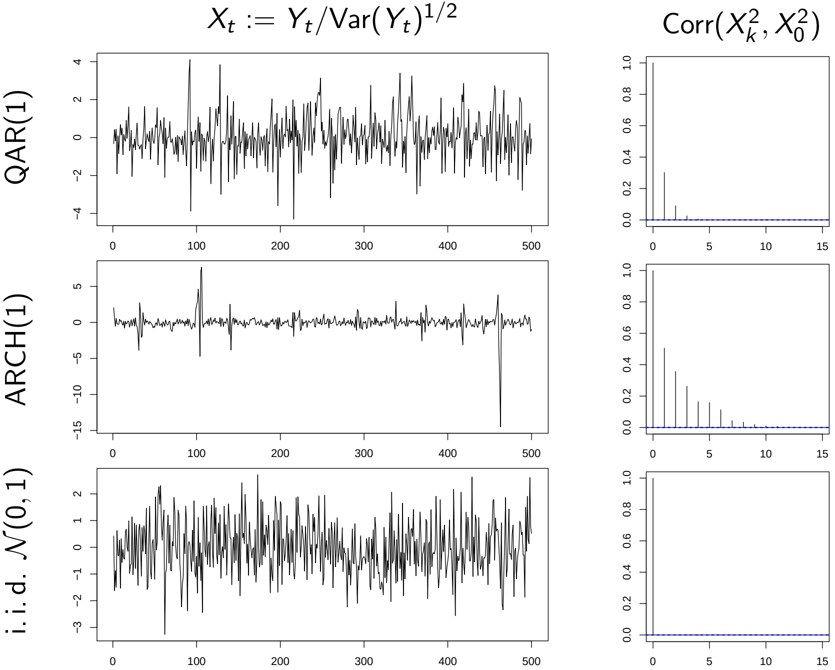



Spectral density:

$$\frac{1}{2\pi}\sum_{k\in\mathbb{Z}} \mathrm{Cov}(X_t, X_{t-k})e^{-ik\omega}$$

$$X_t := Y_t / \mathrm{Var}(Y_t)^{1/2}$$




$X_t := Y_t/\mathrm{Var}(Y_t)^{1/2}$
$\mathrm{Corr}(X_k^2, X_0^2)$
QAR(1)
ARCH(1)
i.i.d. $\mathcal{N}(0,1)$

# Multivariate Example: $\mathbf{X}_t = (X_{t,1}, X_{t,2})$

- *Independence*,

$$X_{t,1} = \varepsilon_t, \; X_{t,2} = \eta_t,$$

  - $(\varepsilon_t)_{t\in\mathbb{Z}}$, $(\eta_t)_{t\in\mathbb{Z}}$ standard normal white noise; independent.

- *Cross-sectional* dependence,

$$X_{t,1} = \varepsilon_t, \; X_{t,2} = 0.5(\varepsilon_t^2 - 1),$$

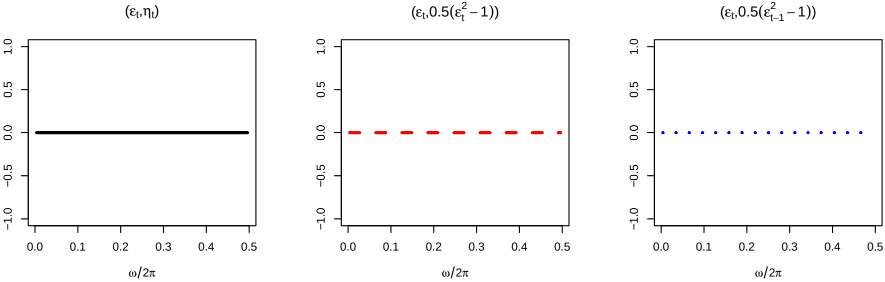
  - $(\varepsilon_t)_{t\in\mathbb{Z}}$ standard normal white noise.

- *Cross-sectional and serial* dependence,

$$X_{t,1} = \varepsilon_t, \; X_{t,2} = 0.5(\varepsilon_{t-1}^2 - 1),$$

  - $(\varepsilon_t)_{t\in\mathbb{Z}}$ standard normal white noise.

# Coherency




Coherency:

$$\frac{\frac{1}{2\pi}\sum_{k\in\mathbb{Z}} \mathrm{Cov}(X_{t,1}, X_{t-k,2})e^{-ik\omega}}{\left(\left(\frac{1}{2\pi}\sum_{k\in\mathbb{Z}} \mathrm{Cov}(X_{t,1}, X_{t-k,1})e^{-ik\omega}\right)\left(\frac{1}{2\pi}\sum_{k\in\mathbb{Z}} \mathrm{Cov}(X_{t,2}, X_{t-k,2})e^{-ik\omega}\right)\right)^{1/2}}$$

# Outline

# Quantification of Serial Dependence

Traditionally: $(\mathbf{X}_t)$, $\mathbf{X}_t = (X_{t,1}, \ldots, X_{t,d})$, (weakly) stationary
Auto/Cross-covariances of lag $k$: $\mathrm{Cov}(X_{t,j_1}, X_{t-k,j_2})$

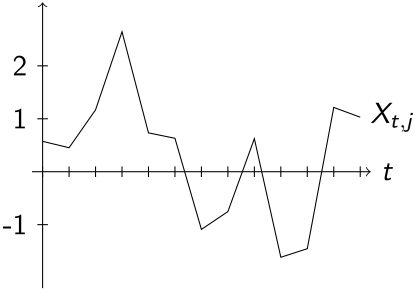
- completely describe centered Gaussian process,
- particularly well suited to measure linear dependencies,
- requires the existence of second order moments,
- invariant with respect to translations $x \mapsto x + b$.

Analysis of the (cross-)spectral density

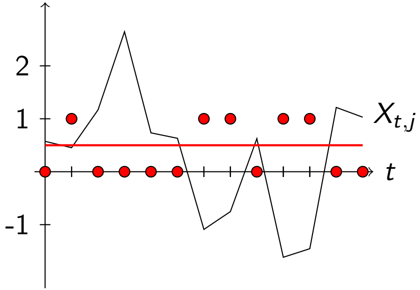
$$f^{j_1,j_2}(\omega) = \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} \mathrm{Cov}(X_{t,j_1}, X_{t-k,j_2}) \mathrm{e}^{-\mathrm{i}k\omega}$$

is analysis of the auto(cross-)covariances.

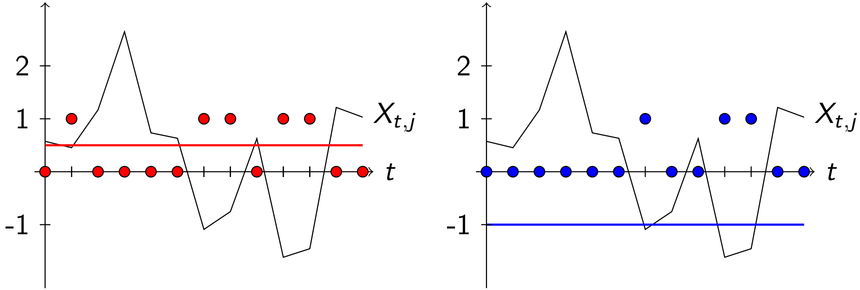
# The Process's $j$th Component $X_{t,j}$



# Clipped Process $(I\{X_{t,j} \leq 0.5\})_{t\in\mathbb{N}}$



# Clipped Processes $(I\{X_{t,j} \le 0.5\})_{t\in\mathbb{N}}$ and $(I\{X_{t,j} \le -1\})_{t\in\mathbb{N}}$



# A Quantile-Based Measure for Serial Dependence

$(\mathbf{X}_t)$, $\mathbf{X}_t = (X_{t,1}, \ldots, X_{t,d})$, stationary

Traditionally: Auto/Cross-covariances of lag $k$:

$$\gamma_k^{j_1,j_2} := \mathrm{Cov}(X_{t,j_1}, X_{t-k,j_2})$$

Analysis of the spectral density

$$\mathfrak{f}^{j_1,j_2}(\omega) := \sum_{k=-\infty}^{\infty} \mathrm{Cov}(X_{t,j_1}, X_{t-k,j_2}) \mathrm{e}^{-\mathrm{i}k\omega}$$

is analysis of the auto/cross-covariances.

# A Quantile-Based Measure for Serial Dependence

$(\mathbf{X}_t)$, $\mathbf{X}_t = (X_{t,1}, \ldots, X_{t,d})$, stationary

Traditionally: Auto/Cross-covariances of lag $k$:

$$\text{Cov}( X_{t,j_1} , X_{t-k,j_2} )$$

Analysis of the spectral density

$$\sum_{k=-\infty}^{\infty} \text{Cov}( X_{t,j_1} , X_{t-k,j_2} ) \mathrm{e}^{-\mathrm{i}k\omega}$$

is analysis of the auto/cross-covariances.

# A Quantile-Based Measure for Serial Dependence

$(\mathbf{X}_t)$, $\mathbf{X}_t = (X_{t,1}, \ldots, X_{t,d})$, strictly stationary,
$F_j$ continuous cdf of $X_{t,j}$, $q_j(\tau) := F_j^{-}(\tau)$

New approach: Copula cross-covariances of lag $k$:

$$\gamma_k^{j_1,j_2}(\tau_1,\tau_2) := \mathrm{Cov}(I\{X_{t,j_1} \leq q_{j_1}(\tau_1)\}, I\{X_{t-k,j_2} \leq q_{j_2}(\tau_2)\})$$

Analysis of the Copula spectral density kernel

$$\mathfrak{f}^{j_1,j_2}(\omega;\tau_1,\tau_2) := \sum_{k=-\infty}^{\infty} \mathrm{Cov}(I\{X_{t,j_1} \leq q_{j_1}(\tau_1)\}, I\{X_{t-k,j_2} \leq q_{j_2}(\tau_2)\}) \mathrm{e}^{-\mathrm{i}k\omega}$$

is analysis of the copula cross-covariances.

## *Covariance* and *Cross-covariance*

- $\gamma_k^{j_1,j_2}(\tau_1,\tau_2)$ is cross-covariance of bivariate time series

$$(I\{F_{j_1}(X_{t,j_1}) \le \tau_1\}, I\{F_{j_2}(X_{t,j_2}) \le \tau_2\}),$$

- $\gamma_k^{j_1,j_2}(\tau_1,\tau_2)$ always exist (no assumptions about moments),
- $\gamma_k^{j_1,j_2}(\tau_1,\tau_2) = \mathbb{P}(F_{j_1}(X_{t,j_1}) \le \tau_1, F_{j_2}(X_{t-k,j_2}) \le \tau_2) - \tau_1\tau_2$
  $\Rightarrow$ Copula: disentangling serial and marginal features
- Invariance of $\gamma_k^{j_1,j_2}$ under continuous monotone transformation
- $\{\gamma_k^{j_1,j_2}(\tau_1,\tau_2) | \ \tau_1,\tau_2 \in (0,1)\}$, $F_{j_1}$, and $F_{j_2}$ entirely characterize the joint distribution of $(X_{t,j_1}, X_{t-k,j_2})$,
- ... if $\mathbb{E}X_{t,j_1}^2 < \infty$ and $\mathbb{E}X_{t,j_2}^2 < \infty$, then this includes the acf/ccf of $(X_{t,j_1}X_{t,j_2})_{t\in\mathbb{Z}}$,

# Related measures

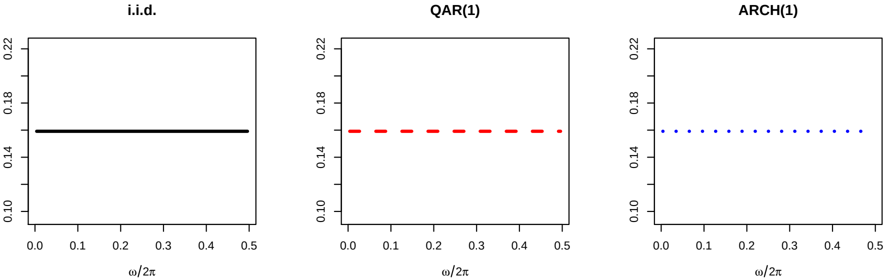
Based on the quantile cross-spectral densities, we define

- quantile cospectrum: $\Re \mathfrak{f}^{j_1,j_2}(\omega; \tau_1, \tau_2)$
- quantile quadrature spectrum: $-\Im \mathfrak{f}^{j_1,j_2}(\omega; \tau_1, \tau_2)$
- quantile amplitude spectrum: $|\mathfrak{f}^{j_1,j_2}(\omega; \tau_1, \tau_2)|$
- quantile phase spectrum: $\arg(\mathfrak{f}^{j_1,j_2}(\omega; \tau_1, \tau_2))$
- quantile coherency:

$$\mathfrak{R}^{j_1,j_2}(\omega; \tau_1, \tau_2) = \frac{\mathfrak{f}^{j_1,j_2}(\omega; \tau_1, \tau_2)}{\left(\mathfrak{f}^{j_1,j_1}(\omega; \tau_1, \tau_1)\mathfrak{f}^{j_2,j_2}(\omega; \tau_2, \tau_2)\right)^{1/2}}$$

- quantile coherence: $|\mathfrak{R}^{j_1,j_2}(\omega; \tau_1, \tau_2)|^2$

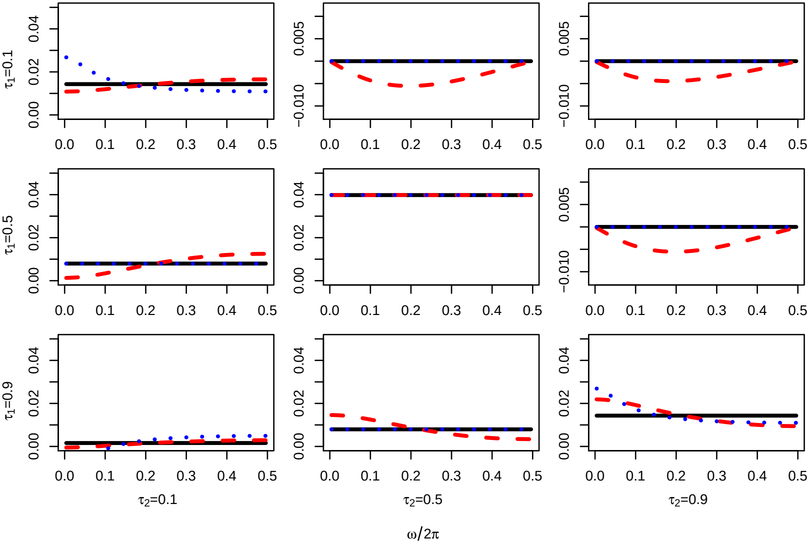
# Spectral densities



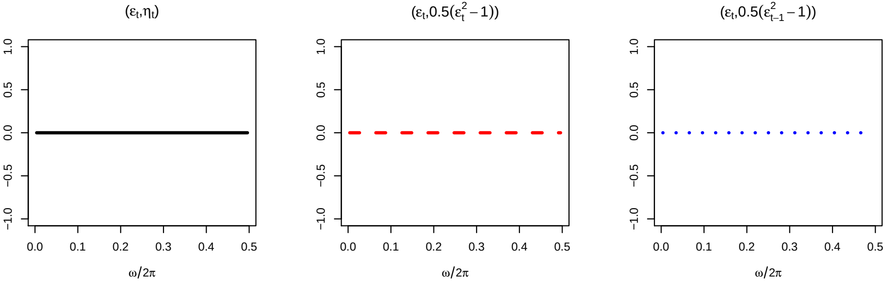

Spectral density:

$$\frac{1}{2\pi}\sum_{k\in\mathbb{Z}} \mathrm{Cov}(X_t, X_{t-k})\mathrm{e}^{-\mathrm{i}k\omega}$$

# Univariate Example: Copula spectral density kernels



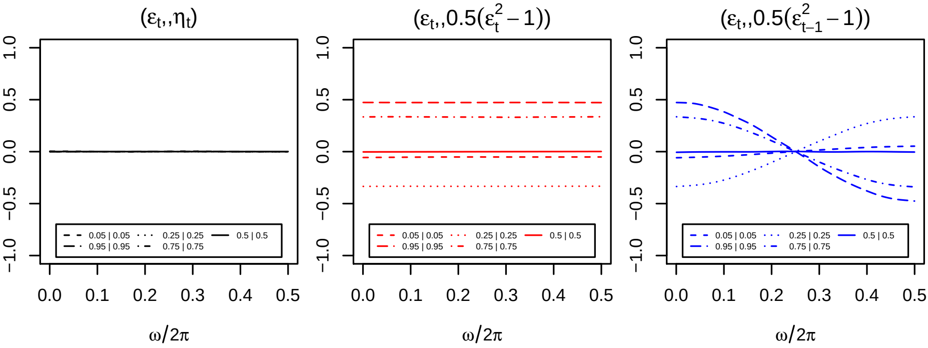
# Coherency




Coherency:

$$\frac{\frac{1}{2\pi}\sum_{k\in\mathbb{Z}} \mathrm{Cov}(X_{t,1}, X_{t-k,2})\mathrm{e}^{-\mathrm{i}k\omega}}{\left(\left(\frac{1}{2\pi}\sum_{k\in\mathbb{Z}} \mathrm{Cov}(X_{t,1}, X_{t-k,1})\mathrm{e}^{-\mathrm{i}k\omega}\right)\left(\frac{1}{2\pi}\sum_{k\in\mathbb{Z}} \mathrm{Cov}(X_{t,2}, X_{t-k,2})\mathrm{e}^{-\mathrm{i}k\omega}\right)\right)^{1/2}}$$

# Multivariate Example: Copula cross spectral density kernels




Figures show real parts of $\mathfrak{R}^{j_1,j_2}(\omega; \tau_1, \tau_2)$.

# Properties

**Copula cross spectral density kernel**

- $$\overline{\mathfrak{f}^{j_1,j_2}(\omega;\tau_1,\tau_2)} = \mathfrak{f}^{j_1,j_2}(-\omega;\tau_1,\tau_2) = \mathfrak{f}^{j_2,j_1}(\omega;\tau_2,\tau_1) = \mathfrak{f}^{j_2,j_1}(2\pi+\omega;\tau_2,\tau_1)$$
- If $\Im\mathfrak{f}^{j_1,j_2}(\,\cdot\,;\tau_1,\tau_2) \equiv 0$, then $\gamma_k^{j_1,j_2}(\tau_1,\tau_2) = \gamma_{-k}^{j_1,j_2}(\tau_1,\tau_2)$, $\forall k$
- **"Pairwise time reversibility":**
  If $\Im\mathfrak{f}^{j_1,j_2} \equiv 0$, then $(X_{t-k,j_1}, X_{t,j_2})$ and $(X_{t+k,j_1}, X_{t,j_2})$ possess the same copula, for all $k$.

# Properties

**Copula cross spectral density kernel**

- $\overline{\mathfrak{f}^{j_1,j_2}(\omega;\tau_1,\tau_2)} = \mathfrak{f}^{j_1,j_2}(-\omega;\tau_1,\tau_2)$
  $= \mathfrak{f}^{j_2,j_1}(\omega;\tau_2,\tau_1) = \mathfrak{f}^{j_2,j_1}(2\pi+\omega;\tau_2,\tau_1)$
- If $\Im\mathfrak{f}^{j_1,j_2}(\,\cdot\,;\tau_1,\tau_2) \equiv 0$, then $\gamma_k^{j_1,j_2}(\tau_1,\tau_2) = \gamma_{-k}^{j_1,j_2}(\tau_1,\tau_2)$, $\forall k$
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# Modifying the (traditional) Cross-Periodogram

From $X_{0,j_1}, \ldots, X_{n-1,j_1}$ and $X_{0,j_2}, \ldots, X_{n-1,j_2}$, for $\omega \in (0, \pi)$, compute

$$I_n^{j_1,j_2}(\omega) := \frac{1}{2\pi n} d_n^{j_1}(\omega) d_n^{j_2}(-\omega),$$

where

$$d_n^j(\omega) := \sum_{t=0}^{n-1} X_{t,j} \mathrm{e}^{-\mathrm{i}t\omega}$$

# Modifying the (traditional) Cross-Periodogram

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$$I_n^{j_1,j_2}(\omega) := \frac{1}{2\pi n} d_n^{j_1} \quad (\omega \quad\quad ) d_n^{j_2} \quad (-\omega \quad\quad ),$$

where

$$d_{n,U}^{j}(\omega, \tau) := \sum_{t=0}^{n-1} I\{F \quad _j(X_t) \leq \tau\} \mathrm{e}^{-\mathrm{i}t\omega}$$

# Modifying the (traditional) Cross-Periodogram

From $X_{0,j_1}, \dots, X_{n-1,j_1}$ and $X_{0,j_2}, \dots, X_{n-1,j_2}$, for $\omega \in (0, \pi)$, compute

$$I_{n,R}(\omega) := \frac{1}{2\pi n} d^{j_1}_{n,R}(\omega, \tau_1) d^{j_2}_{n,R}(-\omega, \tau_2),$$

where

$$d^{\tau}_{n,R}(\omega, \tau) := \sum_{t=0}^{n-1} I\{\hat{F}_{n,j}(X_t) \leq \tau\} \mathrm{e}^{-\mathrm{i}t\omega}$$

$$\hat{F}_{n,j}(x) := \frac{1}{n} \sum_{t=0}^{n-1} I\{X_{t,j} \leq x\}$$

# Related literature

- Hong (2000): joint distributions/copulas (testing independence)
- Lee, Rao (2012+): joint distributions (testing general hypotheses)
- Hagemann (2011+); K., Volgushev, Dette, Hallin (2015+): quantiles, copulas, rank-based estimation
- Mikosch, Zhao (2014, 2015): Indicators of extreme events

**Another approach to estimation: replacing the loss function**

- Katkovnik (1998): use robust loss functions (signal detection)
- Li (2008): replace mean by median
- Li (2012): general quantiles $\tau_1 = \tau_2$
- Dette, Hallin, K., Volgushev (2015): weighted $L^1$ loss, general quantiles $(\tau_1, \tau_2) \in [0, 1]^2$, rank-based estimation

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# Asymptotic properties of $I_{n,R}^{j_1,j_2}(\omega;\tau_1,\tau_2)$

**Theorem (Baruník, K. (2015))**

*Under suitable technical assumptions*

$$\Big(\big(I_{n,R}^{j_1,j_2}(\omega;\tau_1,\tau_2)\big)_{j_1,j_2}\Big)_{(\tau_1,\tau_2)\in[0,1]^2} \rightsquigarrow \Big(\mathbb{I}(\omega;\tau_1,\tau_2)\Big)_{(\tau_1,\tau_2)\in[0,1]^2} \text{ in } \ell^{\infty}_{\mathbb{C}^{d\times d}}([0,1]^2).$$

*The $\mathbb{C}^{d\times d}$-valued limiting processes $\mathbb{I}$, indexed by $(\tau_1,\tau_2)\in[0,1]^2$, is of the form*

$$\mathbb{I}(\omega;\tau_1,\tau_2) = \frac{1}{2\pi}\mathbb{D}(\omega;\tau_1)\overline{\mathbb{D}(\omega;\tau_2)'},$$

*where $\mathbb{D}(\omega;\tau) = (\mathbb{D}^j(\omega;\tau))_{j=1,\dots,d}$, $\underline{\tau\in[0,1]}$, $\omega\in\mathbb{R}$ is a centered, $\mathbb{C}^d$-valued Gaussian processes with $\mathbb{D}(\omega;\tau) = \overline{\mathbb{D}(-\omega;\tau)} = \mathbb{D}(\omega+2\pi;\tau)$ and covariance structure of the following form*

$$\mathrm{Cov}(\mathbb{D}^{j_1}(\omega;\tau_1),\mathbb{D}^{j_2}(\omega;\tau_2)) = 2\pi\mathfrak{f}^{j_1,j_2}(\omega;\tau_1,\tau_2).$$

*The family $\{\mathbb{D}(\omega;\cdot)\ :\ \omega\in[0,\pi]\}$ is a collection of independent processes.*

# Smoothing the CR periodogram kernel

Note that the limit $\mathbb{I}$ from the previous theorem has

- $\mathbb{I}(\omega_1, \cdot), \mathbb{I}(\omega_2, \cdot)$ independent if $\omega_1 - \omega_2, \omega_1 + \omega_2 \notin 2\pi\mathbb{Z}$,
- $\mathbb{E}[\mathbb{I}^{j_1,j_2}(\omega, \tau_1, \tau_2)] = \mathfrak{f}^{j_1,j_2}(\omega, \tau_1, \tau_2)$, and
- $\mathrm{Var}[\mathbb{I}^{j_1,j_2}(\omega, \tau_1, \tau_2)] > 0$

$\Rightarrow$ 'Asymptotic expectation' of $I_{n,R}^{j_1,j_2}(\omega, \tau_1, \tau_2)$ correct, but no consistency

- Classical approach: smoothing

# Smoothing the CR periodogram kernel

## Definition (The smoothed CR periodogram kernel)

*For kernel $W$ and bandwidth $b_n$, let*

$$\hat{G}_{n,R}^{j_1,j_2}(\omega;\tau_1,\tau_2) := \frac{2\pi}{n}\sum_{s=1}^{n-1} W_n\big(\omega - 2\pi s/n\big)\, I_{n,R}^{j_1,j_2}(2\pi s/n,\tau_1,\tau_2),$$

*where* $W_n(u) := \sum_{j=-\infty}^{\infty} b_n^{-1} W(b_n^{-1}[u + 2\pi j])$.

# Asymptotic properties of $\hat{G}_{n,R}^{j_1,j_2}(\omega;\tau_1,\tau_2)$

**Theorem (Baruník and K. (2015+))**

*Under suitable assumptions, for any fixed* $\omega \in (0,\pi)$

$$\sqrt{nb_n}\Big(\big(\hat{G}_{n,R}^{j_1,j_2}(\omega;\tau_1,\tau_2)\big)_{j_1,j_2} - \big(\mathfrak{f}^{j_1,j_2}(\omega,\tau_1,\tau_2)\big)_{j_1,j_2} - \mathbf{B}_n^{(k)}(\omega;\tau_1,\tau_2)\Big)_{\tau_1,\tau_2\in[0,1]} \rightsquigarrow \mathbb{H}(\omega;\cdot,\cdot), \text{ in } \ell^{\infty}_{\mathbb{C}^{d\times d}}([0,1]^2),$$

*where* $\mathbf{B}_n^{(k)}(\omega;\tau_1,\tau_2)$ *is an expression for the bias and* $\mathbb{H}(\omega;\cdot,\cdot)$ *is a centered, complex-valued Gaussian process (details next slide).*

# Details: Asymptotic properties of $\hat{G}^{j_1,j_2}_{n,R}(\omega;\tau_1,\tau_2)$

The elements of the bias matrix $\mathbf{B}_n^{(k)}$ are given by

$$\left\{\mathbf{B}_n^{(k)}(\omega;\tau_1,\tau_2)\right\}_{j_1,j_2} := \sum_{\ell=2}^{k} \frac{b_n^\ell}{\ell!} \int_{-\pi}^{\pi} v^\ell W(v) dv \frac{\mathrm{d}^\ell}{\mathrm{d}\omega^\ell} \mathfrak{f}^{j_1,j_2}(\omega;\tau_1,\tau_2)$$

The process $\mathbb{H}(\omega;\cdot,\cdot) := (\mathbb{H}^{j_1,j_2}(\omega;\cdot,\cdot))_{j_1,j_2=1,\dots,d}$ is a centered, $\mathbb{C}^{d\times d}$-valued Gaussian process characterized by

$$\begin{aligned}
&\mathrm{Cov}\left(\mathbb{H}^{j_1,j_2}(\omega;u_1,v_1),\mathbb{H}^{k_1,k_2}(\lambda;u_2,v_2)\right)\\
&= 2\pi\left(\int_{-\pi}^{\pi} W^2(\alpha)\mathrm{d}\alpha\right)\Big(\mathfrak{f}^{j_1,k_1}(\omega;u_1,u_2)\mathfrak{f}^{j_2,k_2}(-\omega;v_1,v_2)\eta(\omega-\lambda)\\
&\qquad + \mathfrak{f}^{j_1,k_2}(\omega;u_1,v_2)\mathfrak{f}^{j_2,k_1}(-\omega;v_1,u_2)\eta(\omega+\lambda)\Big),
\end{aligned}$$

where $\eta(x) := I\{x = 0 (\mod 2\pi)\}$. The family $\{\mathbb{H}(\omega;\cdot,\cdot),\, \omega\in[0,\pi]\}$ is a collection of independent processes and $\mathbb{H}(\omega;\tau_1,\tau_2) = \overline{\mathbb{H}(-\omega;\tau_1,\tau_2)} = \mathbb{H}(\omega+2\pi;\tau_1,\tau_2)$.

# Some Comments

'Suitable technical assumptions':

1. Weak assumptions on bandwidth and kernel
2. Strict stationarity and continuous marginal distribution
3. Decay condition on cumulants of indicators;
   holds under exponential alpha-mixing

- No assumption on smoothness of (joint) distributions

'Corollary': Consistency of $\hat{G}_{n,R}^{j_1,j_2}(\omega;\tau_1,\tau_2)$ uniform w. r. t $\tau_1,\tau_2$
'Application': rank-based $L^2$ periodogram

Tightness: Restricted chaining and cumulant calculations.

# Estimating the related quantities

Estimators for the quantile cospectrum, quantile quadrature spectrum, quantile amplitude spectrum, quantile phase spectrum, quantile coherency and quantile coherence are then given by $\Re \hat{G}_{n,R}^{j_1,j_2}(\omega;\tau_1,\tau_2)$, $-\Im \hat{G}_{n,R}^{j_1,j_2}(\omega;\tau_1,\tau_2)$, $|\hat{G}_{n,R}^{j_1,j_2}(\omega;\tau_1,\tau_2)|$, $\arg(\hat{G}_{n,R}^{j_1,j_2}(\omega;\tau_1,\tau_2))$,

$$\hat{\mathfrak{R}}_{n,R}^{j_1,j_2}(\omega;\tau_1,\tau_2) := \frac{\hat{G}_{n,R}^{j_1,j_2}(\omega;\tau_1,\tau_2)}{\left(\hat{G}_{n,R}^{j_1,j_1}(\omega;\tau_1,\tau_1)\hat{G}_{n,R}^{j_2,j_2}(\omega;\tau_2,\tau_2)\right)^{1/2}}$$

and $|\hat{\mathfrak{R}}_{n,R}^{j_1,j_2}(\omega;\tau_1,\tau_2)|^2$, respectively.

# Asymptotic properties of $\hat{\mathfrak{R}}^{j_1,j_2}_{n,R}(\omega;\tau_1,\tau_2)$

**Theorem (Barunik, Kley (2015+))**

*Under suitable technical assumptions, for any fixed* $\omega \neq 0 \mod 2\pi$,

$$\sqrt{nb_n}\Big(\big(\hat{\mathfrak{R}}^{j_1,j_2}_{n,R}(\omega;\tau_1,\tau_2)\big)_{j_1,j_2} - \big(\mathfrak{R}^{j_1,j_2}(\omega;\tau_1,\tau_2)\big)_{j_1,j_2} - \mathfrak{B}^{(k)}_n(\omega;\tau_1,\tau_2)\Big)_{(\tau_1,\tau_2)\in[0,1]^2} \rightsquigarrow \mathbb{L}(\omega;\cdot,\cdot) \text{ in } \ell^\infty_{\mathbb{C}^{d\times d}}([\varepsilon,1-\varepsilon]^2),$$

*where*

$$\Big\{\mathbb{L}(\omega;\tau_1,\tau_2)\Big\}_{j_1,j_2} := \frac{1}{\sqrt{\mathfrak{f}_{1,1}\mathfrak{f}_{2,2}}}\Big(\mathbb{H}_{1,2} - \frac{1}{2}\frac{\mathfrak{f}_{1,2}}{\mathfrak{f}_{1,1}}\mathbb{H}_{1,1} - \frac{1}{2}\frac{\mathfrak{f}_{1,2}}{\mathfrak{f}_{2,2}}\mathbb{H}_{2,2}\Big),$$

$$\Big\{\mathfrak{B}^{(k)}_n(\omega;\tau_1,\tau_2)\Big\}_{j_1,j_2} := \frac{1}{\sqrt{\mathfrak{f}_{1,1}\mathfrak{f}_{2,2}}}\Big(\mathbf{B}_{1,2} - \frac{1}{2}\frac{\mathfrak{f}_{1,2}}{\mathfrak{f}_{1,1}}\mathbf{B}_{1,1} - \frac{1}{2}\frac{\mathfrak{f}_{1,2}}{\mathfrak{f}_{2,2}}\mathbf{B}_{2,2}\Big)$$

*with the notation* $\mathfrak{f}_{a,b} = \mathfrak{f}^{j_a,j_b}(\omega;\tau_a,\tau_b)$, $\mathbb{H}_{a,b} = \mathbb{H}^{j_a,j_b}(\omega;\tau_a,\tau_b)$, *and* $\mathbf{B}_{a,b} = \{\mathbf{B}^{(k)}_n(\omega;\tau_a,\tau_b)\}_{j_a,j_b}$ $(a,b=1,2)$.

# R-package quantspec

Earliest version on CRAN from Dec/2011.
Latest version on CRAN (currently version 1.2-0):

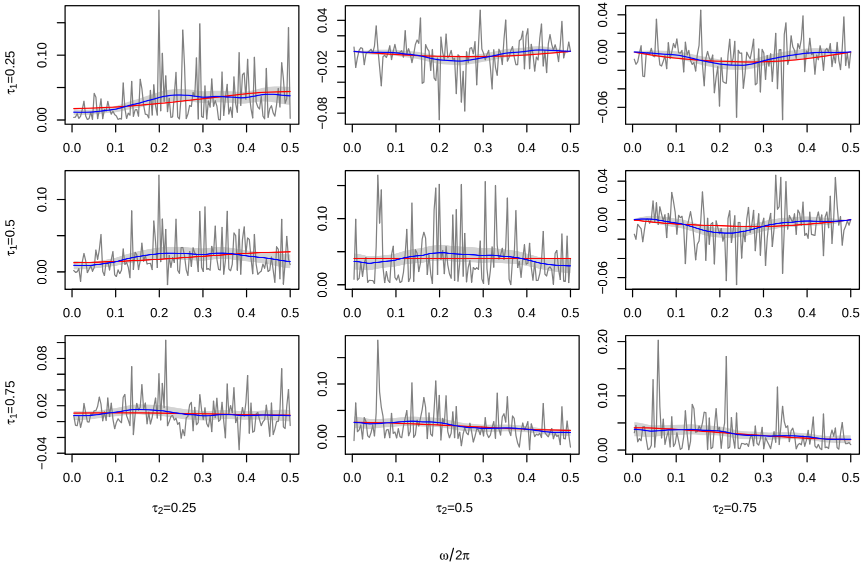
- S4 object-oriented design,
- flexible methods for construction and plotting,
- unit tests using `testthat`,
- thorough technical documentation,
- vignette on the design and with a tutorial.

- Analysis of univariate and multivariate time series
- Computation of $d_n^j(\omega, \tau)$ using FFT (also computation of another estimator),
- Smoothing (convolution via FFT),
- Simulation of copula density kernels.

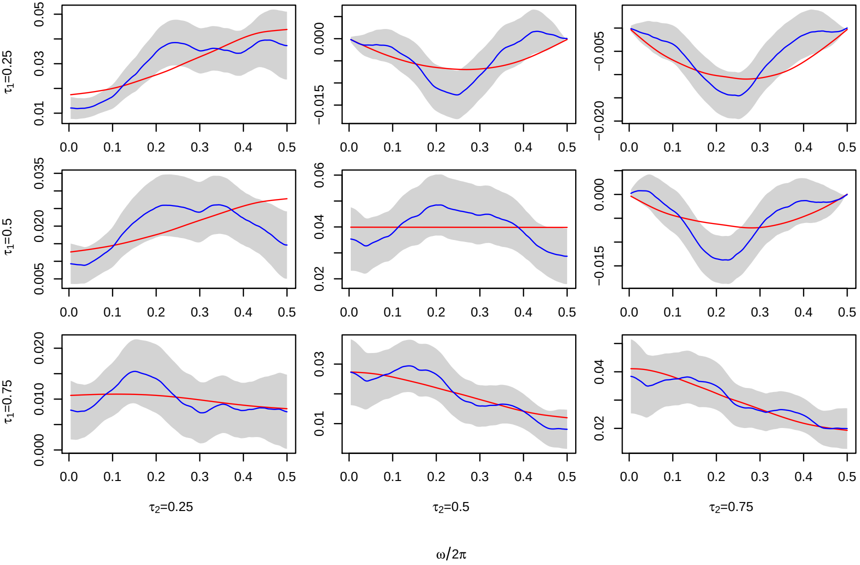
# Example using the R-package quantspec

```
library(quantspec)

Y <- ts3(256)

levels <- c(0.25, 0.5, 0.75)
FF <- 2 * pi * (0:128) / 256
K <- length(levels)

wgt <- kernelWeight(W = W1, b = 0.1)

sPG.cl <- smoothedPG(Y, levels.1 = levels,
  type = "clipped", weight = wgt)

sCSD <- quantileSD(N = 2^9, type = "copula", ts = ts3,
seed.init = 2581, levels.1 = levels, R = 1000)

plot(sPG.cl, plotPG = TRUE, qsd = sCSD, ratio = 1.7,
  frequencies = FF[FF > 0], type.CIs = "naive.sd",
  type.scaling = "individual")
```

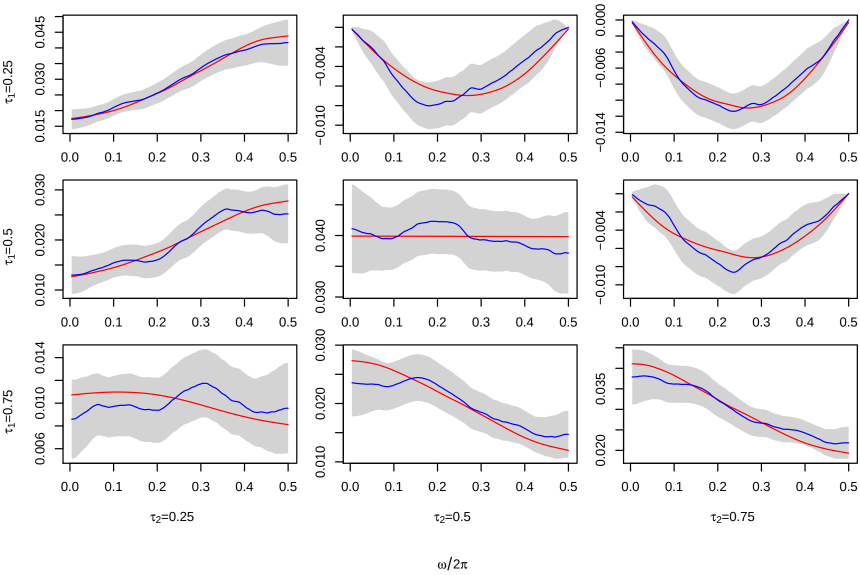
# Plot of the SmoothedPG in the example, $n = 256$



# Plot of the `SmoothedPG` in the example, $n = 256$



# Plot of the SmoothedPG in the example, $n = 1.024$



# Quantile cross-spectral analysis of stock market returns

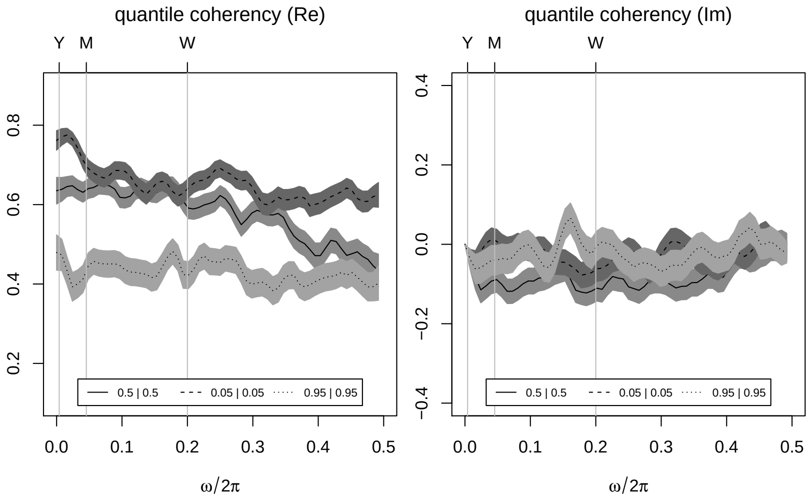
Capital Asset Pricing Model (CAPM)

- We assume investor needs correlated returns in upper quantile (positive returns)
- ... but independence at lower quantile (negative returns).
- An investor also needs to distinguish between long and short runs.

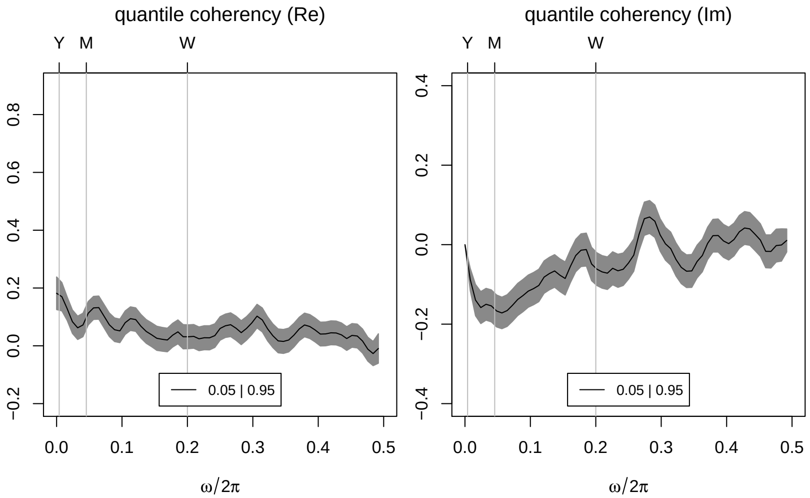
We estimate quantile coherency for time series of returns from

- a *portfolio formed from consumer non-durables* vs. *excess market returns*.
- Data from 1926 to 2015, standardized by estimated volatily.

# Quantile cross-spectral analysis of stock market returns



# Quantile cross-spectral analysis of stock market returns



# “Model-free” and “nonlinear” spectral analysis

Quantile-based measures of serial dependence:

- Separation of serial dependencies and marginal features,
- Invariance under monotone transformations.

Quantile-based periodograms:

- inherit many of the properties of the ordinary periodogram,
- Robustness can be expected,
- Analysis of pair-copulae, not simply covariances,
- Weak convergence in $\ell^{\infty}_{\mathbb{C}^{d \times d}}([0,1]^2)$,
- no linearity, distributional, nor even moment assumptions required.

# Much work remains on the Research Agenda

- Tests based on the CCR periodogram kernel,
- Estimation of higher-order spectra,
- Estimation of integrated spectra,
- Bootstrap,
- Locally stationary processes,
- ...