Probability 1, Autumn 2014, Problem sheet 7

To be discussed on the week 17 Nov…21 Nov.
Mandatory HW's are marked with "**HW**", they are due on the week 24 Nov…28 Nov, the latest.[1]
Solutions will be available on Blackboard on the 29th Nov.

7.1 Consider a roulette wheel consisting of 38 numbers – 1 through 36, 0, and double 0. If Smith always bets that the outcome will be one of the numbers 1 through 12, what is the probability that

(a) Smith will loose his first 5 bets;

(b) his first win will occur on his fourth bet?

7.2 A man has n keys on his keyring, out of which only one opens a door. How many times is he expected to try keys if he tries them completely randomly, without excluding unsuccessful keys from his further trials?

7.3 Married couples in a community only have children until the first boy is born (we neglect cases of twins). What is the ratio of sexes in this community? How is it about independence of sexes of children?

HW 7.4 The suicide rate in a certain country is 1 suicide per 200 000 inhabitants per month.

(a) Find the probability that in a city of 600 000 inhabitants within this country, there will be 6 or more suicides in a given month.

(b) What is the probability that there will be at least 2 months during the year that will have 6 or more suicides?

(c) Counting the present month as month number 1, what is the probability that the first month to have 6 or more suicides will be month number i, $i \geq 1$?

7.5 Which of the following functions can be a distribution function?

(a) $F(x) = \begin{cases} 1 + \mathrm{e}^{1-x} & , \text{ if } x > -1, \\ 0 & , \text{ otherwise} \end{cases}$

(b) $F(x) = \begin{cases} 2 - \dfrac{2}{x+1} & , \text{ if } x \geq 0, \\ 0 & , \text{ otherwise} \end{cases}$

(c) $F(x) = \begin{cases} 1 - \mathrm{e}^{-x} & , \text{ if } x \geq 0, \\ 0 & , \text{ otherwise} \end{cases}$

(d) $F(x) = \begin{cases} 0 & , \text{ if } x \leq 0, \\ \dfrac{x}{4} \cdot (4 - x) & , \text{ if } 0 < x \leq 2, \\ 1 & , \text{ if } x > 2 \end{cases}$

7.6 Which of the following functions can be a probability density function?

(a) $f(x) = \begin{cases} \dfrac{2}{x} & , \text{ if } x > 1, \\ 0 & , \text{ otherwise} \end{cases}$

(b) $f(x) = \begin{cases} \dfrac{\sin(x)}{2} & , \text{ if } 0 < x < 2, \\ 0 & , \text{ otherwise} \end{cases}$

(c) $f(x) = \begin{cases} 3^{x-1} \ln(3) & , \text{ if } x \leq 0, \\ \dfrac{1}{3} \sin\Big(\dfrac{x}{2}\Big) & , \text{ if } 0 < x < \pi, \\ 0 & , \text{ otherwise} \end{cases}$

(d) $f(x) = \begin{cases} 2\mathrm{e}^{-2x} & , \text{ if } x > 0, \\ 0 & , \text{ otherwise} \end{cases}$

[1] Details of how to hand in are to be discussed with your tutor.

HW 7.7 Let X have distribution function

$$F(b) = \begin{cases} 0 & \text{if } b < 0 \\ \dfrac{b}{4} & \text{if } 0 \le b < 1 \\ \dfrac{1}{2} + \dfrac{b-1}{4} & \text{if } 1 \le b < 2 \\ \dfrac{11}{12} & \text{if } 2 \le b < 3 \\ 1 & \text{if } 3 \le b \end{cases}$$

(a) Compute $\mathbf{P}\{X = i\}$ for $i = 1, 2, 3$.

(b) Compute $\mathbf{P}\{\frac{1}{2} < X < \frac{3}{2}\}$.

7.8 A filling station is supplied with gasoline once a week. If its weekly volume of sales in thousands of liters is a random variable with probability density function

$$f(x) = \begin{cases} 5(1-x)^4 & \text{, if } 0 < x < 1, \\ 0 & \text{, otherwise,} \end{cases}$$

what needs the capacity of the tank be so that the probability of the supply's being exhausted in a given week is 0.01?

7.9 For which values of α and c will the function $F(x) = \exp(-ce^{-\alpha x})$ be a distribution function? For such values, what is the corresponding density?

7.10 Compute the expectation and variance of a random variable X with density

$$f(x) = \begin{cases} 2x & \text{, if } 0 < x < 1, \\ 0 & \text{, otherwise.} \end{cases}$$

HW 7.11 Calculate $\mathbf{E}(X)$ for X with density function given by

(a) $f(x) = \begin{cases} -\dfrac{1}{9}xe^{x/3} & \text{, if } x < 0, \\ 0 & \text{, otherwise;} \end{cases}$

(b) $f(x) = \begin{cases} c(1-x^4) & \text{, if } -1 < x < 1, \\ 0 & \text{, otherwise;} \end{cases}$

(c) $f(x) = \begin{cases} \dfrac{3}{x^2} & \text{, if } x > 3, \\ 0 & \text{, otherwise.} \end{cases}$

7.12 Let $X \ge 0$ be a continuous random variable. Prove $\mathbf{E}X = \int_0^\infty \mathbf{P}\{X > t\}\,\mathrm{d}t$.

7.13 Is there a continuous function $g : [0, 1] \to [0, \infty)$ that satisfies $\int\limits_0^1 g(x)\,\mathrm{d}x = 1$, $\int\limits_0^1 x \cdot g(x)\,\mathrm{d}x = a$, $\int\limits_0^1 x^2 \cdot g(x)\,\mathrm{d}x = a^2$? Explain.

7.14 You arrive at a bus stop at 10 o'clock, knowing that the bus will arrive at some time uniformly distributed between 10 and 10:30.

(a) What is the probability that you will have to wait longer than 10 minutes?

(b) If at 10:15 the bus has not yet arrived, what is the probability that you will have to wait at least an additional 10 minutes?

7.15 A bus travels between the two cities A and B, which are 100 km apart. If the bus has a breakdown, the distance from the breakdown to city A has a uniform distribution over (0, 100). There is a bus service station in city A, in B, and in the center of the route between A and B. It is suggested that it would be more efficient to have the three stations located 25, 50, and 75 km, respectively, from A Do you agree? Why? What would be the optimal location of the three stations?

HW 7.16 My friend from IT wants to generate a discrete uniform random variable on the set $\{1, 2, \ldots, n\}$ (that is, one taking on each of the numbers $1, 2, \ldots, n$ with equal chance). To this order he first takes a $U \sim \text{Uniform}(0, 1)$ number as these are generally available from random generators. Then he does

$$X = \lfloor n \cdot U + 1 \rfloor$$

where $\lfloor \cdot \rfloor$ denotes the lower integer part. Is X the random variable he wants? Explain.

7.17 The *base* $\frac{1}{2}$ *expansion* of a number $0 \le x \le 1$ is a 0-1 sequence that tells us if $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots$ are needed or not in a sum that totals to the number x. Examples: $\frac{1}{4} = (01)_{\frac{1}{2}}$, $\frac{11}{16} = (1011)_{\frac{1}{2}}$. Show that an infinite sequence of 0's and 1's that are generated by independent fair coinflips is the base $\frac{1}{2}$ expansion of a Uniform(0, 1) random number. *Remark: The problem is very different and much more involved when the coin is biased. . .*