Probability 1, Autumn 2014, Problem sheet 4

To be discussed on the week 27 Oct. . . 31 Oct.
Mandatory HW's are marked with "**HW**", they are due on the week 3 Nov. . . 7 Nov, the latest.[1]
Solutions will be available on Blackboard on the 8th Nov.

4.1 Once upon a time Odysseus met an intersection of three pathways. One of them lead to Athens, the other lead to Mycenae, and the third lead to Sparta, but he didn't know which route goes to which of these cities. He chose one of the routes by rolling a die, giving equal chance to each of these choices. He knew that, on average, Athenians only tell the truth in one case out of three, Mycenae citizens lie every second time, but people of Sparta are always honest. In the city he arrived, he asked the first man he met how much two times two was, and had four as answer. What is the probability that Odysseus finally reached Athens?

HW 4.2 Suppose that an insurance company classifies people into one of three classes: good risks, average risks, bad risks. Their records indicate that the probabilities that good, average, and bad risk persons will be involved in an accident over a 1-year span are, respectively, 0.05, 0.15, and 0.3. We also know that 20% of the population are good risks, 50% are average risks, and 30% are bad risks.

(a) What proportion of people have accidents next year?

(b) If a policyholder had no accidents last year, what is the probability that (s)he is a good, average, or bad risk?

4.3 A student knows the correct answer for a question with probability p. To test this knowledge, an m-choice test is given to the student (one option is correct, the other $m - 1$ are false). If the answer is known to the student then the correct answer is given, otherwise the student guesses and answers each option with equal chance. Given the answer is correct, what is the probability that the student actually knew the answer? What happens when $m = 1$ and when m is very large?

4.4 Drunkard Druce spends 2/3 of the day in pubs. The village has 5 pubs, and Druce is not choosy, he can be found in any of the 5 pubs with equal chance. Once we set out for finding him. We have looked for him in 4 pubs already, but we haven't found him. What is the probability that he will sit in the fifth pub?

4.5 Alice applied to the university, and a notification mail about whether she has been accepted or not is expected to arrive next week. She will be on holidays next week, and she certainly wouldn't want to receive bad news during her trip. So she instructs her mom not to call her with possible bad news at all during next week. However, missing the call Alice would then guess that she has been rejected. To avoid this she also tells her mom that in case of good news a coin should be flipped, and she only wants a call if the coin lands on head. Thus, even without the call, Alice cannot be certain that she has been rejected. Let α be the probability that she is rejected, and β the probability that she is rejected given her mom doesn't call her.

(a) Which one should be larger α or β?

(b) Calculate β in terms of α and confirm your answer for (a).

4.6 Let $S = \{1, 2, \ldots, n\}$, and suppose that A and B are, independently, equally likely to be any of the 2^n subsets (including the null set and S itself) of S.

(a) Show that $P\{A \subset B\} = \left(\frac{3}{4}\right)^n$.

(b) Show that $P\{A \cap B = \emptyset\} = \left(\frac{3}{4}\right)^n$.

HW 4.7 On a quiz show a married couple is asked a yes-or-no question. Both wife and husband know the correct answer, independently, with probability p. Should they

(a) decide in advance which of them will give the answer, or

(b) think about the question first, and flip a coin if their answers differ to decide which answer they will give?

4.8 Urn 1 contains n red balls and urn 2 contains n blue balls. In each step we pick a random ball from urn 1, throw it away, and move one blue ball from urn 2 into urn 1. We keep doing that until urn 2 empties out. What is the (average) proportion of red balls at that time in urn 1? (In other words, what is the probability that a given red ball is still in urn 1 at this time?) What happens as $n \to \infty$?

[1] Details of how to hand in are to be discussed with your tutor.

4.9 Suppose that we want to generate the outcome of the flip of a fair coin but that all we have at our disposal is a biased coin which lands on heads with some unknown probability p that need not to be equal to $1/2$. Consider the following procedure for accomplishing our task.

1) Flip the coin.
2) Flip the coin again.
3) If both flips land heads or both land tails then return to step 1).
4) Otherwise, if the two flips are different, then let the result of the last flip be the result of the experiment.

(a) Show that the result is equally likely to be either heads or tails.

(b) Could we use a simpler procedure that continues to flip the coin until the last two flips are different and then lets the result be the outcome of the final flip?

4.10 Cities A, B, C, D are located (in this order) on the four corners of a square. Between them, we have the following roads: $A \leftrightarrow B$, $B \leftrightarrow C$, $C \leftrightarrow D$, $D \leftrightarrow A$, $B \leftrightarrow D$. One night each of these roads gets blocked by the snow independently with probability $1/2$. Show that the next morning city C is accessible from city A with probability $1/2$.
Contact Márton if you have or want to see a solution of this problem without computations.

HW 4.11 0.5% of population carry a certain disease. Blood tests performed on infected samples detect the infection with probability 95% independently each time the test is applied. On non-infected samples, however, they have a false positive rate of 1%, independently each time.

(a) What is the chance that a random person will test positive for this disease?

(b) My test comes up positive. Should I be worried? Explain.

(c) So I repeat the test after my first positive test. What is the chance that the second test also comes up positive? Are results of the first and second test independent? Why or why not?

(d) My second test also came up positive, so now I have two positive results in my hands. Should I be worried this time? Explain.

4.12 Die α has four red and two white faces, while die β has two red and four white faces. We flip a fair coin. If it comes head then we use die α, if it comes tail, then we use die β. We then roll the die selected this way n times.

(a) What is the probability that the first roll comes up red?

(b) What is the probability that we used die α, given that the first roll was red?

(c) What is the probability that the second roll comes up red, given that the first roll was red?

(d) What is the probability that the k-th roll will be red, given that all previous rolls were red ($k = 1, 2, \ldots, n$)?

HW 4.13 According to the Etiquette Institute boys can be divided into two categories: $2/3$ of them are polite and $1/3$ are impolite. Polite boys will let girls enter a door first in 90% of cases while impolite buys will only do so with 20% chance, independently for each girl. I have seen that John let Julia, but not Judith, enter before him.

(a) What is the probability that John belongs to the impolite category?

(b) What is the probability that he will let Elizabeth go first?