Probability 1, Autumn 2016, Problem sheet 8

To be discussed on the week 21 Nov. . . 25 Nov.
Problems marked with "PrCl" are discussed in the large problem class on Friday 18 Nov.
Assessed homework 2 will be available on Thu, 24 Nov.
Thus no mandatory HW on this sheet, nevertheless it is an important sheet.
Solutions will be available on Blackboard on the 3rd Dec.

8.1 2% of electric components of a given type break down within 1000 hours of operation. Assume that time before breakdown has an exponential distribution. What is the probability that such a component will work for longer than the average?

8.2 For a memoryless light bulb, the probability that it operates for more than 2000 hours is 2/3. In a city 20 of these light bulbs are installed.

(a) What is the probability that after 1000 hours exactly 15 bulbs are operational?

(b) What is the probability that after 1000 hours exactly 15 bulbs are operational and after an additional 500 hours exactly 13 are operational?

8.3 The lifetime of the *probabilium* radioactive particle is exponentially distributed with mean value 3 years. What is its half-life (the amount of time required for half of the number of atoms of a sample to decay)?

8.4 If X is a normal random variable with parameters $\mu = 10$ and $\sigma^2 = 36$, compute

(a) $\mathbf{P}\{X > 5\}$;

(b) $\mathbf{P}\{4 < X < 16\}$;

(c) $\mathbf{P}\{X < 8\}$;

(d) $\mathbf{P}\{X < 20\}$;

(e) $\mathbf{P}\{X > 16\}$.

8.5 Suppose that X is a normal random variable with mean (that is, expected value) 5. If $\mathbf{P}\{X > 9\} = 0.2$, approximately what is $\mathbf{Var}(X)$?

8.6 The length of the Probability 1 lecture is well approximated by a normal random variable of mean $\mu = 52$ minutes and standard deviation $\sigma = 2$ minutes.

(a) What percentage of classes are over 55 minutes long?

(b) It's 55 past and the lecturer is still speaking. What is the probability that he will finish before 57 past?

PrCl **8.7** What is the probability that the number of outcomes ⚅ is between 970 and 1050 if we roll a die 6 000 times?

8.8 How many times should a coin be tossed so as to having the number of heads between 47% and 53% of the number of all tosses with probability at least 0.95?

8.9 Given are two very similar insurance companies each having 10 000 customers. At the beginning of the year each customer pays their insurance company a fee of £200, and during the year each customer independently puts in a claim for £800 of damages with probability $\frac{1}{4}$. Both companies have a capital of £40 000 from the previous year. An insurance company goes bankrupt if it cannot pay for these claims. Should these two companies unite? Let p_1 be the probability that at least one of the two companies goes bankrupt, and p_2 the probability that the united company goes bankrupt. Find the numerical values of p_1 and p_2 and conclude whether joining the two companies is a good idea.

8.10 A stick of length ℓ is broken randomly. What is the distribution function of the length of the shorter piece?

8.11 Let X be uniformly distributed on the interval $[-3, 4]$, and let $g(x) = |x - 1| + |x + 1|$. Determine the distribution function $F_Y(y)$ of the random variable $Y = g(X)$. Is this variable absolutely continuous? Is it discrete?

8.12 An explosion throws debris at a constant velocity v but in a uniform random direction α on $(0, \frac{\pi}{2})$. Such a piece of debris then lands at distance $R = \frac{v^2}{g} \cdot \sin(2\alpha)$ from the explosion site (g is the gravitational acceleration). Find the density of debris as a function of the distance from the explosion.

PrCl **8.13** Let $X \sim \mathrm{Exp}(\lambda)$, and $c > 0$. Show that $cX \sim \mathrm{Exp}(\frac{\lambda}{c})$.

8.14 Variables Z and $2Z$ have the same distribution, What is it?

8.15 We roll two fair dice, and let

(a) X be the larger of the two numbers and Y be the sum of the two numbers;

(b) X be the first number and Y the larger of the two numbers;

(c) X be the smaller and Y the larger of the two numbers.

In each of these cases determine the joint probability mass function of X and Y.

8.16 We distribute n points uniformly and independently on the circumference of a circle, and want to compute the probability that there is a semicircle in which they each fall. (In other words, the probability that there is a line through the center of the circle such that all n points lie on the same side of this line.) Let E be the event that such a semicircle exists. Denote by $P_1, P_2, \ldots, P_n$ the random points, and by E_i the event that each point is contained in the semicircle that starts from P_i in the counterclockwise direction ($i = 1, 2, \ldots, n$).

(a) Write E in terms of E_i.

(b) Are the E_i's mutually exclusive? (Or "almost mutually exclusive"?)

(c) Calculate $P\{E\}$.

(d) Now answer this question: if we drop n points uniformly and independently on a disk, what is the probability that the center of the disk is contained in the set formed by convex combinations of the n points? (That is, inside the convex polygon of the random points as vertices.) *Mind the previous version with the circumference of the circle.*