Probability 1, Autumn 2014, Problem sheet 5

To be discussed on the week 3 Nov. . . 7 Nov.
Mandatory HW's are marked with "**HW**", they are due on the week 10 Nov. . . 14 Nov, the latest.[1]
Solutions will be available on Blackboard on the 15th Nov.

HW 5.1 A transport company wants to determine the average number of passengers on its buses. They have to decide between two methods:

(a) They entrust n randomly selected passengers to count the total number of passengers on the buses they travel with. Then the company computes the average of the collected n answers.

(b) The company asks n of its bus drivers to count the number of passengers on their buses, and computes the average of these n answers.

Which method would you recommend? Which method will give a larger result?

HW 5.2 Five men and five women are ranked according to their scores on an examination. Assume that no two scores are alike and all 10! possible rankings are equally likely. Let X denote the highest ranking achieved by a woman (for instance, X=1 if the top-ranked person is female). Find the probability mass function $\mathbf{P}\{X = i\}$, $i = 1, 2, \ldots, 10$ of X.

5.3 One of the numbers 1 through 10 is randomly chosen. You are to try to guess the number chosen by asking questions with "yes-no" answers. Compute the expected number of questions you will need to ask in each of the two cases:

(a) Your ith question is to be "Is it i?", $i = 1, 2, \ldots, 10$.

(b) With each question you try to eliminate one-half of the remaining numbers, as nearly as possible. For example, your first question is "Is the number larger than 5?". If yes, then your second question is "Is the number greater than 7?", etc.

5.4 There are two questions on a quiz show, one of them pays £A for the correct answer, the other pays £B. However, the game finishes with no further question asked if the wrong answer is given. If I know the correct answer for these questions with respective probabilities p_A and p_B, in which order should I attempt answering them as to maximising my expected payoff?

5.5 In a game a player bets on a number out of $1, 2, \ldots, 6$. Then three dice are rolled. If the number bet occurs i times, $i = 1, 2, 3$, then the player receives £i. If the number does not show up, then the player looses £1. Is this game fair?

5.6 We randomly place a knight on an empty chessboard. What is the expected number of his possible moves? (A knight placed on the square (i, j) of the chessboard can move to any of the squares $(i + 2, j + 1)$, $(i+1, j+2)$, $(i-1, j+2)$, $(i-2, j+1)$, $(i-2, j-1)$, $(i-1, j-2)$, $(i+1, j-2)$, $(i+2, j-1)$, provided these are on the chessboard.)

HW 5.7 Rolling two dice, what is the expected value of the higher and of the smaller of the two numbers shown?

5.8 **St. Petersburg paradox.** A coin is flipped until the first time it lands on heads. If this happens at the n^{th} flip then we win £2^n. Show that the expected value of our winnings is infinite.

(a) Would you pay 1 million Pounds to pay this game once?

(b) Would you pay 1 million Pounds to play this game as many times as you wish if you only have to settle after the last game?

Explain.

5.9 Every night several meteorologists predict the probability of rain next day. To rate them, we score their predictions as follows. If a meteorologist reports probability p of rain next day, then score

- $1 - (1 - p)^2$ is given if it indeed rains,
- $1 - p^2$ is given if it does not rain

that day. Suppose a meteorologist thinks probability p^* of rain next day, what value of p should she report for maximising her expected score on this rating?

[1]Details of how to hand in are to be discussed with your tutor.

HW 5.10 A man has n keys on his keyring, out of which only one opens a door. How many times is he expected to try keys if he tries them completely randomly, but excludes unsuccessful keys from his further trials?

5.11 Let N be a non-negative integer valued random variable. Show that

$$\mathbf{E}(N) = \sum_{i=1}^{\infty} \mathbf{P}(N \geq i).$$

HINT: $\sum_{i=1}^{\infty} \mathbf{P}(N \geq i) = \sum_{i=1}^{\infty} \sum_{k=i}^{\infty} \mathbf{P}(N = k)$. *Now swap summations (and mind the boundaries).*

5.12 Let N be a non-negative integer valued random variable. Show that

$$\sum_{i=1}^{\infty} i\mathbf{P}(N > i) = \frac{1}{2}\big(\mathbf{E}(N^2) - \mathbf{E}(N)\big).$$