Stationary last passage percolation

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Abstract

An introduction to stationary structures of last passage percolation models; minicourse given in ICTS, July 2022.

1 Introduction

We describe some of the spatial stationarity structures in 2-dimensional last passage percolation. We will define the model, then consider the special case where the distribution satisfies a peculiar distributional identity which can be interpreted as a spatial Markov property with an explicit distribution. As an application we then show how to derive the KPZ scaling order upper bound via probabilistic means.

These notes will certainly not give a full overview of the field; we will not touch the methods of integrable probability, and even the probabilistic side of the field has much more to it than the range we consider. Our modest aim here is to provide an idea, from the probabilistic point of view, how stationarity arises and what it is good for. These notes are largely based on [1]. The reader is referred to *The corner growth model with exponential weights* section of [15] (see also [17]) for a much more comprehensive reading on the topics, and to [16] for a gentle introduction to the integrable probability side; both by Timo Seppäläinen. A few further references will be provided as they naturally arise in the text.

2 Last passage percolation

Last passage percolation models (LPP) concern the largest total weight that can be collected by a path between two points in space, among a random field of weights. As such, this of course doesn't make sense as an infinite excursion, or an infinite number of loops would beat every finite path in such competition. Hence a restriction is imposed on allowed paths.

For this note our space will be the integer lattice $\mathbb{Z}^2$, weights will sit on the vertices of this lattice, and will be non-negative i.i.d. distributed. The restriction will be that paths must be directed: they can only make up or right steps.

As the setup is translation-invariant, we always assume without loss of generality, unless otherwise stated, that our path starts from the origin $(0,\,0)\in\mathbb{Z}^2$.

Definition 2.1. Fixing the endpoint at $(m,\,n)$ with $m>0,\,n>0$ integers, the *set of directed paths* to this point is

$$\Pi_{m,n}:=\bigl\{\pi=\bigl(\pi_0=(0,\,0),\,\pi_1,\,\pi_2,\,\dots,\pi_{m+n}=(m,\,n)\bigr)\,:\,\pi_{k+1}-\pi_k=(1,\,0)\text{ or }(0,\,1)\text{ for }0\le k<m+n\bigr\}.$$

The i.i.d. non-negative weights are denoted by ω_{ij} (at lattice site $(i,\,j)$).

Definition 2.2. The *last passage time* of the point $(m,\,n)$ is

$$G_{m,n}:=\max_{\pi\in\Pi_{m,n}}\sum_{k=1}^{m+n}\omega_{\pi_k}\qquad(m,\,n\ge 0).$$

For $G_{0,0}$ we interpret the empty sum as zero: $G_{0,0}=0$.

In words, this is the largest total weight collected along any directed path π from the origin to $(m,\,n)$.

Due to the directed nature of the paths, any weight outside the non-negative quadrant $\mathbb{Z}_{\ge 0}\times\mathbb{Z}_{\ge 0}$ is irrelevant for $G_{m,n}$. Hence we can equivalently define the past passage percolation on $\mathbb{Z}_{\ge 0}\times\mathbb{Z}_{\ge 0}$ only, instead of $\mathbb{Z}^2$.

By looking at paths ending at $(i-1,\,j)$ or $(i,\,j-1)$, we easily see, with $\vee$ meaning "larger of",

$$G_{i,j}=(G_{i-1,j}\vee G_{i,j-1})+\omega_{i,j}\qquad(i,\,j>0).\tag{2.1}$$

Assuming $G_{-1,j}=G_{i,-1}=0$, we can extend this display to all of $i,\,j\ge 0$.

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2.1 Corner growth model

An alternative representation of LPP considers the set

$$\mathcal{A}(t) := \big\{(i,\, j)\,:\, G_{i,j} \le t\big\} \cap \big(\mathbb{Z}_{\ge 0} \times \mathbb{Z}_{\ge 0}\big)$$

of *occupied points at time* $t \ge 0$. By definition $\mathcal{A}(0) = \{(0,\, 0)\}$, and a new point $(i,\, j)$ can get occupied $\omega_{i,j}$ time after both its South and West neighbours got occupied.

Notice that picking $\omega_{i,j}$ from an Exponential distribution makes $\mathcal{A}(t)$ into a continuous time Markov chain. This forms the basis of further representations of such LPP in terms of M/M/1 queues and the *Totally Asymmetric Simple Exclusion Process*, which we do not explore here.

2.2 Increments of last passage times

Definition 2.3. The *horizontal and vertical increments of last passage times* are defined respectively by

$$\begin{aligned} I_{i,j} &:= G_{i,j} - G_{i-1,j} \qquad && i > 0 \le j,\\ J_{i,j} &:= G_{i,j} - G_{i,j-1} \qquad && i \ge 0 < j. \end{aligned}$$

In the corner growth representation this is how much time it takes for the occupied region to extend one step to the East in a given row, or to the North in a given column, respectively.

Lemma 2.4. *The increments satisfy*

$$\begin{aligned} I_{i,j} &= (I_{i,j-1} - J_{i-1,j})^+ + \omega_{i,j},\\ J_{i,j} &= (J_{i-1,j} - I_{i,j-1})^+ + \omega_{i,j}. \end{aligned} \tag{2.2}$$

Proof. Plug (2.1) into the definition of $I_{i,j}$:

$$\begin{aligned} I_{i,j} &= (G_{i-1,j} \vee G_{i,j-1}) - G_{i-1,j} + \omega_{i,j}\\ &= (G_{i,j-1} \vee G_{i-1,j}) - G_{i-1,j-1} - (G_{i-1,j} - G_{i-1,j-1}) + \omega_{i,j}\\ &= \big((G_{i,j-1} - G_{i-1,j-1}) \vee (G_{i-1,j} - G_{i-1,j-1})\big) - (G_{i-1,j} - G_{i-1,j-1}) + \omega_{i,j}\\ &= (I_{i,j-1} \vee J_{i-1,j}) - J_{i-1,j} + \omega_{i,j}\\ &= (I_{i,j-1} - J_{i-1,j} \vee 0) + \omega_{i,j} = (I_{i,j-1} - J_{i-1,j})^+ + \omega_{i,j}. \end{aligned}$$

The proof of the second identity is similar. □

2.3 Stationary Exponential LPP

We start with a distributional fact a little bit out of the blue, then comment on how something like this could be discovered.

Proposition 2.5. *Let* $0 < \varrho < 1$ *and* $U \sim \mathit{Exp}(1-\varrho)$, $V \sim \mathit{Exp}(\varrho)$, $\omega \sim \mathit{Exp}(1)$ *be mutually independent. Then*

$$I := (U - V)^+ + \omega \sim \mathit{Exp}(1-\varrho), \qquad J := (V - U)^+ + \omega \sim \mathit{Exp}(\varrho), \qquad X := U \wedge V \sim \mathit{Exp}(1) \tag{2.3}$$

and these three are mutually independent. Here $\wedge$ *means "the smaller of".*

Proof. Consider a rate 1 homogeneous Poisson process and colour each of its marks red or blue independently of everything with respective probabilities $1-\varrho$ and ϱ. Then X is realised as the time of the first mark of any colour and U (V) as the time of the first red (blue, respectively) mark. It already follows that $X \sim \mathrm{Exp}(1)$. Notice that $U - V$ depends on the colour the first mark and the future of the coloured Poisson process after X. Both of these are independent of X itself, hence X and the pair $\big((U-V)^+,\, (V-U)^+\big)$ are independent of each other. This in turn gives that the triplet $\big[X,\ \big((U-V)^+,\, (V-U)^+\big),\ \omega\big]$ is mutually independent. Therefore,

$$\big[X,\ \big((U-V)^+,\, (V-U)^+\big),\ \omega\big] \overset{d}{=} \big[\omega,\ \big((U-V)^+,\, (V-U)^+\big),\ X\big]. \tag{2.4}$$

Trivially,

$$\begin{aligned} (U,\, V,\, \omega) &= \big((U-V)^+ + X,\, (V-U)^+ + X,\, \omega\big), \qquad \text{and}\\ (I,\, J,\, X) &= \big((U-V)^+ + \omega,\, (V-U)^+ + \omega,\, X\big) \end{aligned}$$

and the statement immediately follows from (2.4). □

Homework 2.6. Show this identity the analytic way, using joint moment generating functions.

Independence of the min and the difference characterizes the Exponential and the Geometric distribution, see Crawford [7] for details.

Next we consider Exponential weights $\omega_{i,j}$, but slightly modify the parameters to fit the setup in Proposition 2.5.

Definition 2.7. The *stationary exponential LPP model* with parameter $0 < \varrho < 1$ is defined by mutually independent weights

$$\omega_{i,0} \sim \mathrm{Exp}(1-\varrho), \quad \omega_{0,j} \sim \mathrm{Exp}(\varrho), \quad \omega_{i,j} \sim \mathrm{Exp}(1), \qquad (i,\, j > 0)$$

and last passage times as in Definition 2.2 as before but with these modified weights.

Definition 2.8. Take a doubly infinite South-East path $(\sigma_\ell)_{\ell\in\mathbb{Z}}$ indexed by ℓ. South-East means that $\sigma_{\ell+1} - \sigma_\ell = (0,\,-1)$ or $(1,\,0)$. We require σ in the non-negative quadrant: $\sigma_\ell \in \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0}$. This also includes the possibility that σ is just the union of the two coordinate axes.

The *interior of the set enclosed by* σ is

$$\mathcal{B}(\sigma) := \{(i,\, j) \,:\, 0 \leq (i,\, j) < \sigma_\ell \text{ for some } \ell \in \mathbb{Z}\},$$

where "$<$" is domination in both coordinates, $(i,\, j) < (a,\, b)$ iff $i < a$ and $j < b$. $\mathcal{B}(\sigma)$ is empty when σ is the union of the two coordinate axes.

Definition 2.9. We now introduce the auxiliary random variables

$$X_{i-1,j-1} := I_{i,j-1} \wedge J_{i-1,j} \qquad (i,\, j > 0) \tag{2.5}$$

with $\wedge$ meaning "the smaller of".

With all this preparation, we can now prove stationarity of the stationary LPP.

Theorem 2.10. *In the stationary LPP, for any coordinates $(i,\, j)$ where we defined the below,*

- $I_{i,j} \sim \mathit{Exp}(1-\varrho)$,
- $J_{i,j} \sim \mathit{Exp}(\varrho)$,
- $X_{i,j} \sim \mathit{Exp}(1)$.

Moreover, for any South-East path σ, all of the $I_{i,j}$ variables when part of σ (that is, $(i-1,\, j)$ and $(i,\, j)$ are both in σ), the $J_{i,j}$ variables when part of σ (that is, $(i,\, j-1)$ and $(i,\, j)$ are both in σ), and $X_{i,j}$ variables for $(i,\, j) \in \mathcal{B}(\sigma)$ are mutually independent.

Remark 2.11. *It is very important to notice that independence of the last passage time increments is lost at the moment σ fails to be South-East, in other words if $\sigma_\ell < \sigma_k$ for any two $\ell,\, k \in \mathbb{Z}$. In particular, there is no way to access $(m,\, n) > (0,\, 0)$ from the origin along independent last passage time increments.*

Remark 2.12. *The mapping of LPP to M/M/1 queues or totally asymmetric exclusion takes this theorem to Burke's theorem for reversibility of M/M/1 queues, or the marginal Poisson property of a tagged particle (as well as of a tagged hole) in totally asymmetric exclusion. Hence this theorem is sometimes referred to as Burke's theorem even in the LPP setting.*

Proof. A.s. there is a unique path from the origin to any point $(i,\, 0)$ or $(0,\, j)$. Hence the last passage time to such a point is just the sum of ω's along one of the axes. The statement therefore immediately follows when σ is the union of the coordinate axes.

We proceed by induction from here. Assume a South-East path σ as in Figure 1 and that the theorem holds for this path. Find a South-West corner in σ that is, some $(i,\, j)$ with $(i-1,\, j)$, $(i-1,\, j-1)$, $(i,\, j-1)$ each in σ. The I and J increments along σ, as well as the X variables in $\mathcal{B}(\sigma)$, only depend on the ω weights in $\mathcal{B}(\sigma)$ and along σ. Hence ω_{ij} is independent of all of these variables. Moreover, by the inductive assumption, the pair $(I_{i,j-1},\, J_{i-1,j})$ is also mutually independent of the rest of the I's and J's along σ, and the X's in $\mathcal{B}(\sigma)$.

Now comes Proposition 2.5 with

$$U = I_{i,j-1}, \qquad V = J_{i-1,j}, \qquad \omega = \omega_{i,j}, \qquad I = I_{i,j}, \qquad J = J_{i,j}, \qquad X = X_{i-1,j-1}.$$

Notice that (2.3) then exactly matches the constructions (2.2) and (2.5). Hence the triplet $(I_{i,j},\, J_{i,j},\, X_{i-1,j-1})$ has the desired joint distribution. Furthermore, these variables are fixed via (2.2) and (2.5) by $(I_{i,j-1},\, J_{i-1,j},\, \omega_{ij})$, which was independent of the rest of the I's and J's along σ, and the X's in $\mathcal{B}(\sigma)$, hence the independence structure is maintained.

We see that we successfully flipped a South-West corner into a North-East one, obtaining an extended South-East path from σ this way. As any σ with finite $\mathcal{B}(\sigma)$ can be reached by a number of such corner flips from the path of the coordinate axes, the proof is done for those. For a σ not touching one or both of the axes, notice that any finite segment of it with a finite subset of $\mathcal{B}(\sigma)$ is identical to the same parts of a South-East path with finite $\mathcal{B}$, for which the proof holds. □

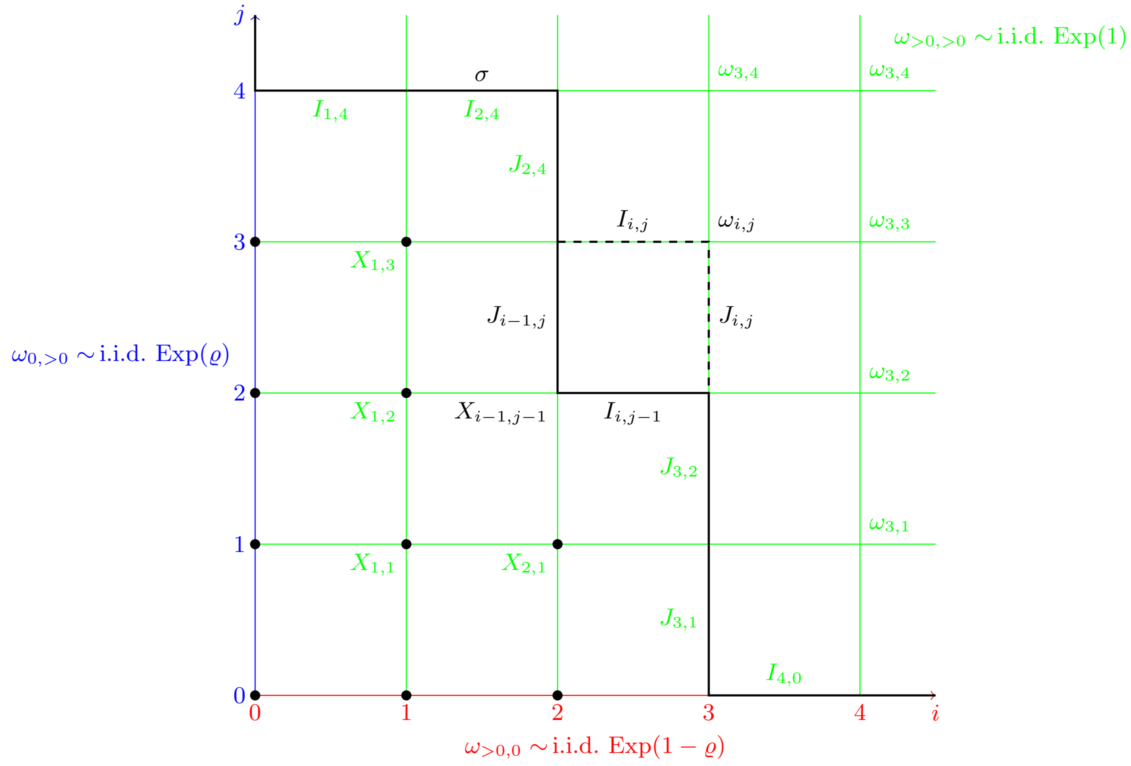

Figure 1: The corner flip induction step. Vertices in $\mathcal{B}(\sigma)$ are marked $\bullet$.

The i.i.d. increments structure immediately gives the Law of Large Numbers for the last passage times in the stationary model.

Corollary 2.13. *For any* $m,\, n \ge 0$,

$$\mathbb{E}\, G_{m,n} = \frac{m}{1-\varrho} + \frac{n}{\varrho}. \tag{2.6}$$

Fix real parameters $0 \le x,\, y$ *and a sequence* $\big(m(N),\, n(N)\big)$ *of endpoints with* $\frac{m(N)}{N} \to x$ *and* $\frac{n(N)}{N} \to y$. *Then*

$$\lim_{N\to\infty} \frac{G_{m(N),n(N)}}{N} \xrightarrow{a.s.} \frac{x}{1-\varrho} + \frac{y}{\varrho}. \tag{2.7}$$

Proof. By the definition of the increments,

$$G_{m,n} = \sum_{i=1}^{m} I_{i,0} + \sum_{j=1}^{n} J_{m,j}. \tag{2.8}$$

Each of these sums features i.i.d. Exponential variables, with respective parameters $1-\varrho$ and ϱ. The mean identity follows immediately, and classical Strong Law of Large Numbers (SLLN) applies on the first sum in the above limit. The second sum is slightly tricky as the summands change in m as well as j. A fourth moment proof of the SLLN applies though. $\square$

Homework 2.14. Work out the details of proving (2.7).

Remark 2.15. *It is very important to notice (again) that the two sums in* (2.8) *are very* dependent, *hence such arguments are not helpful towards proving any finer limits than the law of large numbers.*

We see from (2.7) that the rescaled last passage time is constant among rescaled endpoints x and y if they are on the curve $(1-\varrho)y + \varrho x = \text{const}$. In some sense this plays the role of a "ball" of fixed "radius": it is equally hard to reach points on this line in this hydrodynamic scaling. This is the shape for the stationary model, a boring, straight line. To see something interesting, one removes the artificial boundary and keeps i.i.d. Exp(1) weights $\omega_{i,j}$ all over $\mathbb{Z}_{\ge 0} \times \mathbb{Z}_{\ge 0}$ including the boundaries.

The boundary weights $\omega_{i,0} \sim \text{Exp}(1-\varrho)$ and $\omega_{0,j} \sim \text{Exp}(\varrho)$ are heavier than the bulk Exp(1) weight distribution. Hence the longest LPP path can be expected to stick to the boundary for some time after departing from the origin. With a fixed endpoint (m, n), tuning the parameter ϱ changes whether the South or the West boundary is more preferable. There should be a particular value ϱ_0 where the path is "undecided" as to which of the two boundaries it should favour. At this parameter one would expect the smallest improvement to the last passage time from the boundary, as the path is not really sticking to it for a long time. Minimising (2.7) in ϱ, one finds

$$\frac{y}{x} = \frac{\varrho_0^2}{(1-\varrho_0)^2}, \tag{2.9}$$

and solving this for ϱ_0 gives $\varrho_0 = \frac{\sqrt{y}}{\sqrt{x}+\sqrt{y}}$. Plugging this back to (2.7) indeed recovers the shape $\left(\sqrt{x}+\sqrt{y}\right)^2$ for the model without modified boundaries. [17] discusses such connections in rigour and great detail, in these short notes we do not pursue this further.

Reverting point of view, for a fixed parameter value ϱ_0 the direction (2.9) is called the *characteristic direction* for the stationary LPP. Once pushed through the connection between LPP and either M/M/1 queues or the Totally Asymmetric Simple Exclusion Process, (2.9) maps to the so-called *characteristic velocity* in those models.

To finish this section, we mention that much of this research was inspired back then by Cator and Groeneboom's seminal paper [3] for the so-called *Hammersley process*, based on similar stationarity structures. Other stationary LPP models exist as well, see Ciech and Georgiou [5, 6]. For a systematic exploration of stationarity in polymer models, see Chaumont and Noack [4]. The exponential increments I and J also arise without the stationary boundaries as *Busemann functions* i.e., limits of differences of last passage times from a far away common starting point in direction (2.9). See e.g. [17] for details.

3 Fluctuation bounds

To demonstrate ways of using stationarity, we provide parts of the arguments towards KPZ-scaling of last passage times in the Exponential LPP. These are old (hence relatively simple) probabilistic arguments from [1]. Notice that such techniques have recently been significantly improved by Emrah, Georgiou, Janjigian, Ortmann and Seppäläinen [9, 8].

There exists a vast amount of literature of KPZ scaling limits using *methods of integrable probability*, tools that evolved from combinatorial, algebraic and random matrix theory techniques. We do not consider this area here, just notice that, while the nature of these arguments is different, integrable probability methods often apply for models that exhibit stationary properties as the one seen above.

As noticed before, $G_{m,n}$ is two sums of i.i.d. Exponentials added together. However, these two sums are highly dependent on each other, making the fluctuations of $G_{m,n}$ difficult to handle. We proceed with showing some steps towards finding the order of these fluctuations.

3.1 A coupling of boundary weights

For later use, we perturb the South boundary as follows. Recall that the variables $\omega_{i,0}$ are i.i.d. Exponential$(1-\varrho)$ distributed with our parameter $0 \le \varrho \le 1$. Fix $\varrho < \lambda < 1$ and let, for each $i > 0$,

$$\xi_{i,0} := \frac{1-\varrho}{1-\lambda} \cdot \omega_{i,0}. \tag{3.1}$$

Homework 3.1. Show that $\xi_{i,0}$ are i.i.d. Exponential$(1-\lambda)$ distributed, and that

$$\mathbb{V}\mathrm{ar}(\xi_{i,0} - \omega_{i,0}) = \Big(\frac{1}{1-\lambda} - \frac{1}{1-\varrho}\Big)^2. \tag{3.2}$$

The i.i.d. Exponential(ϱ) (West) and Exponential$(1-\varrho)$ (South) boundary is stationary in the sense that the increments $I_{i,1}$ are again i.i.d. Exponential$(1-\varrho)$ distributed. One might ask whether the above coupling also keeps stationarity for the LPP with the $\xi_{i,0}$ boundaries. This can be asked in a marginal sense for $\xi_{i,0}$ only. The answer is no, as we have not changed the West boundary from parameter ϱ to λ accordingly. This can be helped however by pushing the origin to the far left, leaving only a South boundary, in which case the answer is yes. A more interesting question is whether we have stationarity between increments $I_{i,0}$ and $I_{i,1}$ *jointly for the LPP using the $\omega_{i,0}$ boundary weights, and the one using $\xi_{i,0}$.* The answer is no, not with this simple coupling. The correct coupling to achieve this is known but is more complicated, see Fan and Seppäläinen [10] and, for a scaling limit, Busani [2]. We do not go into the details as the above simple coupling will work for our purposes.

3.2 Exit point from the boundary

We already saw a taster of arguments that consider the effects of the boundary weights on the last passage times. We will exploit these further now. The longest path from the origin to $(m,\, n)$ proceeds along the heavier weights of the boundary for some time, then makes a brave step and exits into the bulk $\mathbb{Z}_{>0} \times \mathbb{Z}_{>0}$. Hence its optimal property can be split into two parts: it optimises over where to exit the boundary, and then over its geometry in the bulk. This motivates the definitions below.

Definition 3.2. For an integer $-n \le x \le m$, let

$$U_x := G_{x^+,x^-} = \begin{cases} \displaystyle\sum_{i=1}^{x} \omega_{i,0}, & \text{if } x \ge 0, \\ \displaystyle\sum_{j=1}^{-x} \omega_{0,j}, & \text{if } x \le 0 \end{cases}$$

with the convention that empty sums are zero.

This is the weight collected on the boundary up to $(x,\, 0)$ on the horizontal axis when $x > 0$, and to $(0,\, -x)$ on the vertical axis when $x < 0$.

For a path that happens to exit the boundary at $(x^+,\, x^-) \ne (0,\, 0)$, we then note the most possible weight collected in the bulk by

Definition 3.3.

$$A_x := \max_{\pi} \sum_{k=1}^{m+n-|x|} \omega_{\pi_k}, \tag{3.3}$$

where the maximum is over North-East paths π starting at $\pi_0 = (x^+,\, x^-)$, ending at $\pi_{m+n-|x|} = (m,\, n)$ and satisfying $\pi_1 > (0,\, 0)$ (i.e., the first step's both coordinates are positive; this π really exits the boundary in the first step).

The last passage time then satisfies $G_{m,n} = \max_{x\ne0}(U_x+A_x)$. Moreover, by the continuity of the Exponential distribution, there is an a.s. unique maximiser $x \ne 0$, which we denote by Z:

$$G_{m,n} = \max_{x\ne0}(U_x + A_x) = U_Z + A_Z.$$

The next lemma is a fundamental building block in proving KPZ-scaling of fluctuations, and is here to demonstrate the power of stationarity in this business.

Lemma 3.4.

$$\begin{aligned} \mathbb{V}\mathrm{ar}\, G_{m,n} &= \frac{n}{\varrho^2} - \frac{m}{(1-\varrho)^2} + \frac{2}{1-\varrho} \cdot \mathbb{E}\, U_{Z^+} \\ &= \frac{m}{(1-\varrho)^2} - \frac{n}{\varrho^2} + \frac{2}{\varrho} \cdot \mathbb{E}\, U_{-Z^-}. \end{aligned}$$

Proof. For brevity, define the increments along the sides of the box as

$$\mathcal{W} := G_{0,n}, \qquad \mathcal{N} := G_{m,n} - G_{0,n}, \qquad \mathcal{E} := G_{m,n} - G_{m,0}, \qquad \mathcal{S} := G_{m,0}.$$

$\mathcal{W}$ and $\mathcal{S}$ are just the i.i.d. $\mathrm{Exp}(\varrho)$ and i.i.d. $\mathrm{Exp}(1-\varrho)$ boundaries respectively, which are also independent of each other. By definition, $G_{m,n} = \mathcal{W} + \mathcal{N} = \mathcal{S} + \mathcal{E}$, and by Theorem 2.10, $\mathcal{N}$ and $\mathcal{E}$ are independent. Notice, however, that every other pair of $\mathcal{W}$, $\mathcal{N}$, $\mathcal{E}$ and $\mathcal{S}$ is dependent. We have

$$\begin{aligned} \mathbb{V}\mathrm{ar}\, G_{m,n} &= \mathbb{V}\mathrm{ar}(\mathcal{W} + \mathcal{N}) = \mathbb{V}\mathrm{ar}\, \mathcal{W} + \mathbb{V}\mathrm{ar}\, \mathcal{N} + 2\,\mathbb{C}\mathrm{ov}(\mathcal{W},\, \mathcal{N}) \\ &= \mathbb{V}\mathrm{ar}\, \mathcal{W} + \mathbb{V}\mathrm{ar}\, \mathcal{N} + 2\,\mathbb{C}\mathrm{ov}(\mathcal{S},\, \mathcal{N}) + 2\,\mathbb{C}\mathrm{ov}(\mathcal{E},\, \mathcal{N}) - 2\,\mathbb{C}\mathrm{ov}(\mathcal{N},\, \mathcal{N}) \\ &= \mathbb{V}\mathrm{ar}\, \mathcal{W} - \mathbb{V}\mathrm{ar}\, \mathcal{N} + 2\,\mathbb{C}\mathrm{ov}(\mathcal{S},\, \mathcal{N}). \end{aligned}$$

Next we fix $1 - \varrho > \varepsilon > 0$, and introduce the new South boundary according to (3.1) with $\lambda = \varrho + \varepsilon$. While all other $\omega_{\ge0,>0}$'s are left unchanged, we now have two LPP models on the same probability space, the original one and this new one which we will denote by a super-index $^{\varepsilon}$, having the modified South boundary.

The total increment along the new South boundary, $\mathcal{S}^{\varepsilon}$, is the sum of m i.i.d. $\mathrm{Exp}(1-\varrho-\varepsilon)$ variables which is Gamma distributed. Hence its density for $s > 0$ is

$$f_{\mathcal{S}^{\varepsilon}}(s) = \frac{(1-\varrho-\varepsilon)^m s^{m-1} \cdot \mathrm{e}^{-(1-\varrho-\varepsilon)s}}{(m-1)!}$$

with its derivative w.r.t. ε

$$\partial_\varepsilon f_{\mathcal{S}^\varepsilon}(s) = s f_{\mathcal{S}^\varepsilon}(s) - \frac{m}{1-\varrho-\varepsilon} \cdot f_{\mathcal{S}^\varepsilon}(s).$$

The sum of Exponential variables could as well be seen as a Poisson process. A particular feature of the Poisson process is the *order statistics property*: given the sum $\mathcal{S}$, the $(\omega_{i,0})_{0<i\le m})$ variables have the same joint distribution for all parameter values. Hence fixing the value $\mathcal{S}$, the original LPP model and the one modified by ε are indistinguishable. That implies $\mathbb{E}(\mathcal{N}^\varepsilon \,|\, \mathcal{S}^\varepsilon = s) = \mathbb{E}(\mathcal{N} \,|\, \mathcal{S} = s)$, which we can use as

$$\begin{aligned}
\partial_\varepsilon \, \mathbb{E}\mathcal{N}^\varepsilon\big|_{\varepsilon=0} &= \partial_\varepsilon \, \mathbb{E}\,\mathbb{E}(\mathcal{N}^\varepsilon \,|\, \mathcal{S}^\varepsilon)\big|_{\varepsilon=0} \\
&= \partial_\varepsilon \int_0^\infty \mathbb{E}(\mathcal{N}^\varepsilon \,|\, \mathcal{S}^\varepsilon = s) f_{\mathcal{S}^\varepsilon}(s)\,\mathrm{d}s\Big|_{\varepsilon=0} \\
&= \partial_\varepsilon \int_0^\infty \mathbb{E}(\mathcal{N} \,|\, \mathcal{S} = s) f_{\mathcal{S}^\varepsilon}(s)\,\mathrm{d}s\Big|_{\varepsilon=0} \\
&= \int_0^\infty \mathbb{E}(\mathcal{N} \,|\, \mathcal{S} = s) \partial_\varepsilon f_{\mathcal{S}^\varepsilon}(s)\big|_{\varepsilon=0}\,\mathrm{d}s \\
&= \int_0^\infty \mathbb{E}(\mathcal{N} \,|\, \mathcal{S} = s) s f_{\mathcal{S}^0}(s)\,\mathrm{d}s - \int_0^\infty \mathbb{E}(\mathcal{N} \,|\, \mathcal{S} = s) \frac{m}{1-\varrho} \cdot f_{\mathcal{S}^0}(s)\,\mathrm{d}s \\
&= \mathbb{E}(\mathcal{N}\mathcal{S}) - \frac{m}{1-\varrho}\,\mathbb{E}\mathcal{N} = \mathbb{C}\mathrm{ov}(\mathcal{N},\,\mathcal{S}).
\end{aligned}$$

Next we concentrate on this derivative using the exit point Z. When modifying the South boundary from $\mathcal{S}$ to $\mathcal{S}^\varepsilon$, only $G_{m,n}$ can change in $\mathcal{N} = G_{m,n} - G_{0,n}$, and this can happen in two ways: the exit point Z for the longest path doesn't move, but the longest path might collect more weight on the South boundary, or the exit point changes as well. This is the split we make below.

$$\begin{aligned}
\mathcal{N}^\varepsilon - \mathcal{N} &= (\mathcal{N}^\varepsilon - \mathcal{N}) \cdot \mathbb{1}\{Z^\varepsilon = Z\} + (\mathcal{N}^\varepsilon - \mathcal{N}) \cdot \mathbb{1}\{Z^\varepsilon \ne Z\} \\
&= (U^\varepsilon_{Z^\varepsilon} - U_Z) \cdot \mathbb{1}\{Z^\varepsilon = Z\} + (\mathcal{N}^\varepsilon - \mathcal{N}) \cdot \mathbb{1}\{Z^\varepsilon \ne Z\} \\
&= (U^\varepsilon_{Z} - U_Z) \cdot \mathbb{1}\{Z^\varepsilon = Z\} + (\mathcal{N}^\varepsilon - \mathcal{N}) \cdot \mathbb{1}\{Z^\varepsilon \ne Z\} \\
&= (U^\varepsilon_{Z} - U_Z) + (\mathcal{N}^\varepsilon - \mathcal{N} - U^\varepsilon_Z + U_Z) \cdot \mathbb{1}\{Z^\varepsilon \ne Z\}.
\end{aligned}$$

By (3.1), the first term is

$$U^\varepsilon_Z - U_Z = U^\varepsilon_{Z^+} - U_{Z^+} = \Big(\frac{1-\varrho}{1-\varrho-\varepsilon} - 1\Big) U_{Z^+} = \frac{\varepsilon}{1-\varrho-\varepsilon} \cdot U_{Z^+}.$$

For the second term, notice that $\mathcal{N}^\varepsilon - \mathcal{N} \le \mathcal{S}^\varepsilon - \mathcal{S}$ and, since the modified South is heavier, that $U^\varepsilon_Z \ge U_Z$. Hence

$$\mathbb{E}\big[(\mathcal{N}^\varepsilon - \mathcal{N} - U^\varepsilon_Z + U_Z) \cdot \mathbb{1}\{Z^\varepsilon \ne Z\}\big] \le \mathbb{E}\big[(\mathcal{S}^\varepsilon - \mathcal{S}) \cdot \mathbb{1}\{Z^\varepsilon \ne Z\}\big] \le \big(\mathbb{E}(\mathcal{S}^\varepsilon - \mathcal{S})^2\big)^{\frac{1}{2}} \cdot \big(\mathbb{P}\{Z^\varepsilon \ne Z\}\big)^{\frac{1}{2}}.$$

The first factor is, as above, $\frac{\varepsilon}{1-\varrho-\varepsilon} \cdot \big(\mathbb{E}\,\mathcal{S}^2\big)^{\frac{1}{2}}$, which is $\mathcal{O}(\varepsilon)$.

Finally, we show that the probability is $\mathcal{O}(\varepsilon)$. The first observation is that Z is the a.s. unique optimiser of the exit point in the original LPP, hence for any $k \ne Z$, $A_Z + U_Z > A_k + U_k$ a.s. (see (3.3)).

The modified South is heavier, hence more preferable for the longest path (notice that the bulk passage times A_k are not modified). This implies $Z^\varepsilon \ge Z$. The event $Z^\varepsilon \ne Z$ occurs exactly when there is an index $Z < k \le m$ that achieves the optimum in the modified LPP. That is, a.s.

$$\begin{aligned}
\{Z^\varepsilon \ne Z\} &= \{A^\varepsilon_Z + U^\varepsilon_Z < A^\varepsilon_k + U^\varepsilon_k \text{ for some } Z < k \le m\} \\
&= \{A_Z + U^\varepsilon_Z < A_k + U^\varepsilon_k \text{ for some } Z < k \le m\} \\
&= \{A_Z + U^\varepsilon_Z < A_k + U^\varepsilon_k,\ A_k + U_k < A_Z + U_Z \text{ for some } Z < k \le m\} \\
&= \{U_k - U_Z < A_Z - A_k < U^\varepsilon_k - U^\varepsilon_Z \text{ for some } Z < k \le m\} \\
&\subseteq \{U_k - U_i < A_i - A_k < U^\varepsilon_k - U^\varepsilon_i \text{ for some } 0 \le i < k \le m\}.
\end{aligned}$$

A union bound hence gives $\mathbb{P}\{Z^\varepsilon \ne Z\} \le \sum_{0\le i<k\le m} \mathbb{P}\{U_k - U_i < A_i - A_k < U^\varepsilon_k - U^\varepsilon_i\}$. Looking at one of these probabilities, and using that the boundary is independent of the bulk passage times A_k,

$$\begin{aligned}\mathbb{P}\{U_k - U_i < A_i - A_k < U_k^\varepsilon - U_i^\varepsilon\} &= \mathbb{E}\,\mathbb{P}\{U_k - U_i < A_i - A_k < U_k^\varepsilon - U_i^\varepsilon \mid A_i - A_k\}\\ &\le \sup_{x>0}\mathbb{P}\{U_k - U_i < x < U_k^\varepsilon - U_i^\varepsilon\}\\ &= \sup_{x>0}\mathbb{P}\Big\{U_k - U_i < x < \frac{1-\varrho}{1-\varrho-\varepsilon}\cdot(U_k - U_i)\Big\}\\ &= \sup_{x>0}\mathbb{P}\Big\{x\Big(1-\frac{\varepsilon}{1-\varrho}\Big) < U_k - U_i < x\Big\},\end{aligned}$$

which is $\mathcal{O}(\varepsilon)$ due to the bounded density of the Gamma distribution.

Combining the above displays proves the first line of the lemma. The second line is proved similarly by looking at $\mathbb{C}\mathrm{ov}(\mathcal{W},\,\mathcal{E})$. □

3.3 Comparison of last passage times and a Taylor step

Next comes a sketch of the main steps of proving the upper bound for the order of $\mathbb{V}\mathrm{ar}\,G_{m,n}$ in the stationary model. This follows [1] so the gaps can be filled from there. A much more powerful version was recently constructed by Emrah, Georgiou and Ortmann [8] based on a genius insight to refine Lemma 3.4 by Emrah, Janjigian and Seppäläinen [9] (also noticed earlier by Rains [14]).

To concentrate on the essence of the arguments, we omit integer parts and pretend at many calculations that coordinates $(i,\,j)$ are continuous.

Fix $0<\varrho<1$ and recall the characteristic direction $\frac{\varrho^2}{1-\varrho^2}$ of (2.9). This is where the longest path is not expected to spend a long time on the boundaries. In other directions there will be a macroscopic distance travelled on the boundary, which results in picking up normal scaling of fluctuations. We introduce the scaling parameter t and coordinates

$$m(t) = (1-\varrho)^2 t \qquad\text{and}\qquad n(t) = \varrho^2 t \tag{3.4}$$

(notice how we already lost integer parts), hence quantities G (last passage time to $(m,\,n)$), Z (exit point of the longest path to $(m,\,n)$), A_x (last passage time to $(m,\,n)$ in the bulk, after exiting the boundary at x) will all just receive an argument (t).

The proof works with comparison to LPP with another density $\lambda>\varrho$, while the coordinates (3.4) are fixed with ϱ. Such comparisons are already familiar from (3.1), which is used on the South boundary as before. This time, we also couple the weights on the West boundary in a similar way,

$$\xi_{0,j} := \frac{\varrho}{\lambda}\omega_{0,j} \sim \text{i.i.d. Exp}(\lambda) \qquad (j>0).$$

Hence the λ parameter LPP model that uses weights $\xi_{>0,0}$, $\xi_{0,>0}$ and $\omega_{>0,>0}$ is also stationary. However, the corner (3.4) is *not* characteristic for this one as (3.4) uses the parameter ϱ, rather than λ, for finding the characteristic position. Quantities of the λ-LPP will be denoted by a superscript $^\lambda$. Notice that the bulk last passage times $A_x(t)$ are common between the two models, they do not notice the boundary values.

Since $G^\lambda(t)$ optimizes its path, for any $1\le z\le m(t)$ we have $U_z^\lambda + A_z(t)\le G^\lambda(t)$. This is used below, where we start looking into the distribution of the exit point $Z(t)$ (in the ϱ-LPP model). For any $u>0$,

$$\begin{aligned}\{Z(t)>u\} &= \{\exists z>u\,:\,U_z + A_z(t) = G(t)\}\\ &\subseteq \{\exists z>u\,:\,U_z + G^\lambda(t) - U_z^\lambda \ge G(t)\}\\ &= \{\exists z>u\,:\,U_z^\lambda - U_z \le G^\lambda(t) - G(t)\}\\ &\subseteq \{U_u^\lambda - U_u \le G^\lambda(t) - G(t)\}.\end{aligned}$$

In the last step we used that the South weights are heavier in the λ process than with the ϱ parameter, hence $U_z^\lambda - U_z - (U_u^\lambda - U_u)\ge 0$.

We now center our random variables, for any X with a finite mean, $\widetilde{X} := X - \mathbb{E}\,X$. We have

$$\{Z(t)>u\}\subseteq\big\{\widetilde{U}_u^\lambda - \widetilde{U}_u \le \widetilde{G}^\lambda(t) - \widetilde{G}(t) - \mathbb{E}[U_u^\lambda - U_u - G^\lambda(t) + G(t)]\big\}. \tag{3.5}$$

The expectations are known by U's being the sum of i.i.d. variables, and via (2.6). Disregarding integer parts and using (3.4),

$$\begin{aligned}\mathbb{E}[U_u^\lambda - U_u - G^\lambda(t) + G(t)] &= \frac{u}{1-\lambda} - \frac{u}{1-\varrho} - \frac{(1-\varrho)^2 t}{1-\lambda} - \frac{\varrho^2 t}{\lambda} + \frac{(1-\varrho)^2 t}{1-\varrho} + \frac{\varrho^2 t}{\varrho}\\ &= t - \frac{u}{1-\varrho} - \frac{(1-\varrho)^2 t - u}{1-\lambda} - \frac{\varrho^2 t}{\lambda}.\end{aligned} \tag{3.6}$$

The case $u \geq (1-\varrho)^2 t$ might give some trouble, but it is not important and can be handled so we do not deal with this here. To make our bounds the sharpest, we want this expectation to be as large as possible. To this order, we maximise this display in λ for a given $u < (1-\varrho)^2 t$. The maximum is achieved when

$$\frac{\varrho^2 t}{\lambda^2} = \frac{(1-\varrho)^2 t - u}{(1-\lambda)^2}$$

for a $0 < \lambda < 1$. When $\varrho^2 t = (1-\varrho)^2 t - u$, the solution is $\lambda = \frac{1}{2}$, otherwise

$$\lambda_{1,2} = \frac{-2\varrho^2 t \pm \sqrt{4\varrho^2 t^2 + 4\varrho^2 t(t - 2\varrho t - u)}}{2(t - 2\varrho t - u)} = \varrho \cdot \frac{\varrho \mp \sqrt{(1-\varrho)^2 - \frac{u}{t}}}{\varrho^2 - \left((1-\varrho)^2 - \frac{u}{t}\right)} = \frac{\varrho}{\varrho \pm \sqrt{(1-\varrho)^2 - \frac{u}{t}}}.$$

The solution in the interval (0, 1) is obtained by the + sign. Hence from now on we'll use

$$\lambda = \frac{\varrho}{\varrho + \sqrt{(1-\varrho)^2 - \frac{u}{t}}} > \frac{\varrho}{\varrho + \sqrt{(1-\varrho)^2}} = \varrho, \qquad \frac{1}{\lambda} = 1 + \frac{\sqrt{(1-\varrho)^2 - \frac{u}{t}}}{\varrho}$$

$$1 - \lambda = \frac{\sqrt{(1-\varrho)^2 - \frac{u}{t}}}{\varrho + \sqrt{(1-\varrho)^2 - \frac{u}{t}}}, \qquad \frac{1}{1-\lambda} = 1 + \frac{\varrho}{\sqrt{(1-\varrho)^2 - \frac{u}{t}}}.$$

Plugging this back to (3.6),

$$\begin{aligned}
\mathbb{E}[U_u^\lambda - U_u - G^\lambda(t) + G(t)] &= t - \frac{u}{1-\varrho} - \left[(1-\varrho)^2 t - u\right] - \varrho t\sqrt{(1-\varrho)^2 - \frac{u}{t}} - \varrho^2 t - \varrho t\sqrt{(1-\varrho)^2 - \frac{u}{t}} \\
&= 2\varrho(1-\varrho)t - u\frac{\varrho}{1-\varrho} - 2\varrho t\sqrt{(1-\varrho)^2 - \frac{u}{t}} \\
&= 2\varrho(1-\varrho)t\left(1 - \frac{1}{2}\frac{1}{(1-\varrho)^2}\frac{u}{t} - \sqrt{1 - \frac{1}{(1-\varrho)^2}\frac{u}{t}}\right).
\end{aligned}$$

The next lemma is essentially a second order Taylor expansion. These always appear with fluctuation arguments, recall e.g. the proof of the Central Limit Theorem. Somewhere deep, it is the convexity of the shape of LPP that allows this Taylor expansion.

Lemma 3.5. *For any* $0 < a < 1$,

$$1 - \frac{1}{2}a - \sqrt{1-a} \geq \frac{1}{8}a^2.$$

Proof. We want to show $\sqrt{1-a} \leq 1 - \frac{1}{2}a - \frac{1}{8}a^2$. This is equivalent to

$$1 - a \leq 1 + \frac{1}{4}a^2 + \frac{1}{64}a^4 - a - \frac{1}{4}a^2 + \frac{1}{8}a^3 \qquad \text{or} \qquad 0 \leq \frac{1}{64}a^4 + \frac{1}{8}a^3.$$

□

With $a = \frac{1}{(1-\varrho)^2}\frac{u}{t}$ in this lemma we arrive to

$$\mathbb{E}[U_u^\lambda - U_u - G^\lambda(t) + G(t)] \geq \frac{\varrho}{4(1-\varrho)^3} \cdot \frac{u^2}{t}.$$

With this to our help we split (3.5):

$$\begin{aligned}
\{Z(t) > u\} &\subseteq \left\{\widetilde{U}_u^\lambda - \widetilde{U}_u \leq \widetilde{G}^\lambda(t) - \widetilde{G}(t) - \frac{\varrho}{4(1-\varrho)^3} \cdot \frac{u^2}{t}\right\} \\
&\subseteq \left\{\widetilde{U}_u^\lambda - \widetilde{U}_u \leq -\frac{\varrho}{8(1-\varrho)^3} \cdot \frac{u^2}{t}\right\} \cup \left\{\widetilde{G}^\lambda(t) - \widetilde{G}(t) \geq \frac{\varrho}{8(1-\varrho)^3} \cdot \frac{u^2}{t}\right\}.
\end{aligned} \tag{3.7}$$

3.4 A loop of inequalities

We further restrict u from $u < (1-\varrho)^2 t$ to $u < \frac{3}{4}(1-\varrho)^2 t$; again, the complement can be handled and is not relevant. On both events of the union in (3.7), Chebyshev's inequality applies. For the first one, by (3.2),

$$\begin{aligned}\mathbb{P}\Big\{\widetilde{U}^\lambda_u - \widetilde{U}_u \le -\frac{\varrho}{8(1-\varrho)^3}\cdot\frac{u^2}{t}\Big\} &\le \mathbb{V}\mathrm{ar}\big(U^\lambda_u - U_u\big)\cdot\frac{64(1-\varrho)^6}{\varrho^2}\cdot\frac{t^2}{u^4}\\ &= u\Big(\frac{1}{1-\lambda}-\frac{1}{1-\varrho}\Big)^2\cdot\frac{64(1-\varrho)^6}{\varrho^2}\cdot\frac{t^2}{u^4}\\ &= u\Big(1+\frac{\varrho}{\sqrt{(1-\varrho)^2-\frac{u}{t}}}-\frac{1}{1-\varrho}\Big)^2\cdot\frac{64(1-\varrho)^6}{\varrho^2}\cdot\frac{t^2}{u^4}\\ &\le u\Big(\frac{\varrho}{\sqrt{\frac{1}{4}(1-\varrho)^2}}-\frac{\varrho}{1-\varrho}\Big)^2\cdot\frac{64(1-\varrho)^6}{\varrho^2}\cdot\frac{t^2}{u^4} = 64(1-\varrho)^4\frac{t^2}{u^3}.\end{aligned}$$

Chebyshev's inequality on the second event of (3.7) gives

$$\mathbb{P}\Big\{\widetilde{G}^\lambda(t) - \widetilde{G}(t) \ge \frac{\varrho}{8(1-\varrho)^3}\cdot\frac{u^2}{t}\Big\} \le \mathbb{V}\mathrm{ar}\big(G^\lambda(t) - G(t)\big)\cdot\frac{64(1-\varrho)^6}{\varrho^2}\cdot\frac{t^2}{u^4}.$$

Homework 3.6. For any random variables X and Y with finite second moments, $\mathbb{V}\mathrm{ar}(X+Y) \le 2\,\mathbb{V}\mathrm{ar}\,X + 2\,\mathbb{V}\mathrm{ar}\,Y$.

Using Lemma 3.4, it turns out that $\mathbb{V}\mathrm{ar}\,G^\lambda(t)$ can be bounded by $\mathbb{V}\mathrm{ar}\,G(t)$ plus an error term that's smaller order than the quantities we are dealing with. Hence we can proceed with the above display as

$$\le \mathrm{const}\cdot\mathbb{V}\mathrm{ar}\,G(t)\cdot\frac{t^2}{u^4} + \mathrm{error} = \mathrm{const}\cdot\mathbb{E}\,U_{Z^+(t)}\cdot\frac{t^2}{u^4} + \mathrm{error}.$$

In the last step we used Lemma 3.4 with the characteristic position (3.4).

Combining it all, we now have

$$\mathbb{P}\{Z^+(t) > u\} = \mathbb{P}\{Z(t) > u\} \le \mathrm{const}\cdot\frac{t^2}{u^3} + \mathrm{const}\cdot\mathbb{E}\,U_{Z^+(t)}\cdot\frac{t^2}{u^4} + \mathrm{error}.$$

The weight $U_{Z^+(t)}$ collected on the South axis and the exit point $Z^+(t)$ are not that much different, as the former is the sum of the latter many i.i.d. exponentials. A large deviation argument connects these two rather strongly. Hence one can further transform the last display into

$$\mathbb{P}\{U_{Z^+(t)} > y\} \le \mathrm{const}\cdot\frac{t^2}{y^3} + \mathrm{const}\cdot\mathbb{E}\,U_{Z^+(t)}\cdot\frac{t^2}{y^4} + \mathrm{error}.$$

Now, let us abbreviate $E = \mathbb{E}\,U_{Z^+(t)}$. With $v = \frac{y}{E}$,

$$\begin{aligned}E = \int_0^\infty \mathbb{P}\{U_{Z^+(t)} > y\}\,\mathrm{d}y = E\int_0^\infty \mathbb{P}\{U_{Z^+(t)} > Ev\}\,\mathrm{d}v \le E\int_{\frac{1}{2}}^\infty \mathbb{P}\{U_{Z^+(t)} > Ev\}\,\mathrm{d}v + \frac{E}{2}\\ \le \mathrm{const}\cdot E\int_{\frac{1}{2}}^\infty \frac{t^2}{E^3v^3}\,\mathrm{d}v + \mathrm{const}\cdot E\int_{\frac{1}{2}}^\infty E\frac{t^2}{E^4v^4}\,\mathrm{d}v + \mathrm{error} + \frac{E}{2} = \mathrm{const}\cdot\frac{t^2}{E^2} + \mathrm{error} + \frac{E}{2}.\end{aligned}$$

Believe me that the error is not relevant, and rearrange to $E^3 \le \mathrm{const}\cdot t^2$. Lemma 3.4 then transfers the result to $\mathbb{V}\mathrm{ar}\,G(t)$.

4 A few more homeworks on stationarity

Homework 4.1 (The reversed LPP)**.** Fix $(m,\,n)$ and define $H_{i,j} = G_{m,n} - G_{i,j}$ for all $0 \le i \le m$ and $0 \le j \le n$. Show that the $X_{i,j}$ variables of (2.5) act as i.i.d. weights for the last passage times $H_{i,j}$ looked backward from the corner $(m,\,n)$. Notice that in the stationary version even the boundary and the bulk weight distributions work perfectly for this reversed process.

Homework 4.2 (Competition interface)**.** The competition interface was explored by Ferrari, Pimentel and Martin [13, 11, 12]. The idea is this: colour site (1, 0) red and (0, 1) blue. The longest path to any $(i,\,j)$ will first visit one of these two sites. Colour $(i,\,j)$ accordingly. Doing this for all sites $(i,\,j)$ partitions $\mathbb{Z}^+\times\mathbb{Z}^+$ into two sets of vertices. The boundary between these sets is called the *competition interface*. Describe the local evolution of this path in terms of the last passage times $G_{i,j}$.

Homework 4.3 (Longest path and competition interface)**.** Show that the competition interface is exactly the longest path in the reversed LPP.

Homework 4.4 (Stationarity on steroids)**.** This one is from Emrah, Janjigian and Seppäläinen [9] in this context, although it first appeared in Rains [14]. Consider the model of Section 3.1:

- the South boundary is i.i.d. $\mathrm{Exp}(1-\lambda)$;
- the West boundary is i.i.d. $\mathrm{Exp}(\varrho)$;
- the bulk weights are i.i.d. $\mathrm{Exp}(1)$.

This model is *not* stationary unless $\lambda=\varrho$. Call its last passage times $K_{i,j}$. Prove

$$\mathbb{E}\,\mathrm{e}^{(\varrho-\lambda)K_{m,n}}=\Big(\frac{1-\lambda}{1-\varrho}\Big)^m\cdot\Big(\frac{\varrho}{\lambda}\Big)^n.$$

It is very difficult to say anything about a non-stationary last passage time like $K_{m,n}$. And yet here is a completely explicit form of its moment generating function at $\varrho-\lambda$. Unfortunately this function cannot be evaluated this way at any other point. Nevertheless, this identity is the starting point of a fundamental improvement of the probabilistic method for last passage time fluctuations.

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