Probability 1, Autumn 2014, Problem sheet 10

To be discussed on the week 8 Dec. . . 12 Dec.
Mandatory HW's are marked with "**HW**", they are due on the week 15 Dec. . . 19 Dec, the latest.[1]
Solutions will be available on Blackboard on the 20th Dec.

HW 10.1 N people arrive to a business dinner one after the other. Upon arrival each of them looks for friends. If they find a friend then they sit to this friend's table, otherwise they open a new table. Suppose that any two people are friends, independently, with probability p and determine the expected number of tables opened by the N people. *HINT: let X_i be the indicator that the i^{th} person to arrive opens a new table upon arrival.*

10.2 A biased coin, that comes up Heads with probability p, is flipped ten times. Let X be the number of runs. (That is, the number of sequences of H's only or of T's only. Example: TTTHHTHHHH has four runs.) Find $\mathbf{E}X$. *HINT: use a clever indicator.*

10.3 We roll a fair die until each of the six numbers occur at least once. What is the expected number of rolls we make?

10.4 In a lecture room of 300 students (that's almost the figure for Probability 1), find the expected number of distinct birthdays that is, the expected number of days with at least one birthday on them. Forget about leap years, make and state the natural assumptions. Use indicators.

10.5 Let X and Y be i.i.d. positive random variables. Find the numerical value of $\mathbf{E}\frac{X}{X+Y}$. *HINT: Symmetry!*

10.6 Let $A_1 \dots A_n$ be events. Their indicator variables will be denoted by $\mathbf{1}\{A_i\}$ $(i = 1 \dots n)$ for this problem. Which event does

$$\prod_{i=1}^{n} \big(1 - \mathbf{1}\{A_i\}\big)$$

indicate? On one hand, taking expectation will give the probability of this event. On the other hand, expand the above product according to a 1, singletons of $\mathbf{1}\{A_i\}$'s, products of pairs $\mathbf{1}\{A_i\}\mathbf{1}\{A_j\}$, products of triplets $\mathbf{1}\{A_i\}\mathbf{1}\{A_j\}\mathbf{1}\{A_k\}$, etc. Apply an expectation on each of these terms and conclude the inclusion-exclusion formula.

10.7 If X and Y are independent and identically distributed with mean μ and variance σ^2, find $\mathbf{E}[(X - Y)^2]$.

HW 10.8 In my purse the number of one penny, two pence, five pence, ten pence, twenty pence, fifty pence, one pound and two pounds coins are i.i.d. Poisson random variables with parameter λ. Use linearity of expectations to compute the expectation and variance of the joint value of my coins.

10.9 We roll two fair dice, and let X to be the sum of the numbers shown, Y to be the number on the first die minus the number on the second die. Find $\mathbf{Cov}(X, Y)$. Are X and Y independent?

10.10 **The matching problem.** Recall problem **3.5**: n gentlemen go out for dinner, and they leave their hats in the cloakroom. After the dinner (and several glasses of wine) they pick their hats completely randomly. Denote by X the number of gen's who take their own hats. This time find $\mathbf{E}X$ and $\mathbf{Var}X$.

HW 10.11 5 students enter the elevator on the ground floor of the Maths Building, and they each choose one of the floors 1. . . 4 independently and randomly. Find the expectation and variance of the number of stops the elevator makes. Use indicators for the floors.

10.12 We roll a fair die n times. Let X be the number of times we see ⚄ occurring and Y the number of times we see ⚅. Compute the correlation coefficient of these two variables.

10.13 A *graph* consists of *vertices* and *edges* that connect two different vertices. Consider a graph of n vertices, and assume that each of the $\binom{n}{2}$ possible edges between them are independently present with probability p and absent with probability $1 - p$. (This is called the Erdős-Rényi random graph.) The *degree* D_i of vertex i is the number of edges with one end being i $(i = 1, 2, \dots, n)$.

(a) What is the distribution of D_i?

(b) Compute the correlation coefficient of D_i and D_j for $i \neq j$. *HINT: Write D_i and D_j in terms of indicators $I_{\ell k}$ of the presence of the edge between vertices ℓ and k, where ℓ, $k = 1, 2, \dots n$, $\ell \neq k$.*

[1] Details of how to hand in are to be discussed with your tutor.

HW 10.14 Let $X_1, X_2, \ldots$ be i.i.d. variables with mean μ and variance σ^2, and define $Y_n := X_n + X_{n+1} + X_{n+2}$. Calculate $\varrho(Y_n, Y_{n+j})$ for all $j \geq 0$.

10.15 Suppose $Y = aX + b$ and show

$$\varrho(X, Y) = \begin{cases} +1 & \text{if } a > 0, \\ -1 & \text{if } a < 0. \end{cases}$$