Stability of multi-dimensional Markov chains, with applications

Joint work with S. Foss (HWU), A. Turlikov (SUAI), J. Thomas and T. Worrall (UoE)

Seva Shneer

Heriot-Watt University

Bristol, 28 November 2014

Network

- M devices
- Wi-Max (IEEE 802.16) and Wi-Fi (IEEE 802.11) protocols
- Schedule for Wi-Max traffic
- Random access for Wi-Fi traffic
- Data streams of the same standard interfere with each other
- Data streams from different standards generated by different devices do not interfere
- Data streams from different standards generated by the same device do interfere

Model

- M transmitters
- Slotted time
- Each transmission duration equals 1
- High- and low- priority messages
- At most one high- and at most one low-priority messages transmitted in a time slot
- A transmitter cannot transmit both high- and low-priority messages in a time slot
- Schedule for high-priority messages, no collisions
- Random access for low-priority messages, collided messages return to their origins

High-priority (red) messages assumptions

- At time slot t, ξ_n^t new ones arrive at device n, $\mathbf{E}\xi_n^t = \lambda_R/M$
- $\{\xi_n^t\}$ are i.i.d. in t
- Symmetrical schedule for transmissions: at time slot t, transmitter number $i(t) = ((t-1) \mod M) + 1$ is scheduled to transmit a red message
- If transmitter $i(t)$ has a red message, its transmission is attempted and is successful
- Otherwise no red message is transmitted in the time slot

Red-messages dynamics

Let R_n^t be the number of red messages in the queue of transmitter n at time t. Then

$$\left(R_1^t, .., R_M^t\right)$$

is a Markov chain which is positive recurrent if $\lambda_R < 1$. Moreover,

$$\mathbf{P}(R_{i(t)}^t = 0) \to 1 - \lambda_R, \quad t \to \infty.$$

Low-priority (green) messages assumptions

- At time slot t, η_n^t new ones arrive at device n, $\mathbf{E}\eta_n^t = \lambda_G/M$
- $\{\eta_n^t\}$ are i.i.d. in t
- Transmission attempts are governed by a random-access ALOHA-type protocol: every transmitter not currently transmitting a red message with a non-empty green queue attempts to transmit a green message with (fixed) probability p
- Three possibilities:
  - No transmission attempted
  - Exactly one transmission attempted. It is successful
  - More than one transmission attempts. All of them unsuccessful, messages stay in their queues

Model dynamics

Let G_n^t be the number of green messages in the queue of transmitter n at time t. Then

$$\left(G_1^t, .., G_M^t, R_1^t, .., R_M^t\right)$$

is a Markov chain and we are interested in its long-time behaviour.
Let first $M = 1$. Then a green message transmission will be attempted (and will always be successful) with probability p every time the red queue is empty. We expect that

$$\lambda_G < (1 - \lambda_R)p$$

leads to stability (positive recurrence).

Model dynamics

Let now $M \geq 2$. Two cases:

- Transmitter $i(t)$ has a red message to transmit. Then probability of a successful transmission of a green message is $(M-1)p(1-p)^{M-2}$
- Transmitter $i(t)$ does not have a red message to transmit. Then probability of a successful transmission of a green message is $Mp(1-p)^{M-1}$

Stability should be achieved if

$$\lambda_G < \lambda_R(M-1)p(1-p)^{M-2} + (1-\lambda_R)Mp(1-p)^{M-1}.$$

General mathematical model

Let $\{X^t\}$ and $\{Y^t\}$ be random sequences taking values in measurable spaces $(\mathcal{X}, \mathcal{B}_\mathcal{X})$ and $(\mathcal{Y}, \mathcal{B}_\mathcal{Y})$, respectively, and assume that $\{(X^t, Y^t)\}$ is a Markov Chain. Assume also that

- $\{X^t\}$ is a Markov Chain with autonomous dynamics: for all $x \in \mathcal{X}$ and $y \in \mathcal{Y}$,
$$\mathbf{P}_{x,y}(X^1 \in \cdot) = \mathbf{P}_x(X^1 \in \cdot).$$
- The X-chain is *Harris ergodic*, so there exists a stationary distribution $\pi = \pi_X$ such that
$$\sup_{B \in \mathcal{B}} \left| \mathbf{P}_x \left(X^t \in B \right) - \pi(B) \right| \to 0$$
as $t \to \infty$, for any $x \in \mathcal{X}$.

Auxiliary chain

Introduce an auxiliary (time-homogeneous) Markov chain $\{\widehat{Y}^t\}$ with transition probabilities

$$\mathbf{P}(\widehat{Y}^{t+1} \in \cdot \mid \widehat{Y}^t = y) = \int_{\mathcal{X}} \pi_X(dx)\mathbf{P}(Y^1 \in \cdot \mid X^1 = x, Y^0 = y).$$

Hypothesis

If $\{\widehat{Y}^t\}$ is positive recurrent, then so is (X^t, Y^t).

Theorem

Natural (Foster-Lyapunov type conditions) imply positive recurrence of both $\{\widehat{Y}^t\}$ and (X^t, Y^t).

Conditions on Y

For the sequence $\{Y^t\}$ we assume that there exists a non-negative measurable function L_2 such that:

- The expectations of the absolute values of the increments of the sequence $\{L_2(Y^t)\}$ are bounded from above by a constant U:

$$\sup_{x\in\mathcal{X},y\in\mathcal{Y}} \mathbf{E}_{x,y} \left| L_2\left(Y^1\right) - L_2\left(Y^0\right)\right| \le U < \infty.$$

- There exist a non-negative and non-increasing function $h(N), N \ge 0$ such that $h(N) \downarrow 0$ as $N \to \infty$, and a measurable function $f : \mathcal{X} \to (-\infty, \infty)$ such that

$$\int_{\mathcal{X}} f(x)\pi(dx) := -\varepsilon < 0$$

and

$$\mathbf{E}_{x,y}\left(L_2\left(Y^1\right) - L_2(y)\right) \le f(x) + h(L_2(y))$$

for all $x \in \mathcal{X}$ and $y \in \mathcal{Y}$.

Main result

Remark

Conditions on Y imply positive recurrence of $\widehat{Y}^t$.

Theorem (Foss, S, Turlikov (2012))

Under assumptions for X and Y, the Markov chain $\{(X^t, Y^t)\}$ is positive recurrent.

Corollary

Stability of the communication network under natural conditions

(Some of the) related models

- *Bin-packing problems* (Gamarnik, 2004; Gamarnik and Squillante, 2005)
- *Cat-and-mouse Markov chain* (Litvak and Robert, 2012)
- *Other communication-networks applications* (Borst, Jonckheere, Leskela, 2008; Shah, Shin, 2012)
- Queueing systems, storage processes, etc.....

Work in progress and research plans

- No knowledge of stationary distribution
- Non-autonomous dynamics, stronger dependence
- Other standard methods for proving stability
- Other applications
- Unstable first component

Stochastic recursive sequences

Under some (very weak) assumptions, every Markov chain may be represented as a *stochastic recursive sequence*

$$X_{t+1} = f(X_t, \xi_t)$$

with a certain function f and an i.i.d. sequence $\{\xi_t\}$.
SRS (with specific functions f) have applications in economics.

Precautionary savings model (Huggett, 1993)

Agents maximise expected discounted utility

$$\mathbf{E}\left[\sum_{t=0}^{\infty}\beta^t u(c_t)\right]$$

subject to a budget constraint

$$c_t + R^{-1}a_{t+1} \leq a_t + e_t,$$

a non-negativity constraint on consumption $c_t \geq 0$ where e_t is the endowment, a_t is current assets, a_{t+1} are assets next period, c_t is consumption, $\beta \in (0,1)$ is the discount factor and R^{-1} is the price of assets.

Precautionary savings model

The individual's maximisation problem can be written recursively as

$$v_e(a) = \max_{(c,a')\in\Gamma(a,e)} u(c) + \beta \mathsf{E}_{e'|e} v_{e'}(a')$$

where

$$\Gamma(a,e) = \left\{(c,a') \mid c + R^{-1}a' \leq a + e, a' \geq \underline{a}, c \geq 0\right\}$$

is the constraint set and $v_e(a)$ are the value functions (one for each realisation e). The policy functions are $c = c(a,e)$ and $a' = f(a,e)$, and f is continuous and non-decreasing in the first argument.

Known results

We have

$$a_{t+1} = f(a_t, e_t), \quad \text{or} \quad X_{t+1} = f(X_t, \xi_t)$$

with a continuous and non-decreasing f.
Question: does $\{X_t\}$ *converge to a stationary regime?*

- Most known results in economics assume $\{\xi_t\}$ to be i.i.d.
- Huggett(1993) assumes that $\{e_t\}$ is a two-state Markov chain with a positive correlation between e_t and e_{t+1}.

Known results

Theorem (Bhattacharya, Majumdar (2007))

Assume a time-homogeneous Markov chain $\{X_t\}$ *is represented as a stochastic recursion with i.i.d. driving sequence* $\{\xi_n\}$, *where function* $f : [0,1] \times \mathcal{V} \to [0,1]$ *is monotone increasing in the first argument. Assume there exist a number* $c \in [0,1]$ *and an integer* $N \geq 1$ *such that*

$$\varepsilon_1 := \mathbf{P}^{(1)}(X_N \leq c) > 0$$

and

$$\varepsilon_2 := \mathbf{P}^{(0)}(X_N \geq c) > 0.$$

Then there exists a unique distribution π *on* $[0,1]$ *such that*

$$\sup_x d(F_t^{(x)}, \pi) \to 0, \quad n \to \infty$$

exponentially fast.

Our model

Introduce an SRS

$$X_{t+1} = f(X_t, Z_t)$$

with a regenerative sequence $\{Z_t\}$ and a non-decreasing and continuous (in the first argument) function f.

Introduce also an auxiliary process $\tilde{X}_n^{(a)}$ that starts from $\tilde{X}_0^{(a)} = a$ at time 0 and follows recursion

$$\tilde{X}_{n+1}^{(a)} = f\left(\tilde{X}_n^{(a)}, Z_{T_0+n}\right) \quad \text{for all} \quad n \geq 0.$$

Main result

Theorem (Foss, S., Thomas, Worrall (2014))

Assume that the function f is monotone increasing in the first argument and the following assumptions hold:

$$\varepsilon_1 := \mathbf{P}\left(\tilde{X}^{(1)}_{T_1 - T_0} \le c\right) > 0,$$

and

$$\varepsilon_2 := \mathbf{P}\left(\tilde{X}^{(0)}_{T_1 - T_0} \ge c\right) > 0.$$

Then there exists a distribution $\widetilde{\pi}$ on $[0,1]$ such that $\sup_x d(G_n^{(x)}, \widetilde{\pi}) \to 0$ as $n \to \infty$ exponentially fast. Here $G_n^{(x)}$ is the distribution of X_{T_n} if $X_{T_0} = x$.
Further, if in addition the function f is continuous in the first argument, then there exists a distribution π such that the distributions of X_t converge weakly to π, for any initial value X_0.

Corollaries

- Stationary regime in the Huggett model with an arbitrary finite state space Markov chain
- Stationary regime in a number of other well-known economics models with Markov driving sequences rather than i.i.d.