Probability 1, Autumn 2014, Problem sheet 11

To be discussed on the week 15 Dec. . . 19 Dec.
Solutions will be available on Blackboard on the 20th Dec.

11.1 Best prediction of X based on Y.

(a) Prove Steiner's theorem: for any $c \in \mathbb{R}$, $\mathbf{E}(X-c)^2 = \mathbf{Var}X + (c - \mathbf{E}X)^2$. *(This is the same Steiner's Theorem as the one you might have seen in Physics about moments of inertia.)*

(b) By considering conditional expectations rather than ordinary ones, conclude that for any $c(Y)$,

$$\mathbf{E}\big((X-c(Y))^2 \,|\, Y\big) = \mathbf{Var}(X \,|\, Y) + \big(c(Y) - \mathbf{E}(X \,|\, Y)\big)^2.$$

In particular, $c(Y) = \mathbf{E}(X \,|\, Y)$ makes the above display minimal.

(c) Apply $\mathbf{E}$ on the above display (this expectation will be with respect to Y) to conclude that the choice $c(Y) = \mathbf{E}(X \,|\, Y)$ makes the square deviation $\mathbf{E}(X - c(Y))^2$ minimal among functions of Y.

11.2 The number of accidents that a person has in a given year is a Poisson random variable with parameter λ. However, suppose that the value of λ changes from person to person, being equal to 2 for 60 percent of the population, and 3 for the other 40 percent. A person is chosen at random. What is the probability that this person

(a) has no accidents this year;

(b) has exactly 3 accidents this year;

(c) has exactly 3 accidents this year, given that in the previous year (s)he had none?

11.3 Z students enter the elevator on the ground floor of the Maths Building, where Z is random with $\mathbf{E}Z > 1$. They each choose one of the floors $1 \dots 4$ independently and randomly. Let X be the number of stops the elevator makes.

(a) Prove $\mathbf{E}X < \mathbf{E}Z$.

(b) Suppose now $Z \sim \text{Poi}(3)$ and find $\mathbf{E}X$.

11.4 Let $X_1, X_2, \dots, X_n$ be i.i.d. random variables. Calculate $\mathbf{E}(X_1 \,|\, X_1 + X_2 + \dots + X_n = x)$. *HINT:* $\mathbf{E}(X_1 + X_2 + \dots + X_n \,|\, X_1 + X_2 + \dots + X_n = x)$.

11.5 Let X be a standard normal variable, and I independent of X with $\mathbf{P}\{I = 1\} = \mathbf{P}\{I = 0\} = 1/2$. Define

$$Y := \begin{cases} X, & \text{if } I = 1, \\ -X, & \text{if } I = 0. \end{cases}$$

(a) Show that Y is also standard normal.

(b) Are I and Y independent?

(c) Are X and Y independent?

(d) Show that $\mathbf{Cov}(X, Y) = 0$.

11.6 Conditional covariance. The *conditional covariance of X and Y, conditioned on Z* is defined as

$$\mathbf{Cov}(X, Y \,|\, Z) = \mathbf{E}\big[\big(X - \mathbf{E}(X \,|\, Z)\big) \cdot \big(Y - \mathbf{E}(Y \,|\, Z)\big) \,|\, Z\big].$$

(a) Show that

$$\mathbf{Cov}(X, Y \,|\, Z) = \mathbf{E}(XY \,|\, Z) - \mathbf{E}(X \,|\, Z) \cdot \mathbf{E}(Y \,|\, Z).$$

(b) Prove the conditional covariance formula

$$\mathbf{Cov}(X, Y) = \mathbf{E}[\mathbf{Cov}(X, Y \,|\, Z)] + \mathbf{Cov}[\mathbf{E}(X \,|\, Z), \mathbf{E}(Y \,|\, Z)].$$

(c) Let $X = Y$ in this display: the conditional variance formula follows.

(d) Suppose that conditioning on Z, X and Y become independent with mean Z. Show that

$$\mathbf{Cov}(X, Y) = \mathbf{Var}Z.$$

(e) We repeatedly flip a biased coin that comes up Heads with probability p, and Tails with probability $q = 1 - p$. Denote by X and Y the length of the first and the second pure sequence, respectively. (E.g., if we flip $HHHTTH\dots$, then $X = 3$, $Y = 2$; or if we get $THHT\dots$, then $X = 1$, $Y = 2$.) Determine the following quantities: $\mathbf{E}X$, $\mathbf{E}Y$, $\mathbf{Var}X$, $\mathbf{Var}Y$, $\mathbf{Cov}(X, Y)$. *HINT: condition on the first flip.*

11.7 Let X have moment generating function $M(t)$, and let $\Psi(t) = \ln M(t)$. Show that

$$\Psi(t)\big|_{t=0} = 0, \qquad \Psi'(t)\big|_{t=0} = \mathbf{E}(X), \qquad \Psi''(t)\big|_{t=0} = \mathbf{Var}(X).$$

11.8 Calculate the moment generating function of the $\mathrm{Geom}(p)$ distribution by direct computation.

11.9 Let $X \sim \mathrm{Geom}(p)$, the number of trials until the first success in a sequence of independent experiments with success probability p. Let I be the indicator of the success of the first trial.

(a) What is the distribution of $(X \mid I = 1)$?

(b) What is the distribution of $(X \mid I = 0)$? (*Mind the memoryless property.*)

(c) By the tower rule,

$$M(t) = \mathbf{E}\mathrm{e}^{tX} = \mathbf{E}\mathbf{E}\big(\mathrm{e}^{tX} \mid I\big).$$

Expand the right hand-side using your previous answers, then solve this equation for $M(t)$, thus determining the moment generating function of the $\mathrm{Geom}(p)$ distribution.

11.10 (a) Let $X \sim \mathrm{U}(\alpha, \beta)$. Determine its moment generating function $M_X(t)$.

(b) Let Y be the number shown after rolling a fair die. Determine its moment generating function $M_Y(t)$.

(c) Now let $Z \sim \mathrm{U}(0, 1)$ independent of the above Y and, using $M_{Y+Z}(t) = M_Y(t) \cdot M_Z(t)$, conclude that $Y + Z \sim \mathrm{U}(1, 7)$.