Probability 1, Autumn 2014, Problem sheet 9

To be discussed on the week 1 Dec. . . 5 Dec.
Mandatory HW's are marked with "**HW**", they are due on the week 8 Dec. . . 12 Dec, the latest.[1]
Solutions will be available on Blackboard on the 13th Dec.

HW 9.1 We roll two fair dice, and let

(a) X be the larger of the two numbers and Y be the sum of the two numbers;

(b) X be the first number and Y the larger of the two numbers;

(c) X be the smaller and Y the larger of the two numbers.

In each of these cases determine the joint probability mass function of X and Y.

9.2 We distribute n points uniformly and independently on the circumference of a circle, and want to compute the probability that there is a semicircle in which they each fall. (In other words, the probability that there is a line through the center of the circle such that all n points lie on the same side of this line.) Let E be the event that such a semicircle exists. Denote by $P_1, P_2, \ldots, P_n$ the random points, and by E_i the event that each point is contained in the semicircle that starts from P_i in the counterclockwise direction ($i = 1, 2, \ldots, n$).

(a) Write E in terms of E_i.

(b) Are the E_i's mutually exclusive? (Or "almost mutually exclusive"?)

(c) Calculate $P\{E\}$.

(d) Now answer this question: if we drop n points uniformly and independently on a disk, what is the probability that the center of the disk is contained in the set formed by convex combinations of the n points? (That is, inside the convex polygon of the random points as vertices.) *Mind the previous version with the circumference of the circle.*

9.3 Alice and Bob roll their own fair die in every second. What is the chance that they see their first ⚅'s at the same time?

9.4 Suppose that in the post office we have a Poisson(40) number of customers in the first hour. Each customer independently of everything is a female with probability 55% and male with probability 45%. If 25 female customers visited the post office in the first hour, find the probability that meanwhile there were 20 male customers. *Recall the example from lectures.*

9.5 Let $X \sim \text{Poi}(\lambda)$ and $Y \sim \text{Poi}(\mu)$ be independent. Identify the distribution of $(X \mid X+Y)$. That is, find and recognise the conditional mass function $\mathfrak{p}_{X\mid X+Y}(i \mid n) = \mathbf{P}\{X = i \mid X + Y = n\}$. Is the answer surprising?

HW 9.6 Let X and Y be i.i.d. Geom(p) random variables.

(a) Can you guess the value of $\mathbf{P}\{X = i \mid X + Y = n\}$? If the second Head of a sequence of independent (biased) coinflips comes at the n^{th} time, what are the probabilities that the first Head comes at the i^{th} flip, $i = 1, 2, \ldots, n-1$?

(b) Verify your intuition by actually calculating $\mathbf{P}\{X = i \mid X + Y = n\}$. *Recall that on lectures we have seen the distribution of* $X + Y$.

9.7 Rolling two dice, let X be the smaller and Y the larger number shown. Determine the conditional mass function $\mathfrak{p}_{X\mid Y}(i \mid j)$ for all relevant i, j values.

9.8 It is of course clear from the probabilistic definition, but now prove by computation that the sum of n i.i.d. Bernoulli(p) variables is of Binomial(n, p) distribution. *Use induction on n.*

HW 9.9 Find the probability mass function of the sum of $X \sim \text{Bernoulli}(p_1)$ and an independent $Y \sim \text{Bernoulli}(p_2)$ variable.

9.10 Let $p_1 \neq p_2$, $X \sim \text{Geom}(p_1)$ and $Y \sim \text{Geom}(p_2)$ be independent. Calculate the probability mass function of $X + Y$.

[1] Details of how to hand in are to be discussed with your tutor.

9.11 Prove that $\Gamma(\frac{1}{2}) = \sqrt{\pi}$. Here is a way to do it: substitute $y = \sqrt{2z}$ into the definition $\Gamma(\frac{1}{2}) = \int_0^\infty z^{-1/2} \mathrm{e}^{-z}\,\mathrm{d}z$, and compare with the standard normal density.

9.12 Compute the k^{th} moment of the Gamma(α, λ) distribution for any $k \geq 1$ integer and $\alpha, \lambda > 0$ reals. In particular, verify $\mathbf{E}X = \frac{\alpha}{\lambda}$, $\mathbf{Var}X = \frac{\alpha}{\lambda^2}$.

HW 9.13 Let $X \sim \text{Exp}(\lambda)$ and $Y \sim \text{Gamma}(\alpha, \lambda)$ be independent. Prove $X + Y \sim \text{Gamma}(\alpha + 1, \lambda)$ (for any α, λ positive reals).