On the forest fires that Bálint ignited

Balázs Ráth (BME, Rényi)

Stochastics and Influences Workshop

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Tracing back the source of the fire chronologically

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Figure: Piero di Cosimo: The forest fire (c. 1505)

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- Kiss, Manolescu, Sidoravicius (2015): nonexistence of planar forest-fire model

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  - $\lambda(n) \to 0$: the fire does no harm small components
  - $n\lambda(n) \to \infty$: giant components burn very fast

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Well-posed system of ODEs!

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- The functions $v_k(\cdot)$ do not depend on the exact decay rate of $\lambda(n)$, i.e., we do not have to fine-tune the model to see permanent criticality in the limit: S.O.C.

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Open question (up to this day): do we have

$$\lim_{t \to \infty} v_k(t) = v_k(\infty)\,?$$

Controlled Burgers' equation

Generating function / Laplace transform:

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Erdős-Rényi $\mapsto$ Burgers' equation

giant component $\mapsto$ shockwave

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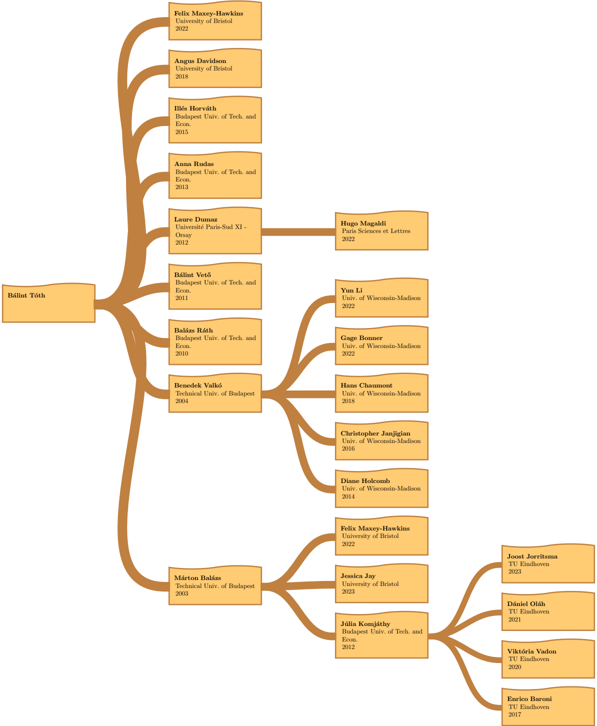
Other supercritical branching processes initiated by Bálint?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Angus Davidson
University of Bristol
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Illés Horváth
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2015
Anna Rudas
Budapest Univ. of Tech. and Econ.
2013
Laure Dumaz
Université Paris-Sud XI - Orsay
2012
Hugo Magaldi
Paris Sciences et Lettres
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2011
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Hans Chaumont
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2018
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Diane Holcomb
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Márton Balázs
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TU Eindhoven
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Dániel Oláh
TU Eindhoven
2021
Viktória Vadon
TU Eindhoven
2020
Enrico Baroni
TU Eindhoven
2017

Many happy returns of the day, Bálint!