Probability 1, Autumn 2016, Problem sheet 10

To be discussed on the week 5 Dec. . . 9 Dec.
Problems marked with "PrCl" are discussed in the large problem class on Friday 2 Dec.
Mandatory HW's are marked with "**HW**", they are due on the week 12 Dec. . . 16 Dec, the latest.[1]
Solutions will be available on Blackboard on the 17th Dec.

10.1 If X and Y are independent and identically distributed with mean μ and variance σ^2, find $\mathbf{E}[(X-Y)^2]$.

HW 10.2 In my purse the number of one penny, two pence, five pence, ten pence, twenty pence, fifty pence, one pound and two pounds coins are i.i.d. Poisson random variables with parameter λ. Use linearity of expectations to compute the expectation and variance of the joint value of my coins.

10.3 We roll two fair dice, and let X to be the sum of the numbers shown, Y to be the number on the first die minus the number on the second die. Find $\mathbf{Cov}(X,\, Y)$. Are X and Y independent?

PrCl **10.4** **The matching problem.** Recall problem **3.5**: n gentlemen go out for dinner, and they leave their hats in the cloakroom. After the dinner (and several glasses of wine) they pick their hats completely randomly. Denote by X the number of gen's who take their own hats. This time find $\mathbf{E}X$ and $\mathbf{Var}X$.

HW 10.5 5 students enter the elevator on the ground floor of the Maths Building, and they each choose one of the floors 1. . . 4 independently and randomly. Find the expectation and variance of the number of stops the elevator makes. Use indicators for the floors.

10.6 We roll a fair die n times. Let X be the number of times we see ⚄ occurring and Y the number of times we see ⚅. Compute the correlation coefficient of these two variables.

10.7 A *graph* consists of *vertices* and *edges* that connect two different vertices. Consider a graph of n vertices, and assume that each of the $\binom{n}{2}$ possible edges between them are independently present with probability p and absent with probability $1-p$. (This is called the Erdős-Rényi random graph.) The *degree* D_i of vertex i is the number of edges with one end being i ($i = 1,\, 2,\, \ldots,\, n$).

(a) What is the distribution of D_i?

(b) Compute the correlation coefficient of D_i and D_j for $i \neq j$. *HINT: Write D_i and D_j in terms of indicators $I_{\ell k}$ of the presence of the edge between vertices ℓ and k, where $\ell,\, k = 1,\, 2,\, \ldots n,\, \ell \neq k$.*

HW 10.8 Let $X_1,\, X_2,\, \ldots$ be i.i.d. variables with mean μ and variance σ^2, and define $Y_n := X_n + X_{n+1} + X_{n+2}$. Calculate $\varrho(Y_n,\, Y_{n+j})$ for all $j \geq 0$.

10.9 Suppose $Y = aX + b$ and show

$$\varrho(X,\, Y) = \begin{cases} +1 & \text{if } a > 0, \\ -1 & \text{if } a < 0. \end{cases}$$

10.10 **Best prediction of X based on Y.**

(a) Prove Steiner's theorem: for any $c \in \mathbb{R}$, $\mathbf{E}(X-c)^2 = \mathbf{Var}X + (c - \mathbf{E}X)^2$. *(This is the same Steiner's Theorem as the one you might have seen in Physics about moments of inertia.)*

(b) By considering conditional expectations rather than ordinary ones, conclude that for any $c(Y)$,

$$\mathbf{E}\big((X - c(Y))^2 \,|\, Y\big) = \mathbf{Var}(X \,|\, Y) + \big(c(Y) - \mathbf{E}(X \,|\, Y)\big)^2.$$

In particular, $c(Y) = \mathbf{E}(X \,|\, Y)$ makes the above display minimal.

(c) Apply $\mathbf{E}$ on the above display (this expectation will be with respect to Y) to conclude that the choice $c(Y) = \mathbf{E}(X \,|\, Y)$ makes the square deviation $\mathbf{E}(X - c(Y))^2$ minimal among functions of Y.

10.11 The number of accidents that a person has in a given year is a Poisson random variable with parameter λ. However, suppose that the value of λ changes from person to person, being equal to 2 for 60 percent of the population, and 3 for the other 40 percent. A person is chosen at random. What is the probability that this person

(a) has no accidents this year;

(b) has exactly 3 accidents this year;

[1]Details of how to hand in are to be discussed with your tutor.

(c) has exactly 3 accidents this year, given that in the previous year (s)he had none?

10.12 Z students enter the elevator on the ground floor of the Maths Building, where Z is random with $\mathbf{E}Z > 1$. They each choose one of the floors $1 \dots 4$ independently and randomly. Let X be the number of stops the elevator makes.

(a) Prove $\mathbf{E}X < \mathbf{E}Z$.

(b) Suppose now $Z \sim \text{Poi}(3)$ and find $\mathbf{E}X$.

10.13 Let $X_1, X_2, \dots, X_n$ be i.i.d. random variables. Calculate $\mathbf{E}(X_1 \,|\, X_1 + X_2 + \cdots + X_n = x)$. *HINT:* $\mathbf{E}(X_1 + X_2 + \cdots + X_n \,|\, X_1 + X_2 + \cdots + X_n = x)$.

PrCl **10.14** Let X be a standard normal variable, and I independent of X with $\mathbf{P}\{I = 1\} = \mathbf{P}\{I = 0\} = 1/2$. Define

$$Y := \begin{cases} X, & \text{if } I = 1, \\ -X, & \text{if } I = 0. \end{cases}$$

(a) Show that Y is also standard normal.

(b) Are I and Y independent?

(c) Are X and Y independent?

(d) Show that $\mathbf{Cov}(X, Y) = 0$.

10.15 **Conditional covariance.** The *conditional covariance of X and Y, conditioned on Z* is defined as

$$\mathbf{Cov}(X, Y \,|\, Z) = \mathbf{E}\big[\big(X - \mathbf{E}(X \,|\, Z)\big) \cdot \big(Y - \mathbf{E}(Y \,|\, Z)\big) \,|\, Z\big].$$

(a) Show that

$$\mathbf{Cov}(X, Y \,|\, Z) = \mathbf{E}(XY \,|\, Z) - \mathbf{E}(X \,|\, Z) \cdot \mathbf{E}(Y \,|\, Z).$$

(b) Prove the conditional covariance formula

$$\mathbf{Cov}(X, Y) = \mathbf{E}[\mathbf{Cov}(X, Y \,|\, Z)] + \mathbf{Cov}[\mathbf{E}(X \,|\, Z), \mathbf{E}(Y \,|\, Z)].$$

(c) Let $X = Y$ in this display: the conditional variance formula follows.

(d) Suppose that conditioning on Z, X and Y become independent with mean Z. Show that

$$\mathbf{Cov}(X, Y) = \mathbf{Var}Z.$$

(e) We repeatedly flip a biased coin that comes up Heads with probability p, and Tails with probability $q = 1 - p$. Denote by X and Y the length of the first and the second pure sequence, respectively. (E.g., if we flip $HHHTTH\dots$, then $X = 3$, $Y = 2$; or if we get $THHT\dots$, then $X = 1$, $Y = 2$.) Determine the following quantities: $\mathbf{E}X$, $\mathbf{E}Y$, $\mathbf{Var}X$, $\mathbf{Var}Y$, $\mathbf{Cov}(X, Y)$. *HINT: condition on the first flip.*