Order of current variance in the simple exclusion process

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(Budapest University of Technology and Economics)

Joint work with

Timo Seppäläinen
(University of Wisconsin - Madison)

Rényi Institute, November 23, 2006.

1. ASEP: Interacting particles
2. ASEP: Surface growth
3. Growth fluctuations
4. The second class particle

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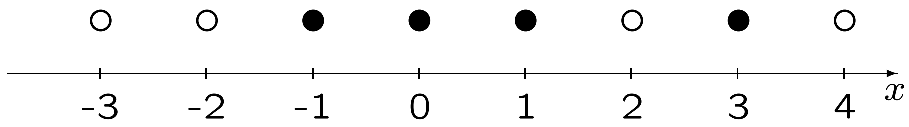

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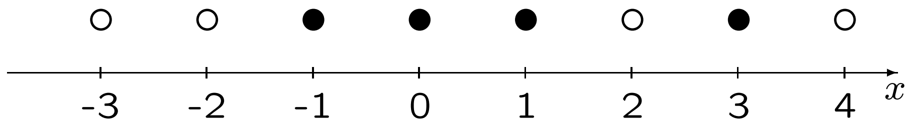

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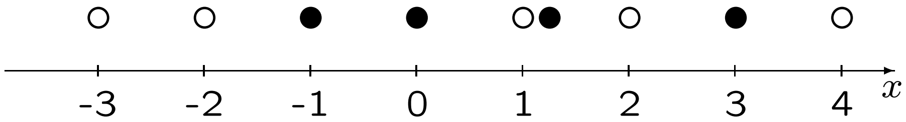

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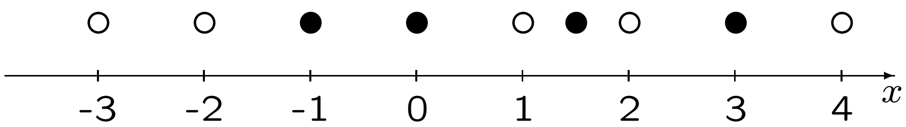

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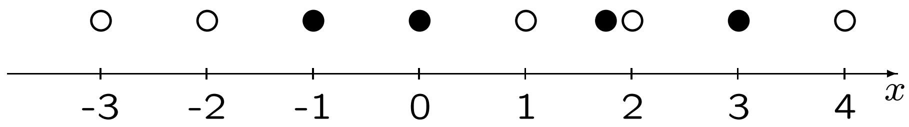

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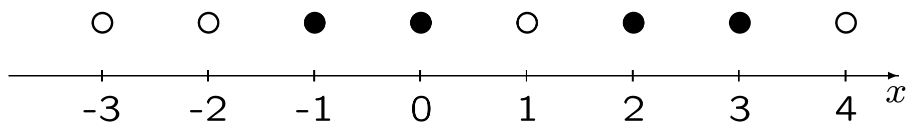

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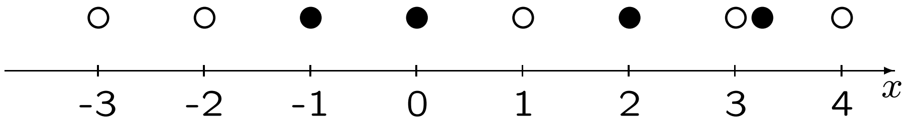

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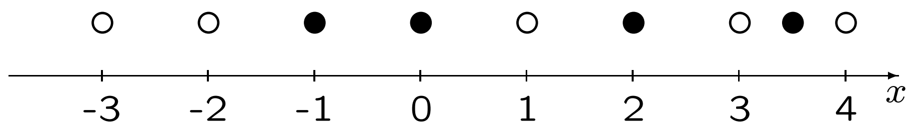

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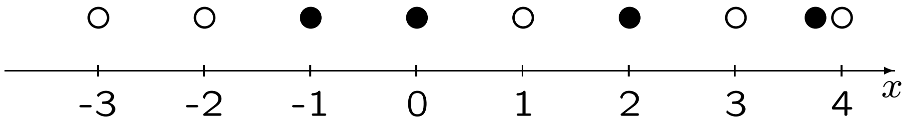

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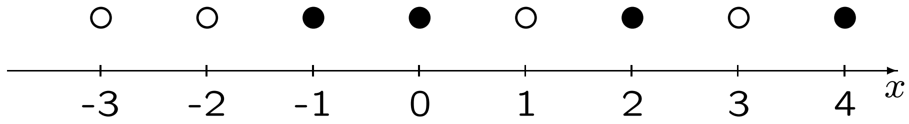

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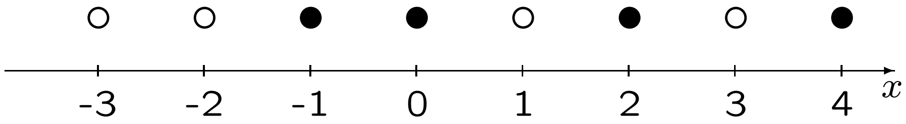

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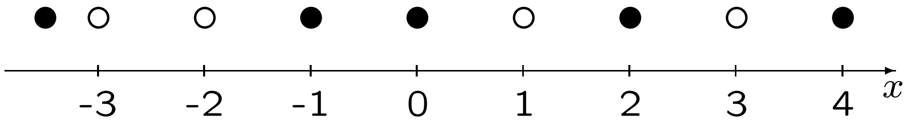

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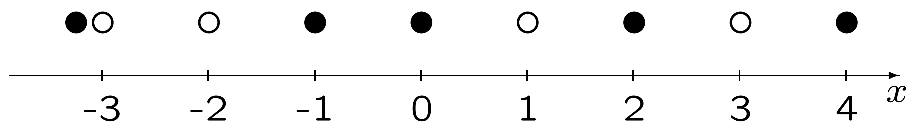

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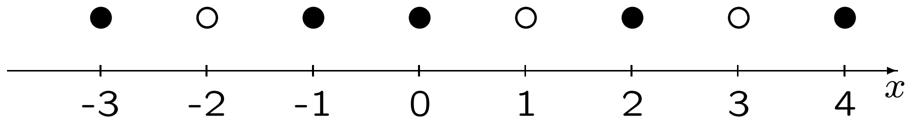

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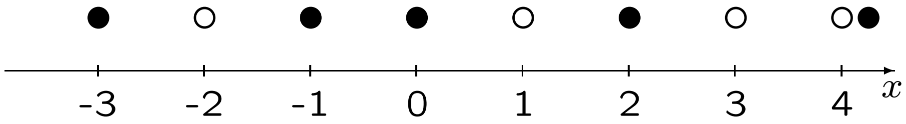

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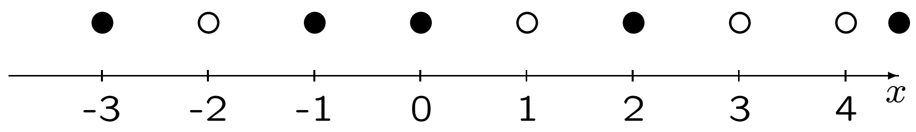

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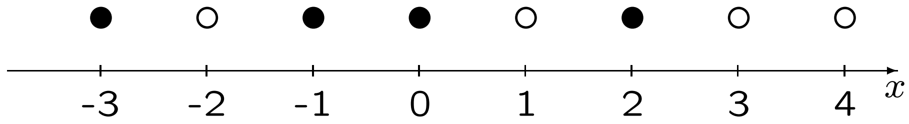

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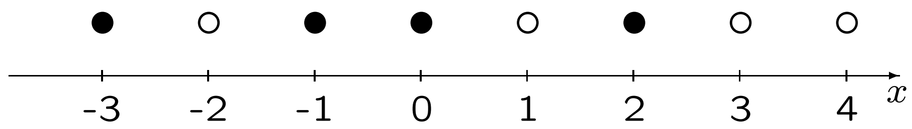

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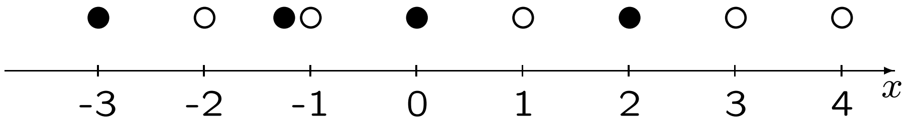

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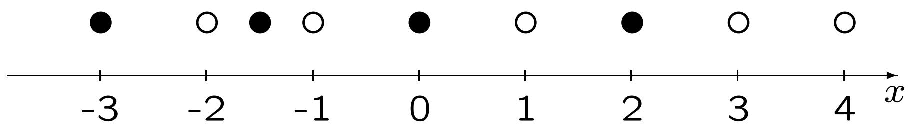

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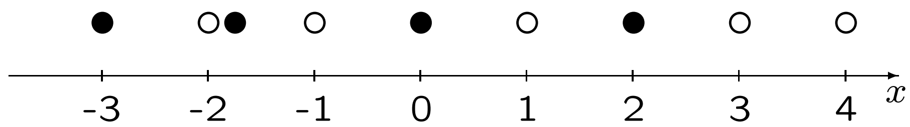

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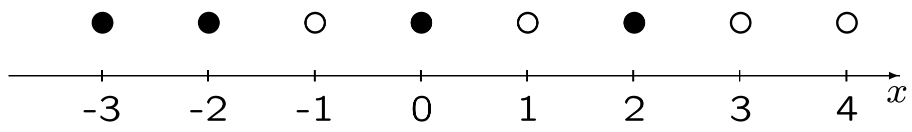

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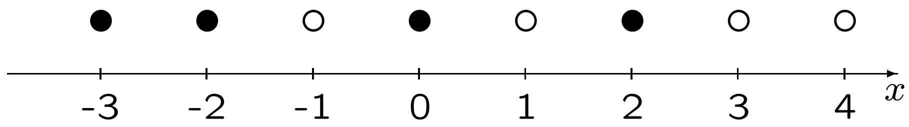

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The Bernoulli(ϱ) distribution is time-stationary for any $(0 \leq \varrho \leq 1)$.

Any translation-invariant stationary distribution is a mixture of Bernoullis.

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$\rightsquigarrow$ The characteristic speed $C(\varrho) := a[1-2\varrho]$.
(ϱ is constant along $\dot{X}(T) = C(\varrho)$.)

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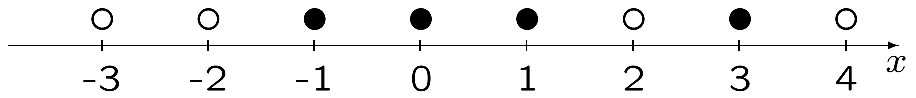

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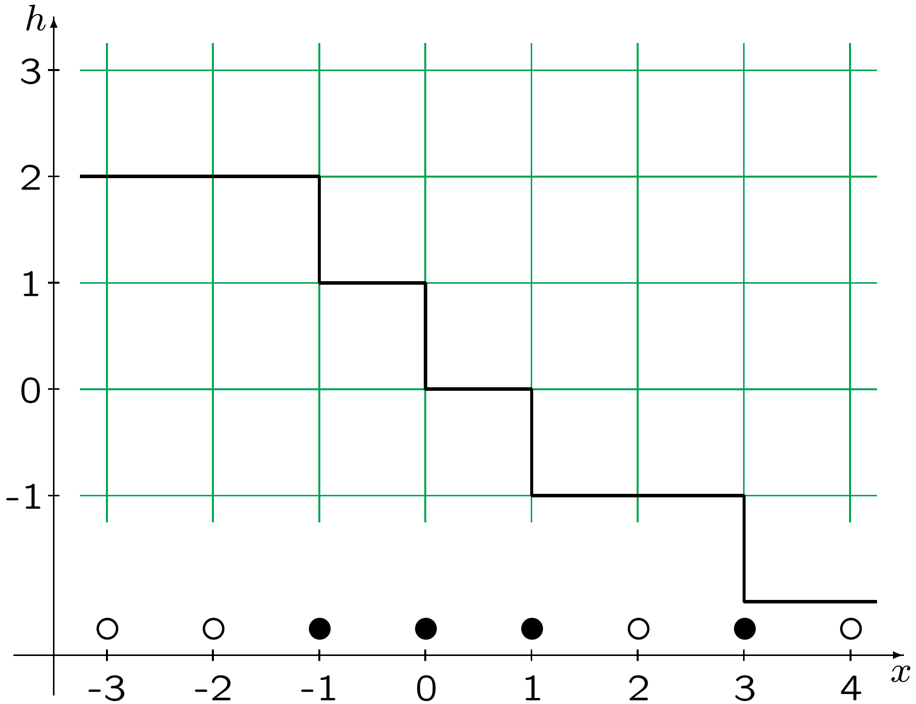

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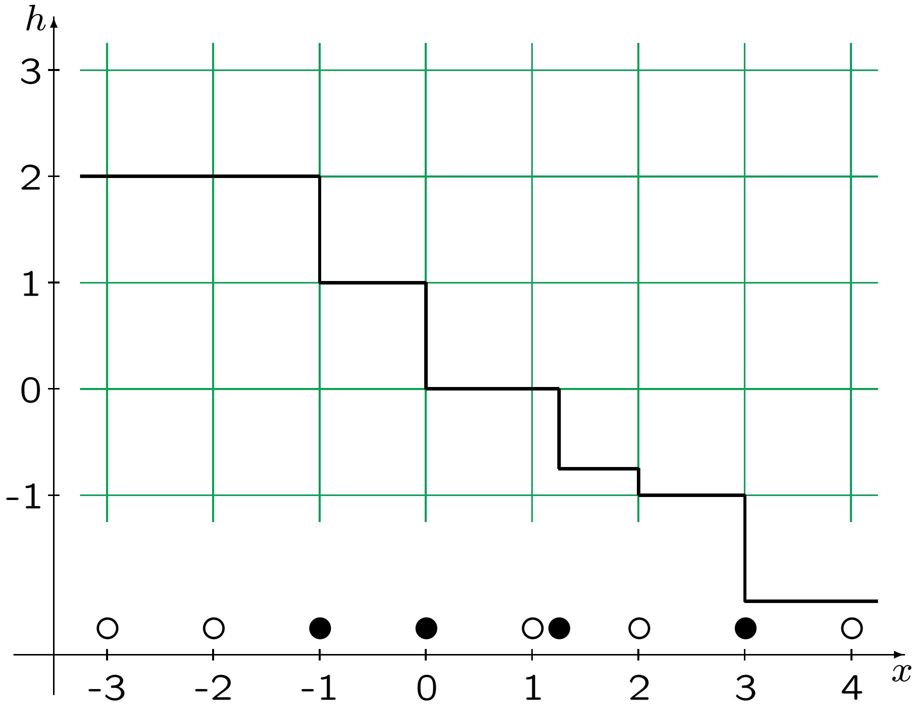

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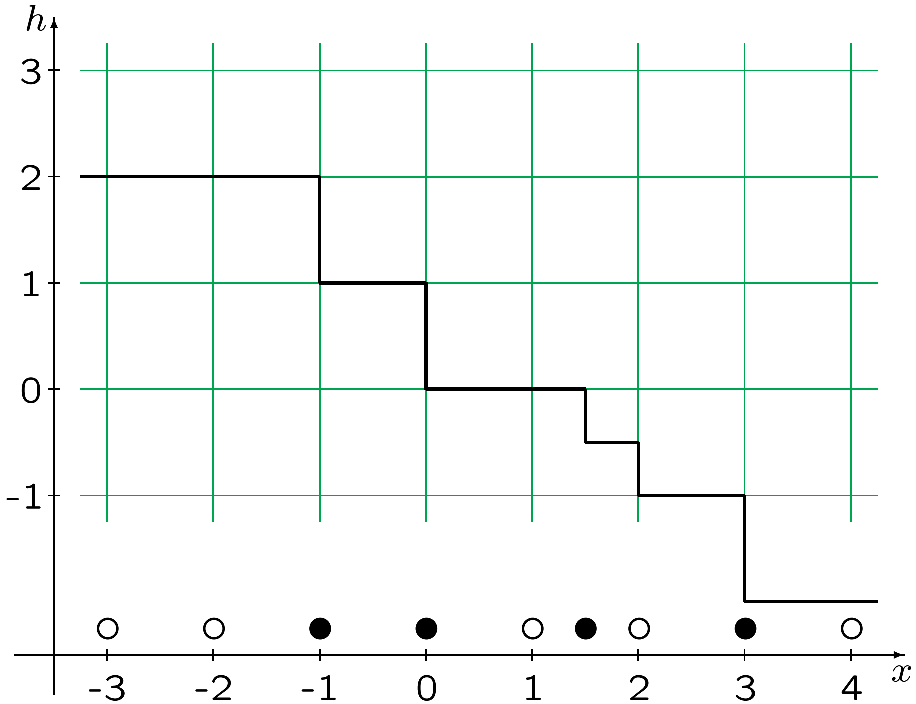

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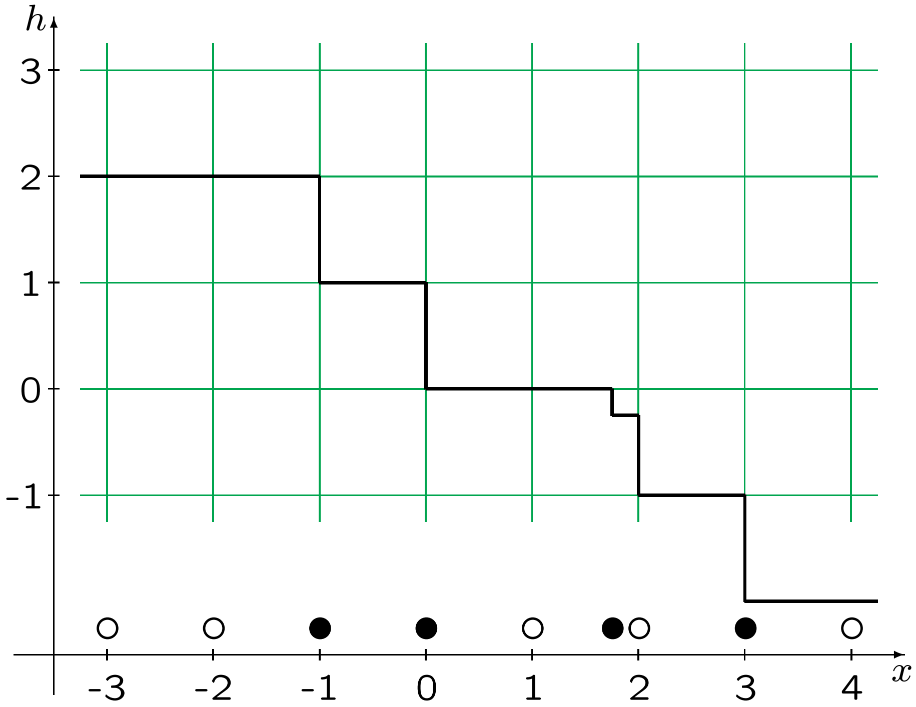

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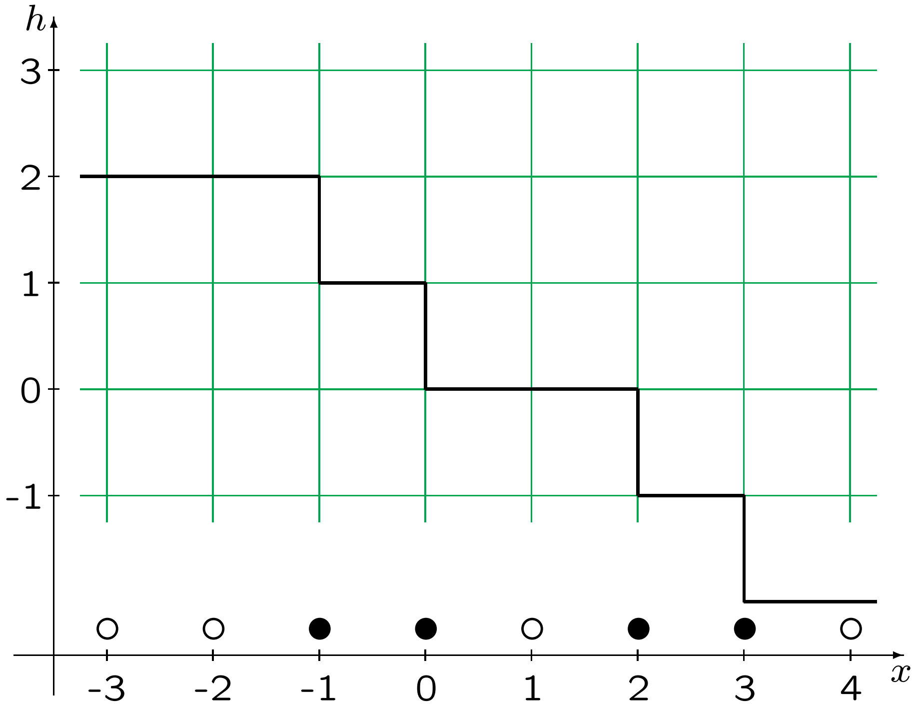

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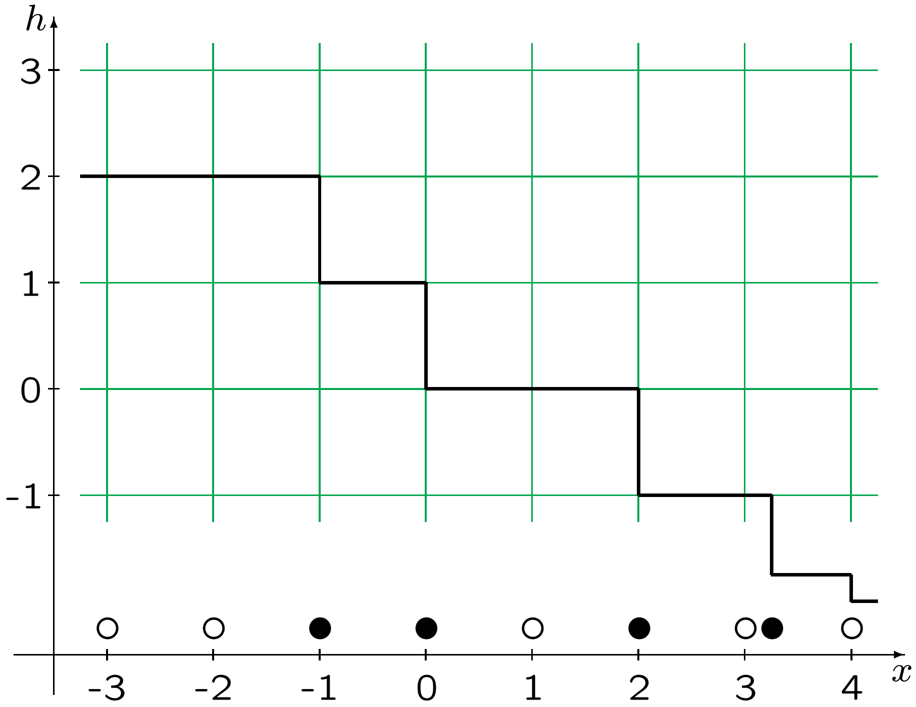

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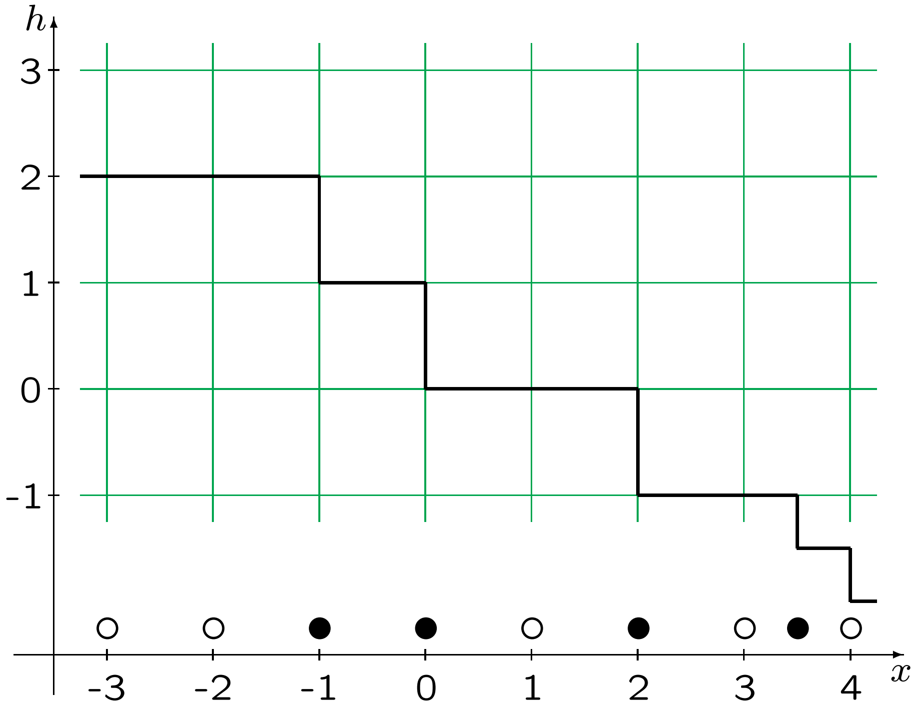

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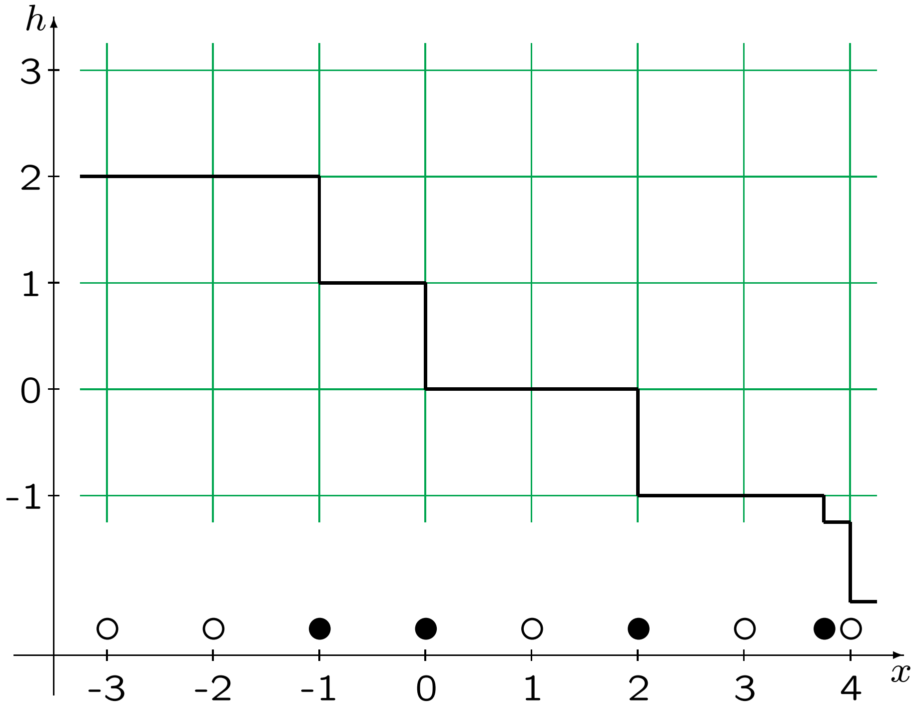

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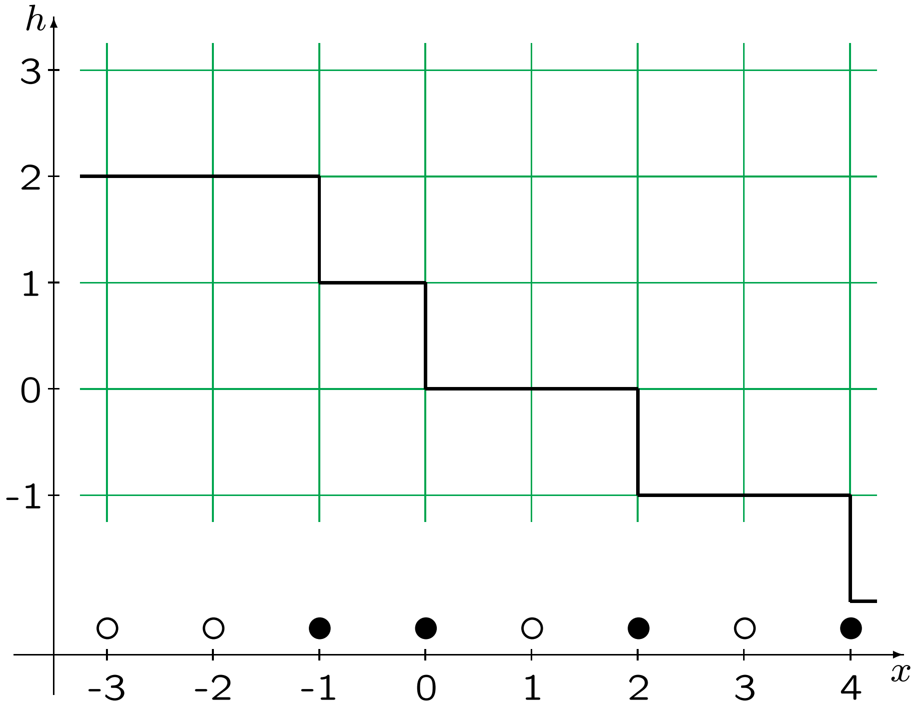

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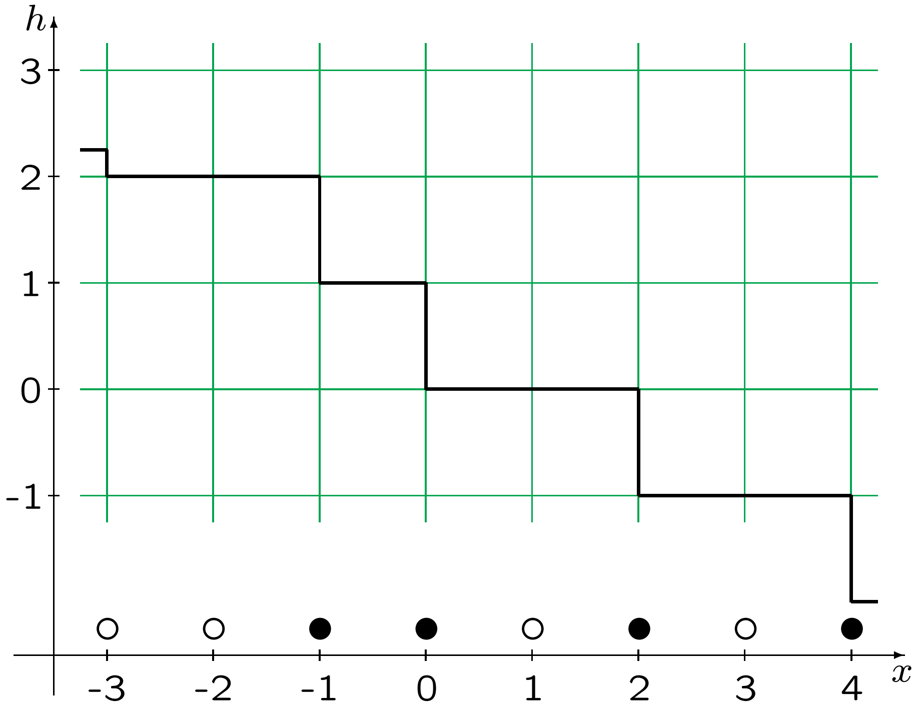

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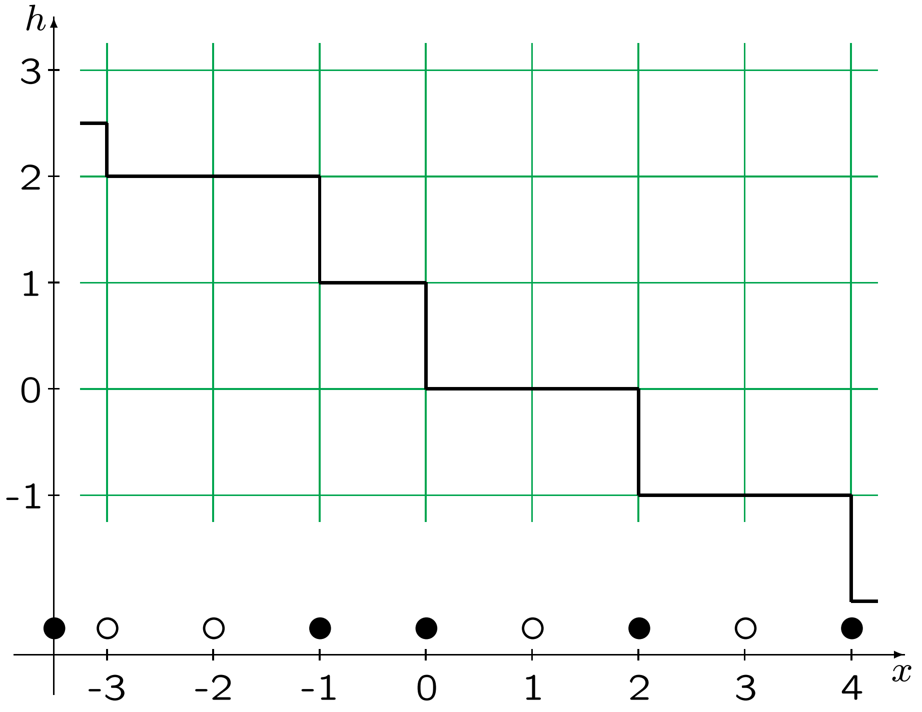

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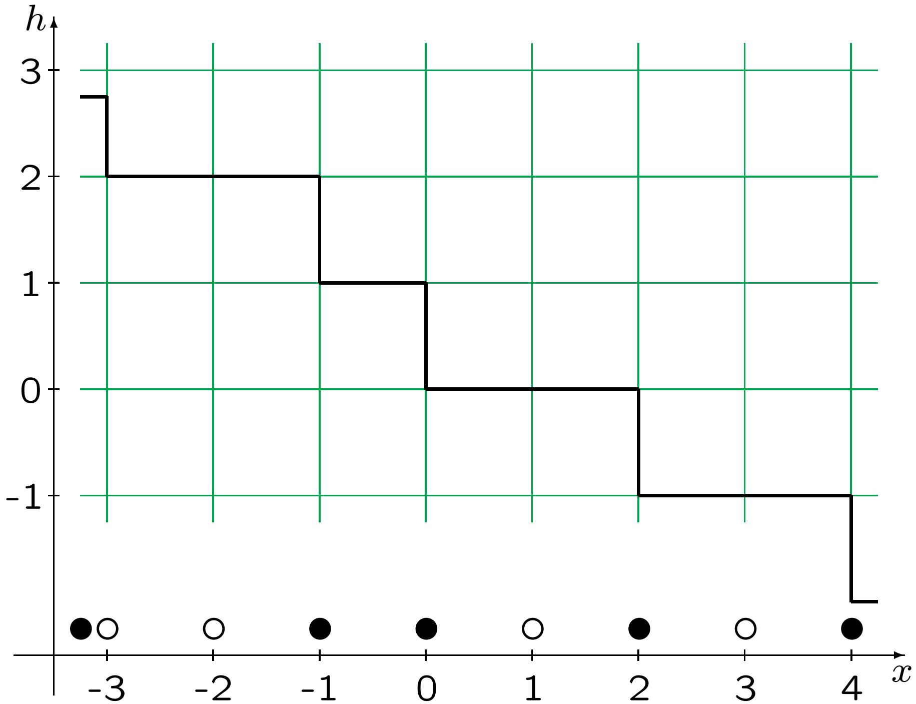

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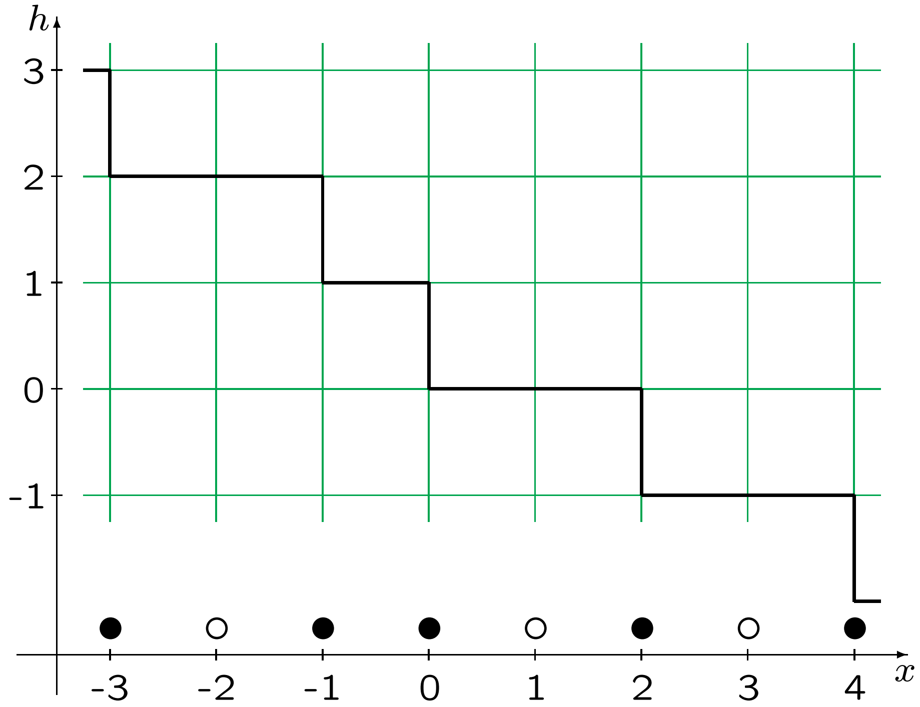

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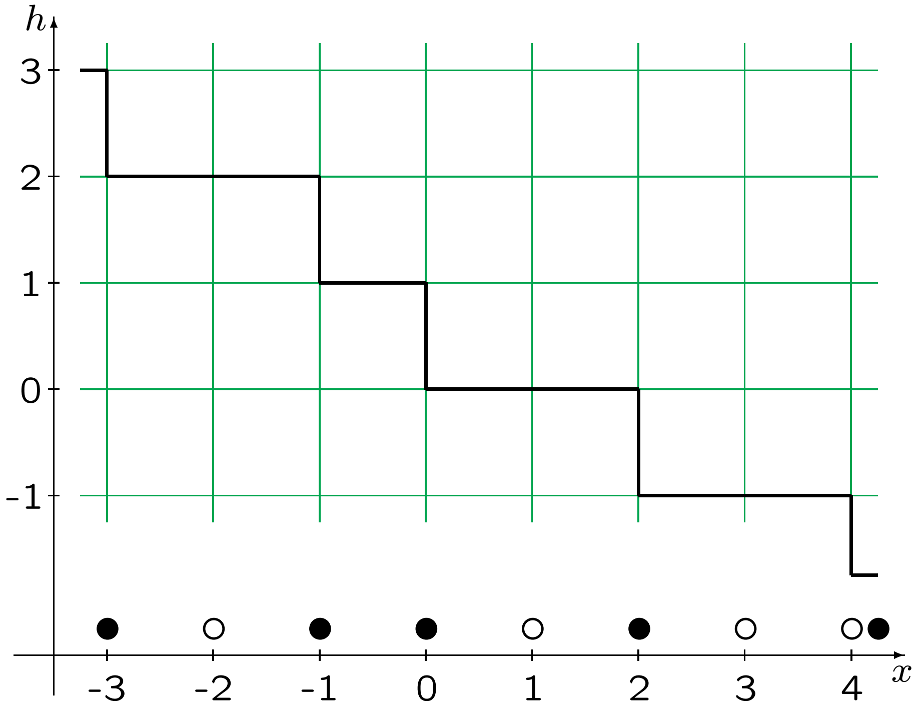

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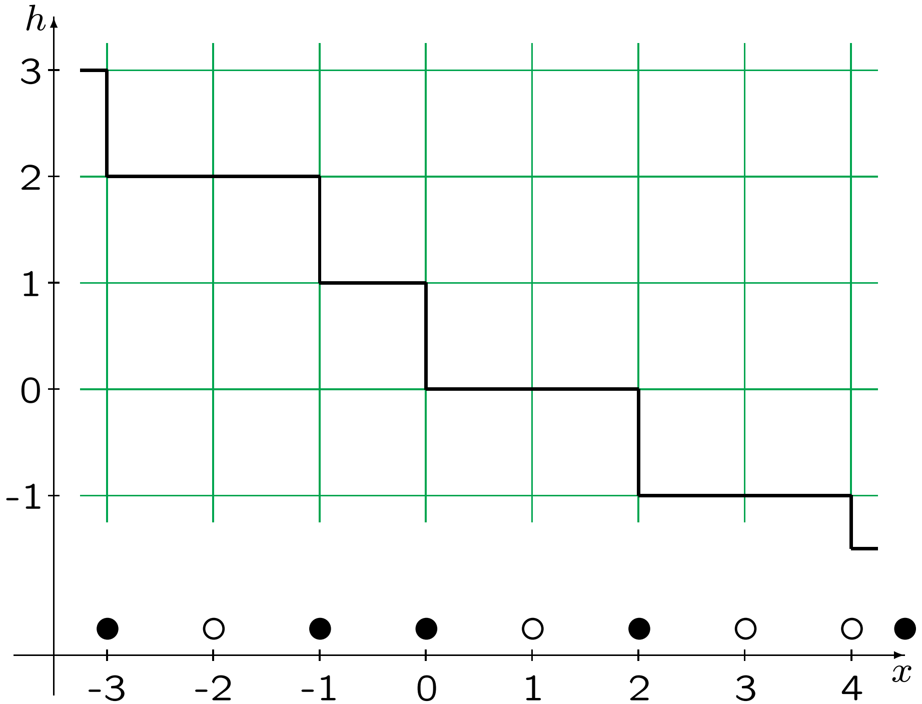

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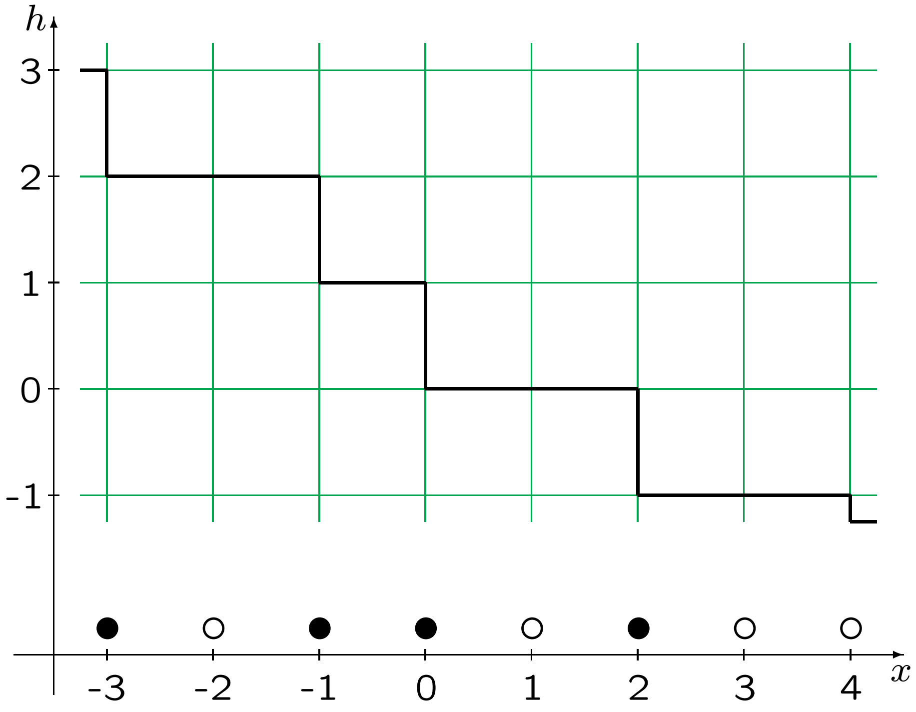

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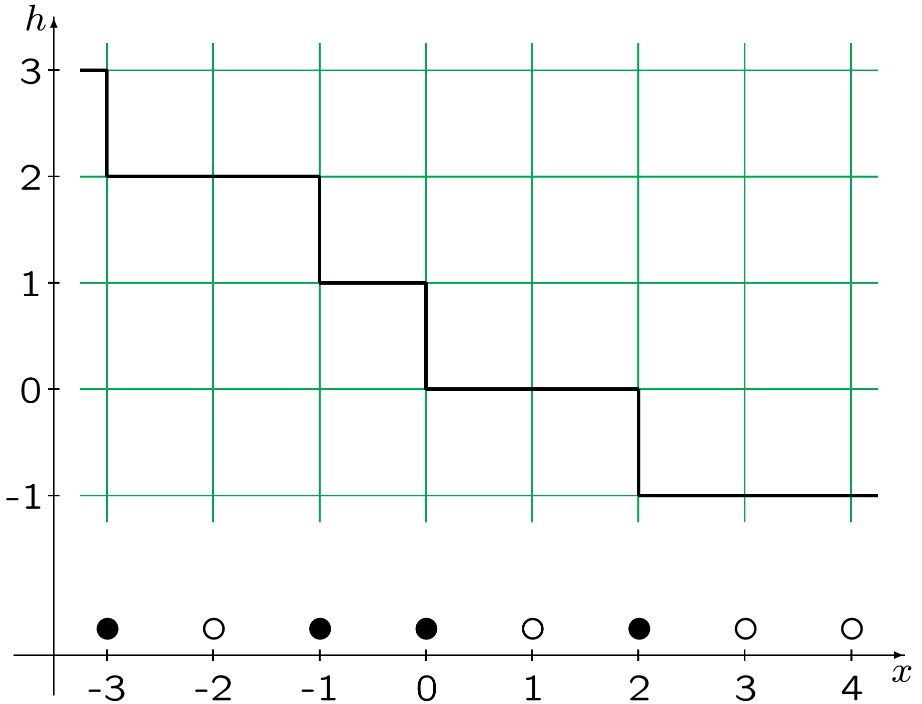

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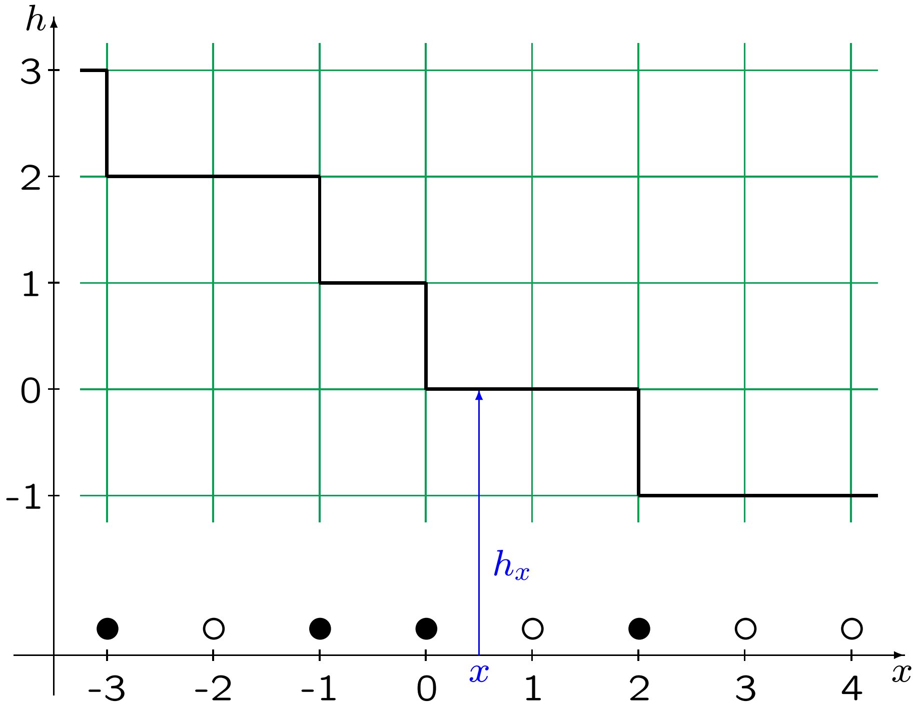

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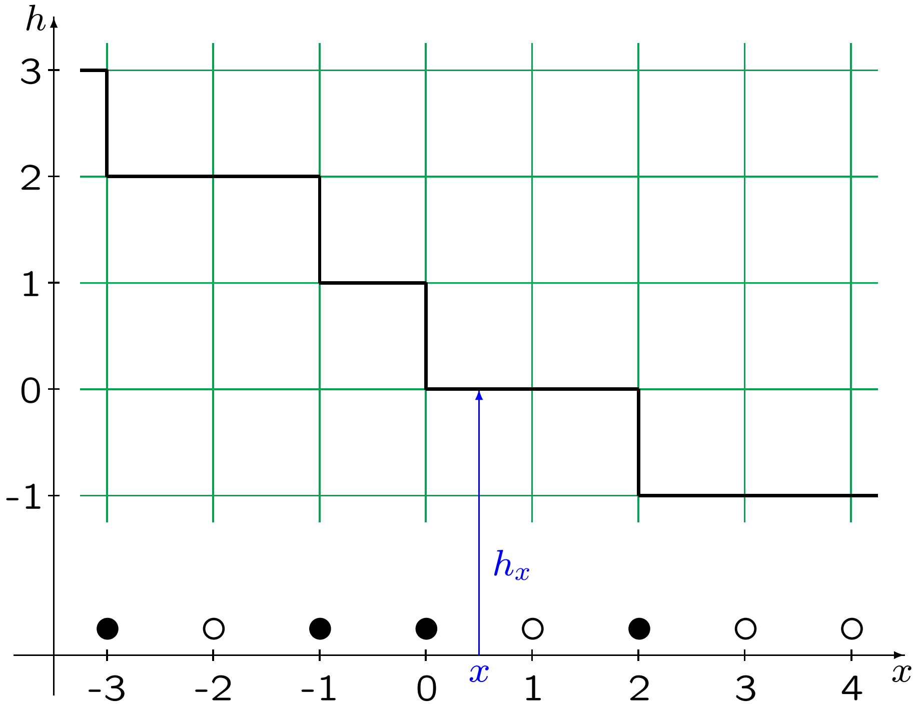

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$h_x(t) - h_x(0)$ = net number of particles passed above x.

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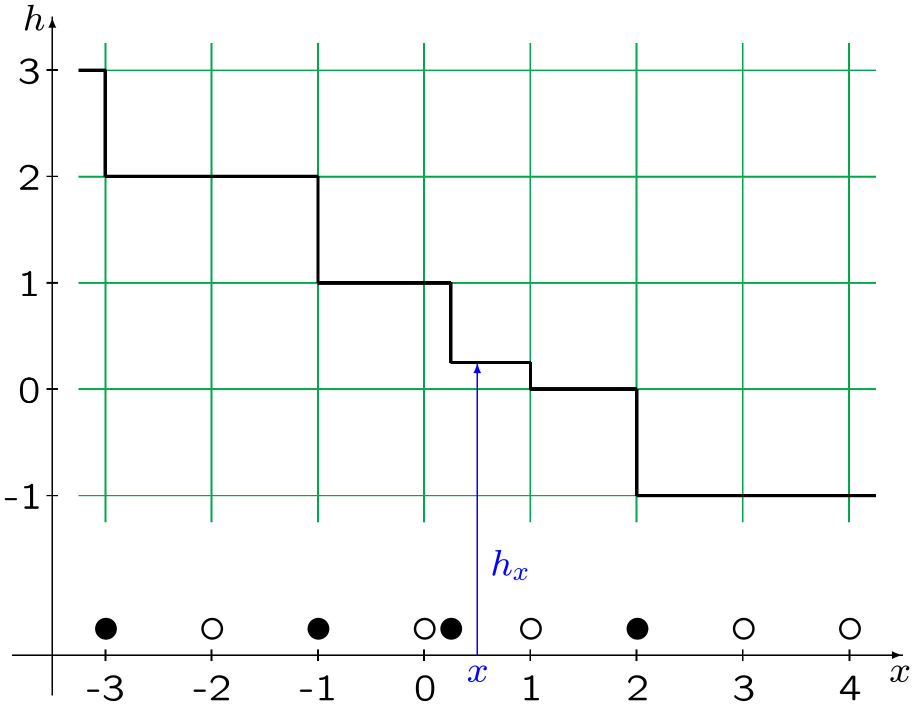

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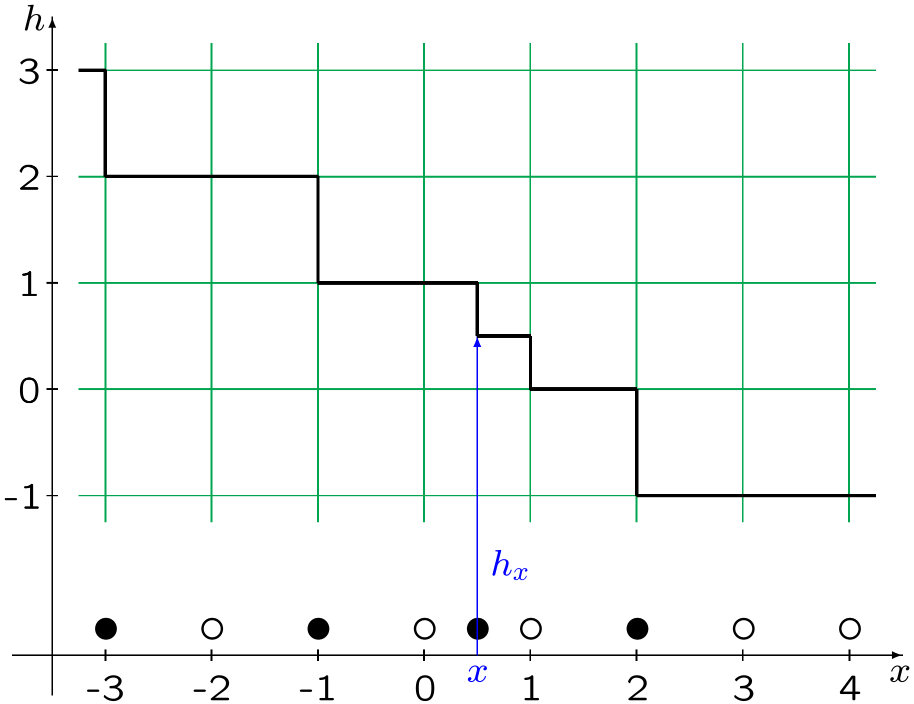
2. ASEP: Surface growth



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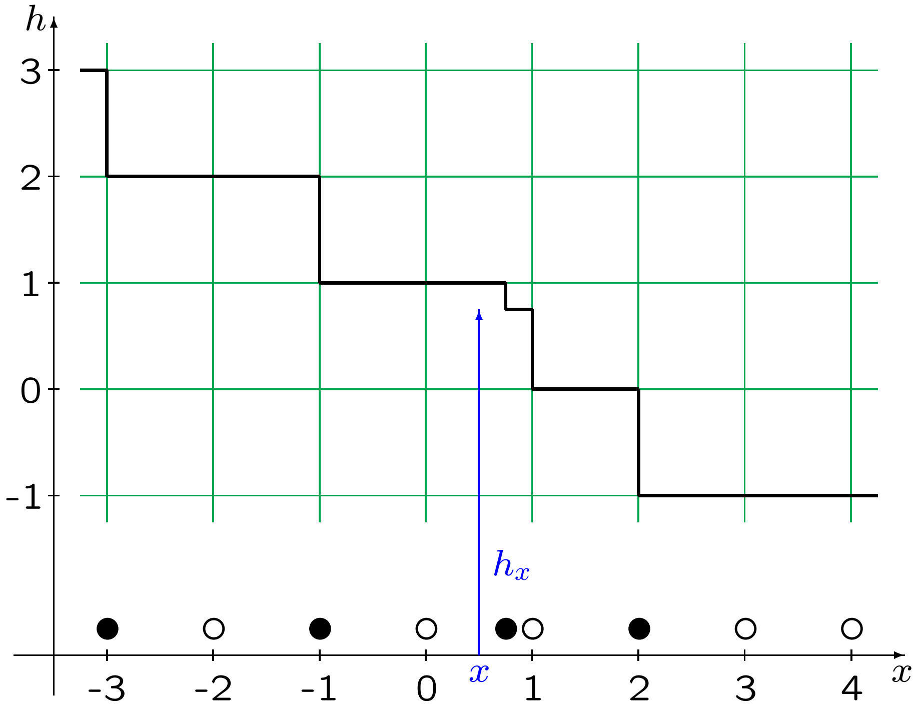

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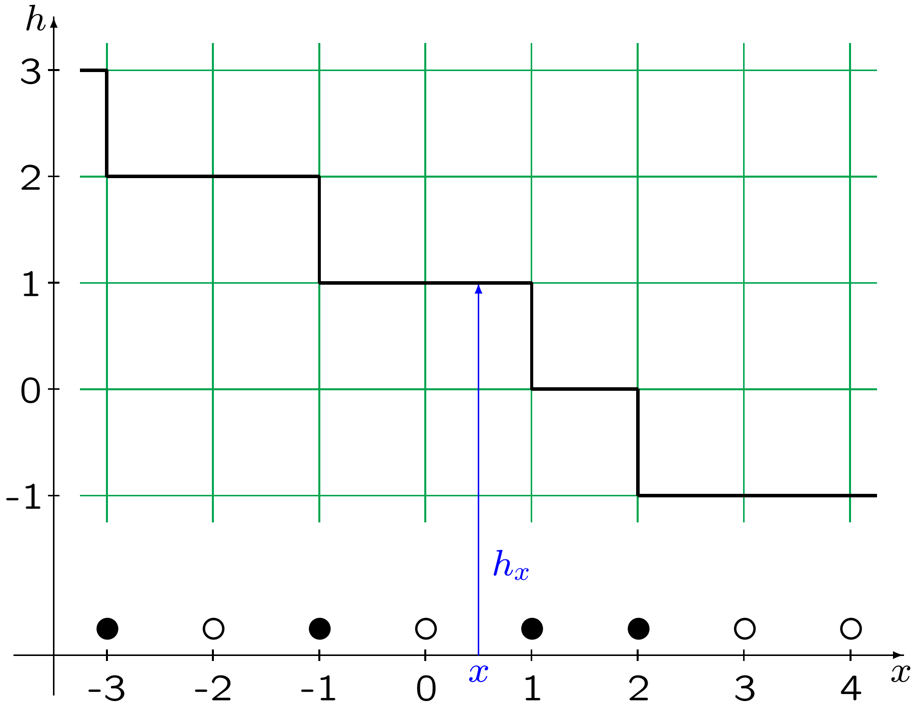

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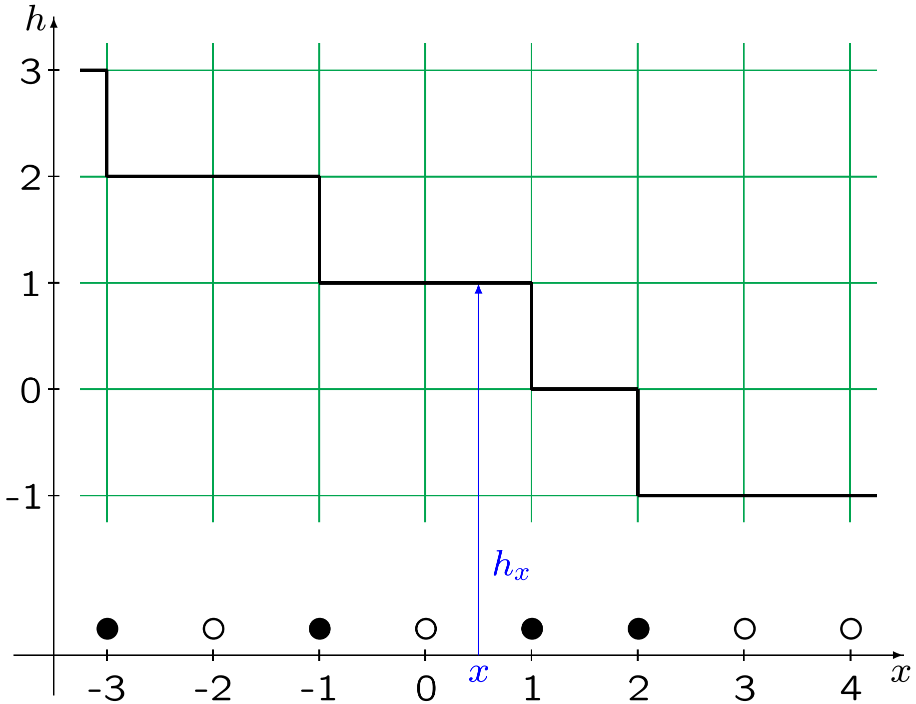

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$h_{Vt}(t)$ = net number of particles passed through the moving window at Vt $(V \in \mathbb{R})$.

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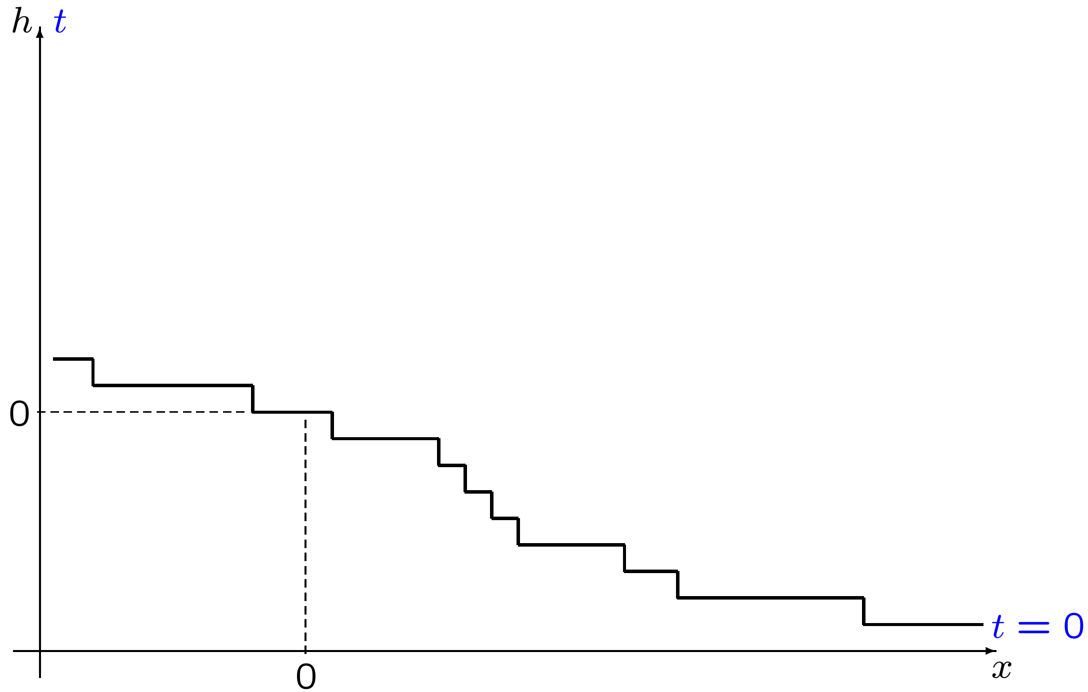

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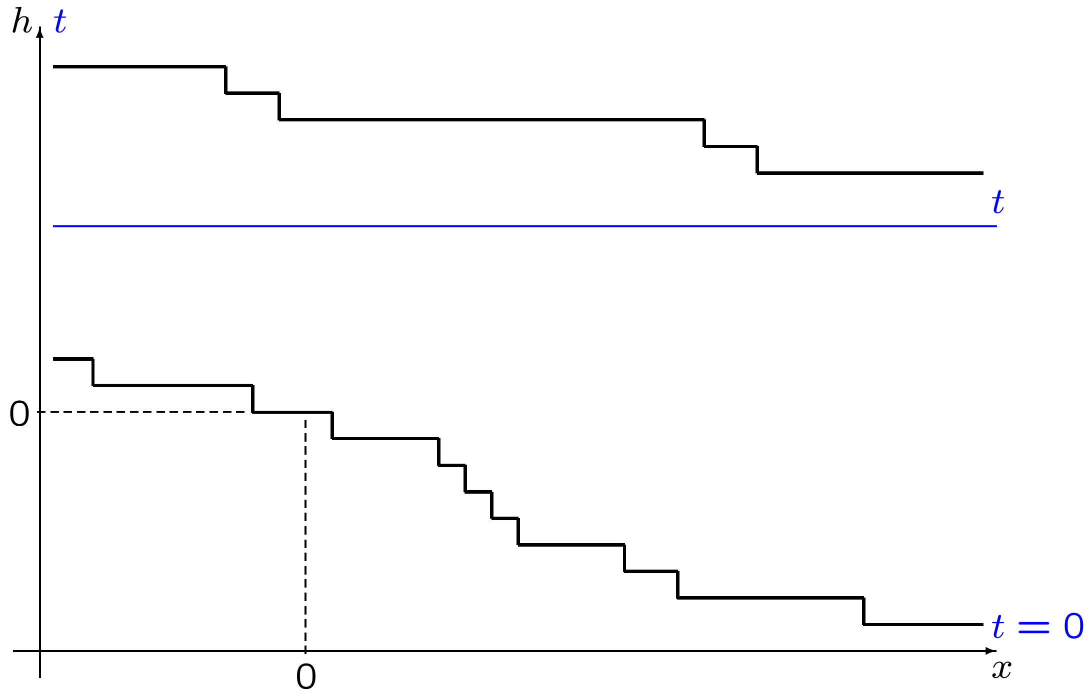

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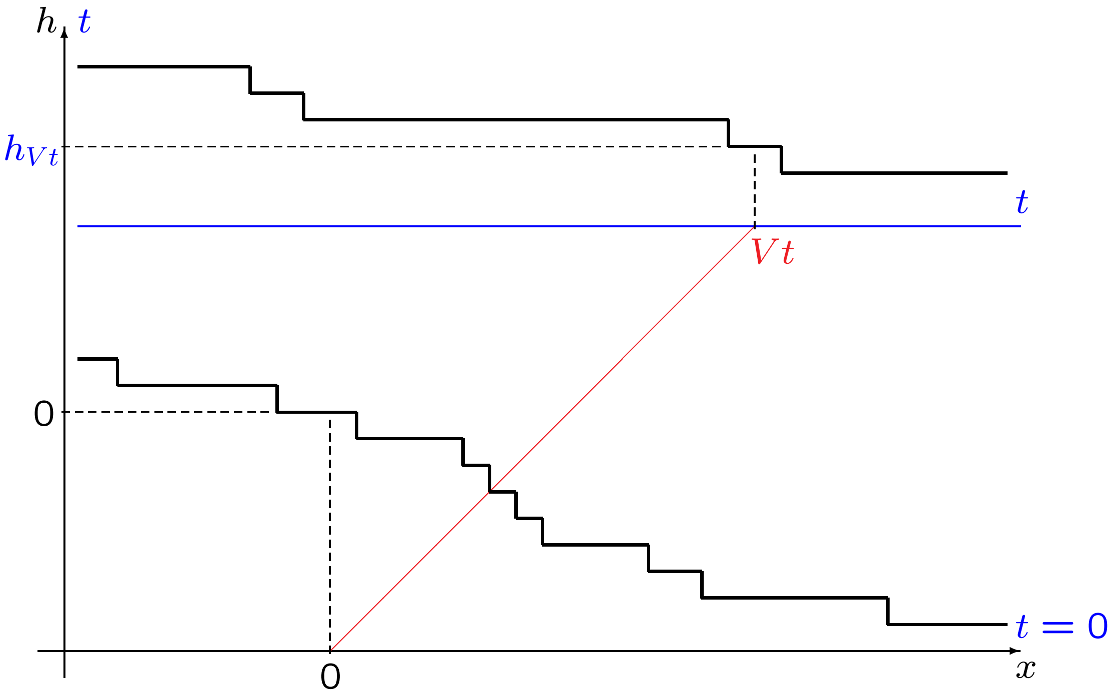

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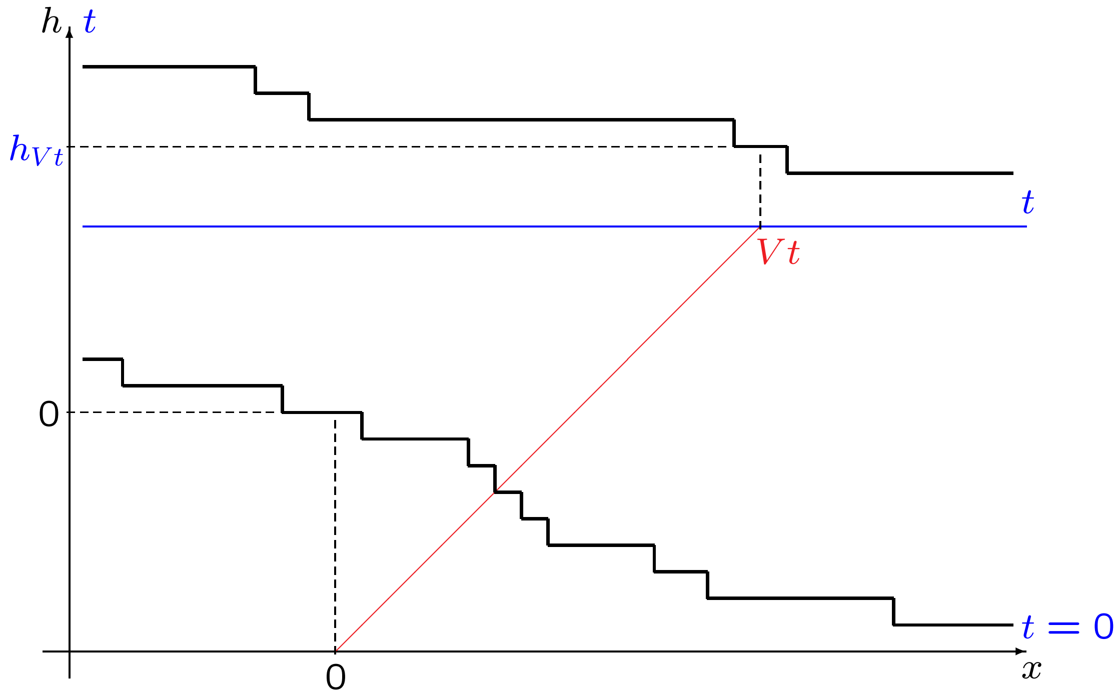

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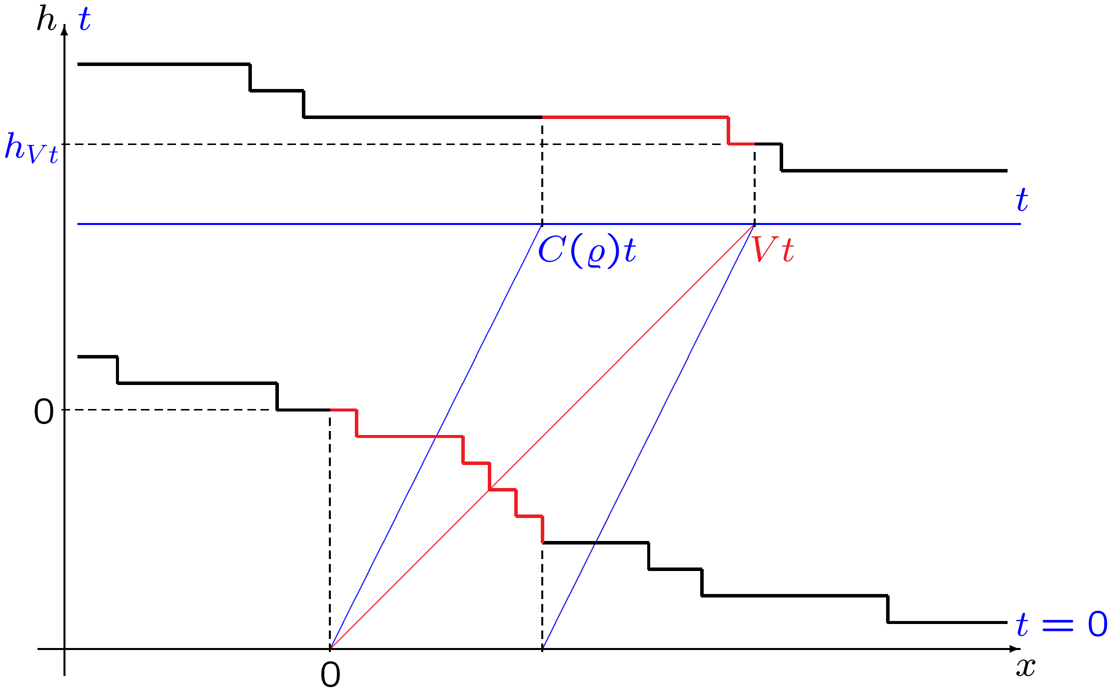

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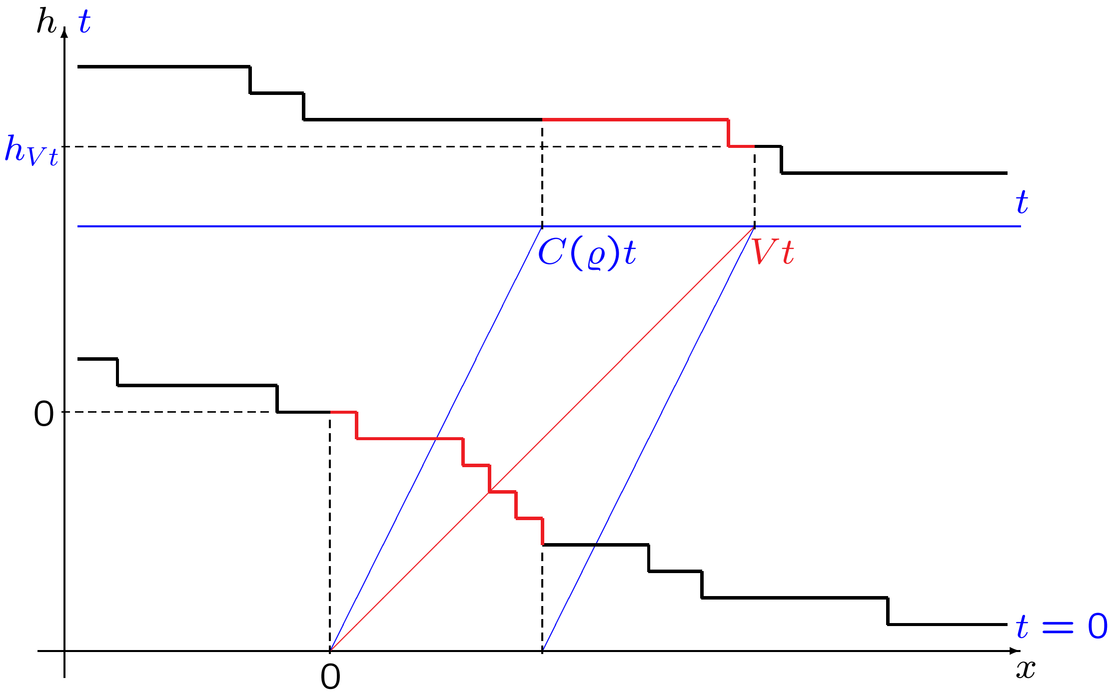

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$\rightsquigarrow$ Initial fluctuations are transported along the characteristics.

$\rightsquigarrow$ How about $V = C(\varrho)$?

3. Growth fluctuations

Conjecture:

$$\lim_{t \to \infty} \frac{\mathbf{Var}(h_{C(\varrho)t}(t))}{t^{2/3}} = [\text{sg. non trivial}].$$

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Corollary: The corresponding scaling of the diffusivity is also proved.

3. Growth fluctuations

Limit distributions (not yet controlling the second moment) in terms of the Tracy-Widom distribution were found by Baik, Deift and Johansson 1999, Johansson 2000, and Ferrari and Spohn 2006 for the *totally* asymmetric exclusion (TASEP: $p = 1,\ q = 0$).

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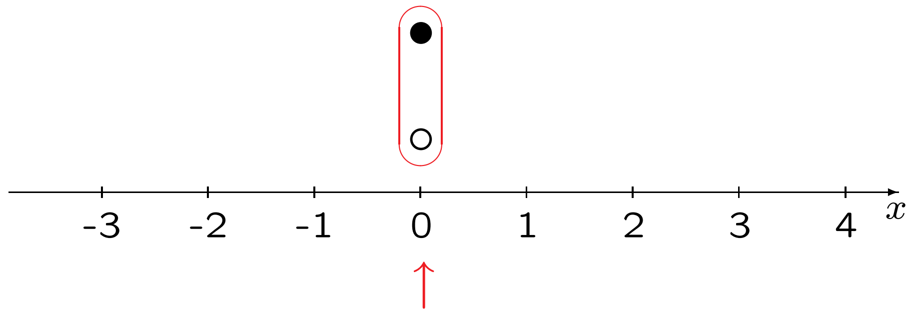
$\rightsquigarrow$ We needed to get rid of these tools. Premises: Cator and Groeneboom 2006 (Hammersley's process), B., Cator and Seppäläinen 2006 (TASEP, last passage).

4. The second class particle



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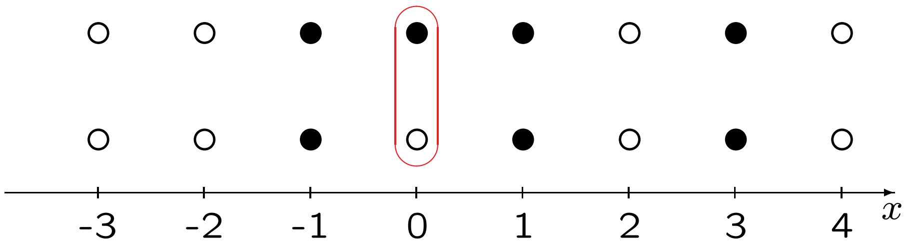

Bernoulli(ϱ) distribution except for 0

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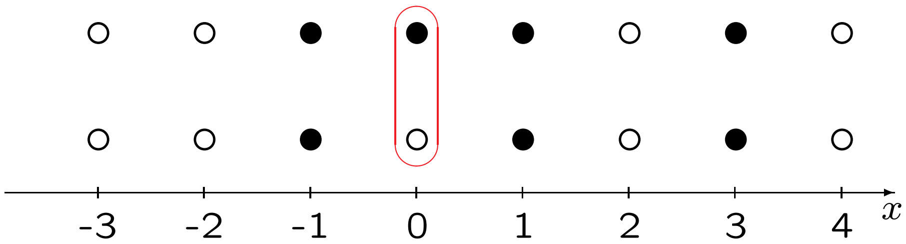

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Coupling: A single discrepancy is always conserved

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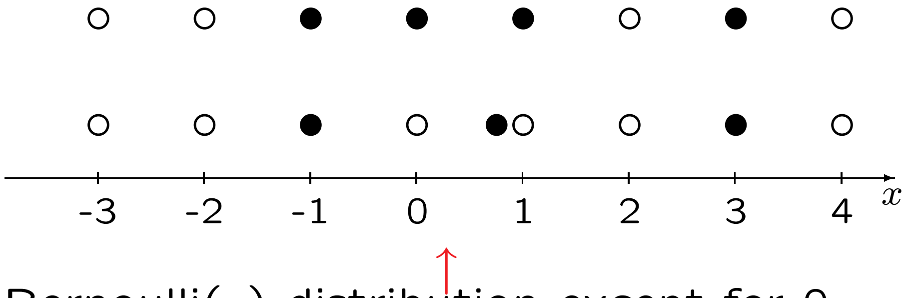

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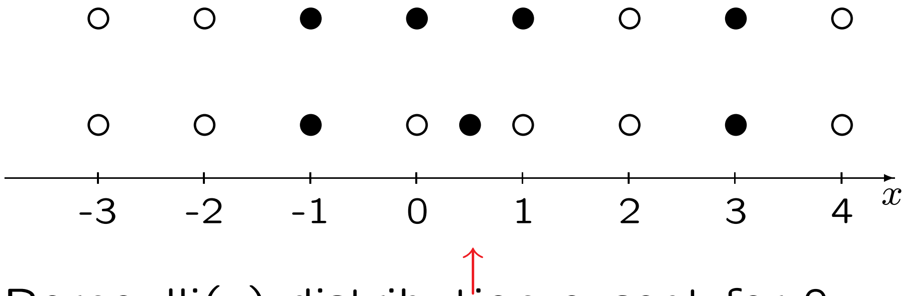

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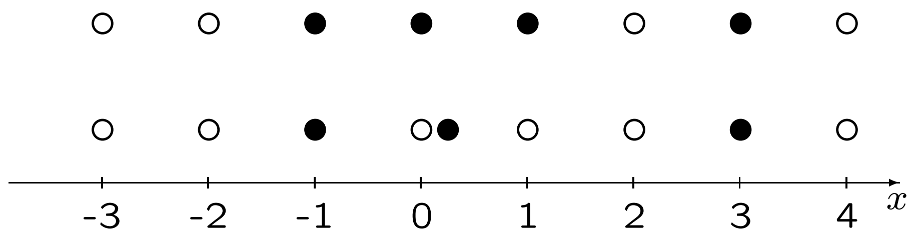

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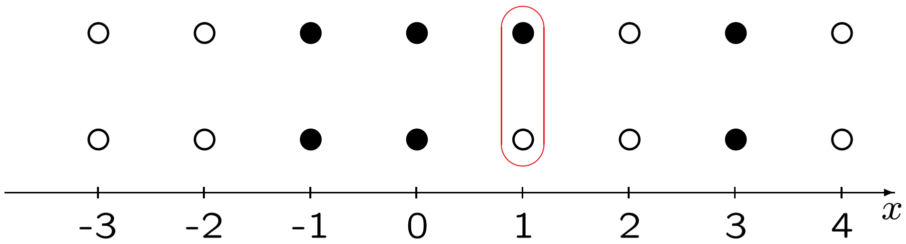

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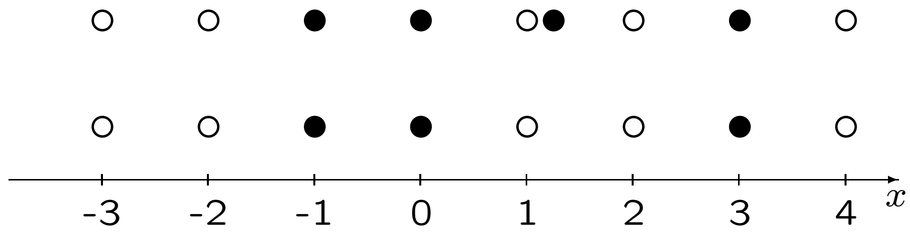

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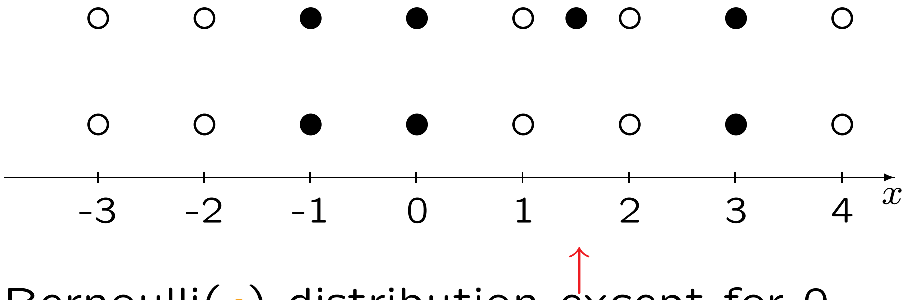

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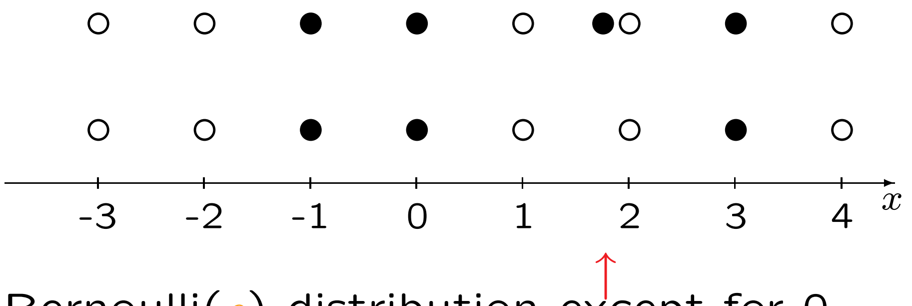

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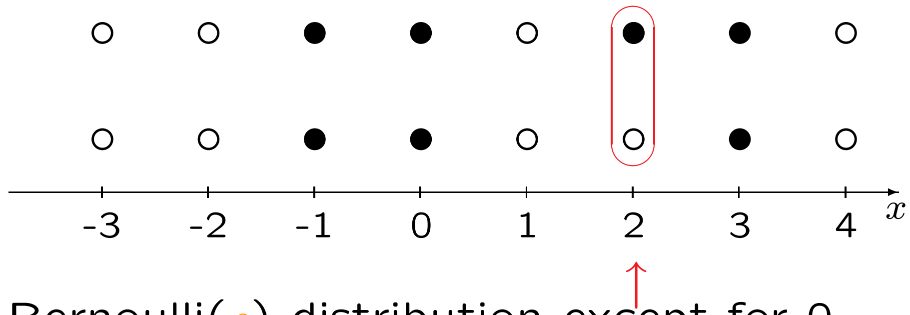

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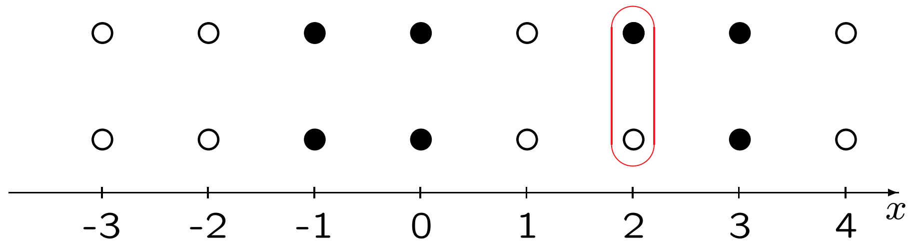

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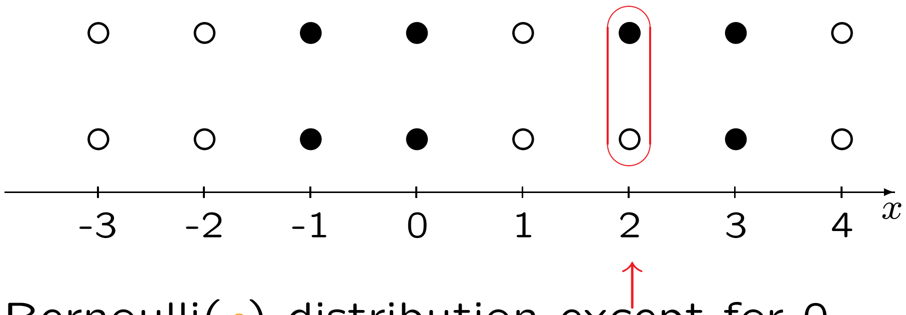

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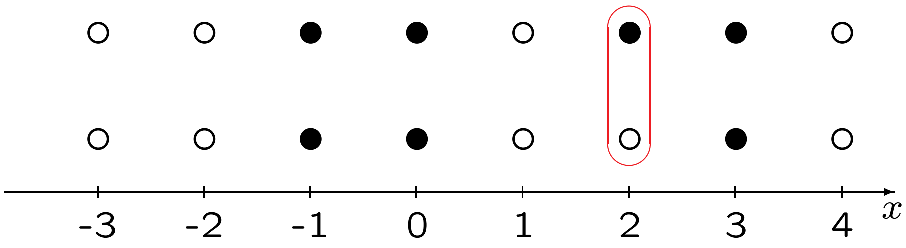

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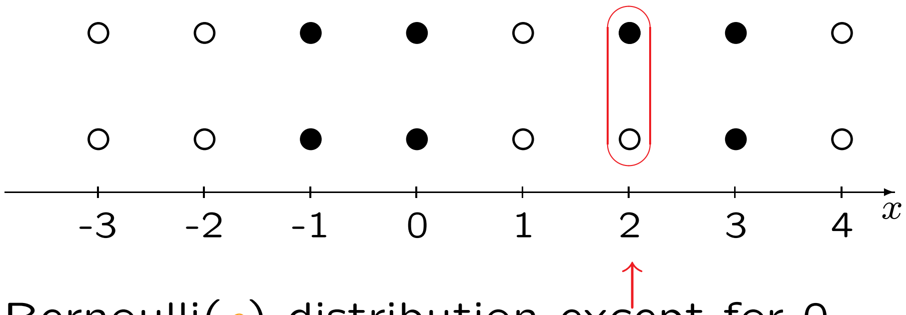

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The proof is based on ideas of Bálint, he said these ideas were standard.

$$\begin{tikzcd} \boxed{\text{Deviations of } Q(t)} \arrow[r, bend left, "\text{Couplings}"] & \boxed{\text{Deviations of } h_{C(\varrho)}(t)} \arrow[l, bend left, "\text{The above thm.}"] \end{tikzcd}$$

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Thank you.