Probability 1, Autumn 2016, Problem sheet 5

To be discussed on the week 31 Oct. . . 4 Nov.
Problems marked with "PrCl" are discussed in the large problem class on Friday 28 Oct.
Mandatory HW's are marked with "**HW**", they are due on the week 7 Nov. . . 11 Nov, the latest.[1]
Solutions will be available on Blackboard on the 12th Nov.

HW 5.1 We have seen in class the definition

$$\mathbf{P}\{B \,|\, A \cap F\} = \mathbf{P}\{B \,|\, F\}$$

for the conditional independence of events A and B, given the event F. Show that this is equivalent to

$$\mathbf{P}\{B \cap A \,|\, F\} = \mathbf{P}\{B \,|\, F\} \cdot \mathbf{P}\{A \,|\, F\} :$$

the conditional probability of A and B equals the product of the respective conditional probabilities. (Assume all conditional probabilities are well-defined.)

PrCl **5.2** Die α has four red and two white faces, while die β has two red and four white faces. We flip a fair coin. If it comes heads then we use die α, if it comes tails, then we use die β. We then roll the die selected this way n times.

(a) What is the probability that the first roll comes up red?

(b) What is the probability that we used die α, given that the first roll was red?

(c) What is the probability that the second roll comes up red, given that the first roll was red?

(d) What is the probability that the k-th roll will be red, given that all previous rolls were red ($k = 1, 2, \ldots, n$)?

HW 5.3 According to the Etiquette Institute boys can be divided into two categories: 2/3 of them are polite and 1/3 are impolite. Polite boys will let girls enter a door first in 90% of cases while impolite buys will only do so with 20% chance, independently for each girl. I have seen that John let Julia, but not Judith, enter before him.

(a) What is the probability that John belongs to the impolite category?

(b) What is the probability that he will let Elizabeth go first?

5.4 A survey aims to determine the average number of children per family in a community. They have to decide between two methods:

(a) They ask n randomly selected children how many children there are in their families, and then average out the answers.

(b) They ask n randomly selected mums how many children they have, and compute the average of these n answers.

Which method would be suitable? Which result would be larger?

5.5 Five men and five women are ranked according to their scores on an examination. Assume that no two scores are alike and all 10! possible rankings are equally likely. Let X denote the highest ranking achieved by a woman (for instance, X=1 if the top-ranked person is female). Find the probability mass function $\mathbf{P}\{X = i\}$, $i = 1, 2, \ldots, 10$ of X.

5.6 One of the numbers 1 through 10 is randomly chosen. You are to try to guess the number chosen by asking questions with "yes-no" answers. Compute the expected number of questions you will need to ask in each of the two cases:

(a) Your ith question is to be "Is it i?", $i = 1, 2, \ldots, 10$.

(b) With each question you try to eliminate one-half of the remaining numbers, as nearly as possible. For example, your first question is "Is the number larger than 5?". If yes, then your second question is "Is the number greater than 7?", etc.

5.7 There are two questions on a quiz show, one of them pays £A for the correct answer, the other pays £B. However, the game finishes with no further question asked if the wrong answer is given. If I know the correct answer for these questions with respective probabilities p_A and p_B, in which order should I attempt answering them as to maximising my expected payoff?

[1] Details of how to hand in are to be discussed with your tutor.

PrCl **5.8** In a game a player bets on a number out of $1, 2, \ldots, 6$. Then three dice are rolled. If the number bet occurs i times, $i = 1, 2, 3$, then the player receives £i. If the number does not show up, then the player looses £1. Is this game fair?

5.9 We randomly place a knight on an empty chessboard. What is the expected number of his possible moves? (A knight placed on the square (i, j) of the chessboard can move to any of the squares $(i+2, j+1)$, $(i+1, j+2)$, $(i-1, j+2)$, $(i-2, j+1)$, $(i-2, j-1)$, $(i-1, j-2)$, $(i+1, j-2)$, $(i+2, j-1)$, provided these are on the chessboard.)

HW 5.10 Rolling two four-sided (tetrahedron) dice, what is the expected value of the higher and of the smaller of the two numbers shown?

5.11 **St. Petersburg paradox.** A coin is flipped until the first time it lands on heads. If this happens at the n^{th} flip then we win £2^n. Show that the expected value of our winnings is infinite.

(a) Would you pay 1 million Pounds to pay this game once?

(b) Would you pay 1 million Pounds to play this game as many times as you wish if you only have to settle after the last game?

Explain.

5.12 Every night several meteorologists predict the probability of rain next day. To rate them, we score their predictions as follows. If a meteorologist reports probability p of rain next day, then score

- $1 - (1-p)^2$ is given if it indeed rains,
- $1 - p^2$ is given if it does not rain

that day. Suppose a meteorologist thinks probability p^* of rain next day, what value of p should she report for maximising her expected score on this rating?

HW 5.13 A man has n keys on his keyring, out of which only one opens a door. How many times is he expected to try keys if he tries them completely randomly, but excludes unsuccessful keys from his further trials?

5.14 Let N be a non-negative integer valued random variable. Show that

$$\mathbf{E}(N) = \sum_{i=1}^{\infty} \mathbf{P}(N \geq i).$$

HINT: $\sum_{i=1}^{\infty} \mathbf{P}(N \geq i) = \sum_{i=1}^{\infty} \sum_{k=i}^{\infty} \mathbf{P}(N = k)$. *Now swap summations (and mind the boundaries).*

5.15 Let N be a non-negative integer valued random variable. Show that

$$\sum_{i=1}^{\infty} i\mathbf{P}(N > i) = \frac{1}{2}\big(\mathbf{E}(N^2) - \mathbf{E}(N)\big).$$