The Brownian web and the Brownian web distance

Bálint Vető

Budapest University of Technology and Economics
HUN–REN Rényi Institute

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Outline

- Introduction
- "True" self-avoiding walk and Ray–Knight type results
- Brownian web
- Random walk web distance (joint work with Bálint Virág)
- Brownian web distance: a new universality class and relation to KPZ
- A weighted version: the Bernoulli-Exponential last passage percolation

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Introduction

I've known Bálint for 25 years.

This is 60% of my lifetime and 36% of his.

What attracted me to become his PhD student:

- concrete and interesting problems
- crystal clear presentation
- enthusiastic explanations
- generous and kind personality

"True" self-avoiding walk with edge repulsion

D. Amit, G. Parisi, L. Peliti, 1983

X_n nearest neighbour random walk on $\mathbb{Z}$ (also defined on $\mathbb{Z}^d$)

Local time on edges:

$$l(n,k) := \left|\, \{ i \in \{0,1,\dots,n-1\} : \{X_i, X_{i+1}\} = \{k, k+1\} \} \,\right|$$

Jump probabilities:

$$\begin{aligned} \mathrm{P}(X_{n+1} = X_n + 1 | X_0, X_1, \dots, X_n) \\ = \frac{\exp(-2\beta l(n, X_n))}{\exp(-2\beta l(n, X_n - 1)) + \exp(-2\beta l(n, X_n))} \end{aligned}$$

for some $\beta > 0$.

The walker is pushed towards less visited areas by its own local time.

Expected to be superdiffusive on $\mathbb{Z}$.

Ray–Knight approach

Jump probabilities with local time difference $D_n(k) = l(n,k) - l(n,k-1)$:

$$\begin{aligned}
\mathrm{P}(X_{n+1} = X_n + 1 | X_0, X_1, \dots, X_n) \\
&= \frac{\exp(-2\beta l(n, X_n))}{\exp(-2\beta l(n, X_n - 1)) + \exp(-2\beta l(n, X_n))} \\
&= \frac{\exp(-\beta D_n(X_n))}{\exp(-\beta D_n(X_n)) + \exp(\beta D_n(X_n))}
\end{aligned}$$

Local time difference is Markovian:

$$\mathrm{P}(D_{n+1}(k) = D_n(k) + 1 | X_n = k) = \frac{\exp(-\beta D_n(k))}{\exp(-\beta D_n(k)) + \exp(\beta D_n(k))}$$

Ray–Knight approach

The Markov chain $D_n(k)$ is close to its explicit symmetric stationary distribution for large n.

Local time differences at different locations are independent.

Local time profile is a random walk: given $l(n,k)=l$,

$$l(n,k+1)=l+D_{\tau_l(k)}(k)$$

where $\tau_j(k)=\min\{n:l(n,k)\geq j\}$ is the jth use of the edge $\{k,k+1\}$ (inverse local time)

After scaling ($n^{2/3}$ in space and $n^{1/3}$ for the local time), local time profile becomes Brownian.

Scaling limit of local times given by the Brownian web, scaling limit of the position is the true self-repelling motion (Tóth and Werner, 1998)

Ray–Knight type results

- Tóth, 1995
  - convergence of the local time profile to a reflected and absorbed Brownian motion
  - convergence of the "true" self-avoiding walk at large independent random times
- Tóth, V., 2008
  - "true" self-avoiding walk with directed edges: local time profile is a random walk with drift
  - deterministic limit shape of local times, uniformly distributed position
- Tóth, V., 2011
  - "true" self-avoiding walk in continuous time, local time on sites
  - Brownian limit of local times, convergence of position at random times
- Kosygina, Peterson, 2025
  - joint convergence of local time profile at multiple points
  - process convergence of the "true" self-avoiding walk to true self-repelling motion

Brownian web and its dual

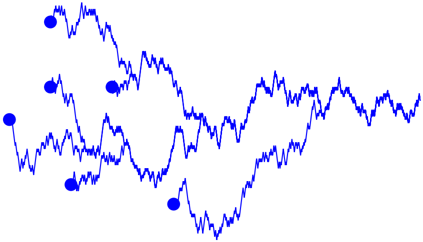
Brownian web: coalescing Brownian motions starting at all $(t, x) \in \mathbb{R}^2$

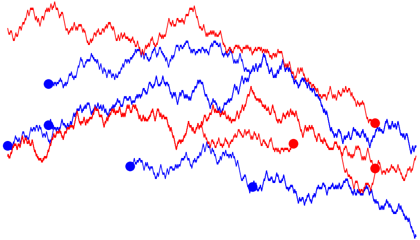
History:

- Arratia, 1979, unpublished
- Tóth, Werner, 1998, construction, special points, local time of true self-repelling motion
- Fontes, Isopi, Newman, Ravishankar, 2004, topology, "Brownian web"

Dual: coalescing backward Brownian motions

Forward and backward paths do intersect but they do not cross





Special points of the Brownian web

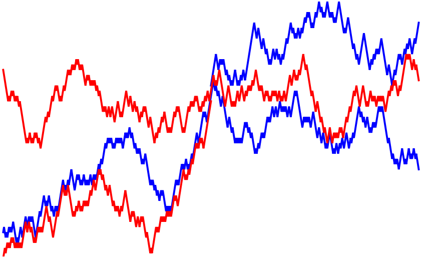
Special points: point of type $(m_{\rm in}, m_{\rm out})$ has $m_{\rm in}$ incoming and $m_{\rm out}$ outgoing paths

Possible types: $(0,1)$, $(0,2)$, $(0,3)$, $(1,1)$, $(1,2)$, $(2,1)$

Almost all points of $\mathbb{R}^2$ are of type $(0,1)$

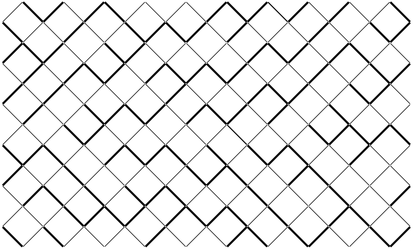
Characterization of $(1,2)$ points (see figure): those hit by a forward and a backward path

Brownian web: unique path starting from almost every point of $\mathbb{R}^2$, an additional path at each $(1,2)$ point



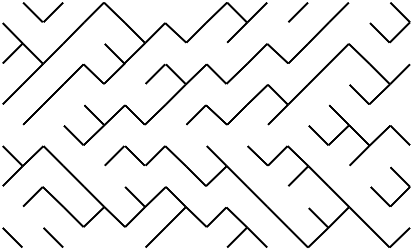
Random walk web

- Lattice:
  $\{(i, n) \in \mathbb{Z}^2 : i + n \text{ is even}\}$
  with directed lattice edges from (i, n) to $(i + 1, n \pm 1)$
- Graph of free edges: exactly one of the outgoing lattice edges with equal probabilities independently, i.e. coalescing random walks to the right



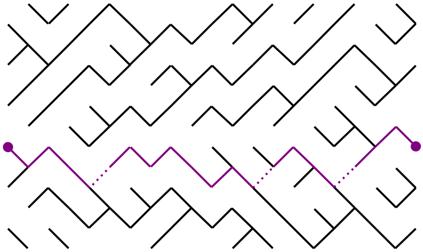
Random walk web

- Lattice: $\{(i,n) \in \mathbb{Z}^2 : i+n \text{ is even}\}$ with directed lattice edges from (i,n) to $(i+1, n \pm 1)$
- Graph of free edges: exactly one of the outgoing lattice edges with equal probabilities independently, i.e. coalescing random walks to the right
- Edge weights: edges of the graph with weight 0, other lattice edges with weight 1
- Distance $D^{\mathrm{RW}}(i,n;j,m)$: weight of the directed path between (i,n) and (j,m) with minimal total weight



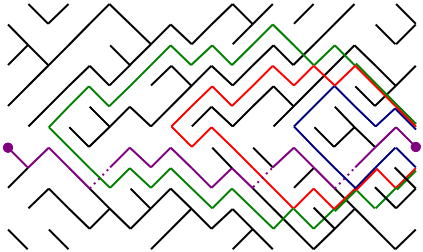
Random walk web distance

- Distance $D^{\mathrm{RW}}(i, n; j, m)$: weight of the directed path between (i, n) and (j, m) with minimal total weight
- In other words: minimal number of jumps to get from (i, n) to (j, m)



Random walk web distance

- Distance $D^{\mathrm{RW}}(i, n; j, m)$: weight of the directed path between (i, n) and (j, m) with minimal total weight
- In other words: minimal number of jumps to get from (i, n) to (j, m)
- Blue, red, green regions: set of starting points with 0, 1 and 2 jumps to the purple target point
- Aim: distance function between remote points, scaling, continuum limit



Brownian web distance

Brownian web distance $D^{\mathrm{Br}}(t,x;s,y)$: minimal number of jumps from (t,x) to (s,y) using Brownian web paths and with jumps at $(1,2)$ points from the incoming path to the additional path

Basic properties:

- D^{Br} is integer valued
- D^{Br} is non-symmetric
- $D^{\mathrm{Br}}(t,x;s,y)=\infty$ for a typical (s,y) which is not hit by a Brownian web path
- $D^{\mathrm{Br}}(t,x;t,x)=0$
- Triangle inequality:

$$D^{\mathrm{Br}}(t,x;s,y)\le D^{\mathrm{Br}}(t,x;u,z)+D^{\mathrm{Br}}(u,z;s,y)$$

Lower semicontinuous version $D^{\mathrm{Br,LSC}}(t,x;s,y)$ differs from $D^{\mathrm{Br}}(t,x;s,y)$ by at most 1 only for special (t,x)

Main results

0:1:2 scale invariance (c.f. 1:2:3 scaling in the KPZ class):

Proposition

For all $\alpha > 0$, it holds that

$$D^{\mathrm{Br}}(\alpha^2 t, \alpha x; \alpha^2 s, \alpha y) \stackrel{\mathrm{d}}{=} D^{\mathrm{Br}}(t, x; s, y).$$

Convergence:

Theorem (B. V., B. Virág, 2023)

- *The Brownian web distance as a function $D^{\mathrm{Br,LSC}} : \mathbb{R}^4 \to \mathbb{R} \cup \{\infty\}$ is almost surely lower semicontinuous.*
- *There is a coupling of the underlying random walk webs and Brownian web such that*

$$D^{\mathrm{RW}}(nt, n^{1/2}x; ns, n^{1/2}y) \to D^{\mathrm{Br,LSC}}(t, x; s, y)$$

as $n \to \infty$ almost surely in the epigraph sense.

KPZ limit after a shear mapping

Theorem (B. V., B. Virág, 2023)

For $\eta \in (0,1)$, we have as $n \to \infty$ that

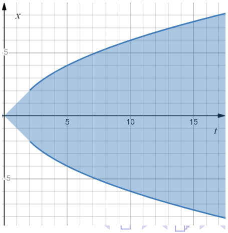
$$\frac{\frac{\eta^2 n}{4} + \frac{\eta^{4/3} z n^{2/3}}{2^{1/3}} - D^{\mathrm{Br}}(-n, \eta n + 2^{2/3}\eta^{1/3} z n^{2/3}; 0, \mathbb{R}_-)}{2^{-2/3}\eta^{2/3} n^{1/3}} \stackrel{\mathrm{d}}{\Longrightarrow} \mathcal{L}(0,0;z,1)$$

and

$$\frac{c_1(\eta)n - c_2(\eta) z n^{2/3} - D^{\mathrm{RW}}(-n, \eta n + c_3(\eta) z n^{2/3}; 0, \mathbb{Z}_-)}{c_4(\eta) n^{1/3}} \stackrel{\mathrm{d}}{\Longrightarrow} \mathcal{L}(0,0;z,1)$$

where $\mathcal{L}$ is the directed landscape and $\mathcal{L}(0,0;z,1) = \mathcal{A}(z) - z^2$ is the parabolic Airy process.

The function $c_1(\eta) = \frac{1-\sqrt{1-\eta^2}}{2}$ for $\eta \in (0,1)$ determines the limit shape of discs in D^{RW} in directions different from horizontal: $t - \sqrt{t^2 - x^2} \leq 2$



Directed landscape and the 1:2:3 scaling

Directed landscape $\mathcal{L}(x,t;y,s)$ constructed by Dauvergne, Ortmann, Virág, 2018

Last passage percolation: let ξ_{ij} for $(i,j)\in\mathbb{Z}^2$ be i.i.d. random variables with geometric or exponential distribution. The last passage value

$$L((i,j),(k,l)) = \sup_{\pi:(i,j)\nearrow(k,l)} \sum_{(a,b)\in\pi} \xi_{ab}$$

Theorem (Dauvergne, Virág, 2022)

$$\frac{L(tn^3u+xn^2v, sn^3u+yn^2v)-\alpha n^3(t-s)}{\chi n} \xrightarrow{\mathrm{d}} \mathcal{L}(x,t;y,s)$$

as $n\to\infty$ where $u=(1,1)$, $v=(1,-1)$ and α,χ are explicit constants.

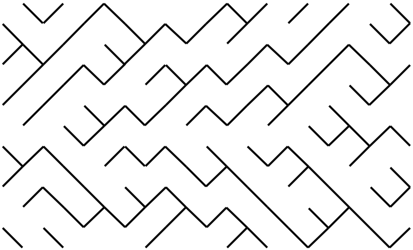
Proposition (1:2:3 scale invariance of directed landscape)

For $\alpha>0$,

$$\alpha^{-1}\mathcal{L}(\alpha^2x,\alpha^3t;\alpha^2y,\alpha^3s) \stackrel{\mathrm{d}}{=} \mathcal{L}(x,t;y,s)$$

Bernoulli-Exponential first passage percolation

- Lattice:
  $\{(i,n) \in \mathbb{Z}^2 : i+n \text{ is even}\}$
  with directed lattice edges from
  (i,n) to $(i+1, n\pm 1)$
- Free edges: coalescing random walks
- Edge weights: 0 for free edges, independent EXP(1) weights for all other edges
- Distance $T(i,n;j,m)$: weight of the directed path between (i,n) and (j,m) with minimal total weight



$$T(i,n;j,[m,\infty)) = \min_{l\in[m,\infty)} T(i,m;j,l)$$

Results on Bernoulli-Exponential first passage percolation

Based on explicit formulas by Barraquand–Corwin, 2017:

Theorem (B.V., 2024)

For any $h \in \mathbb{R}$, as $n \to \infty$,

$$\sqrt{n}T(0,0;n,[h\sqrt{n},\infty)) \stackrel{\mathrm{d}}{\Longrightarrow} T_h$$

where the distribution of T_h can be given explicitly.

Height function $H(n,r) = \max\{k \in \mathbb{Z} : T(0,0;n,k) \leq r\}$: as $n \to \infty$

$$n^{-1/2} H\left(n, sn^{-1/2}\right) \stackrel{\mathrm{d}}{\Longrightarrow} H_s$$

- For $s = 0$, H_0 is Gaussian
- As $s \to \infty$, $2^{4/9}3^{-1/3}s^{1/9}\left(H_s - 2^{-2/3}3s^{1/3}\right) \stackrel{\mathrm{d}}{\Longrightarrow} \mathrm{TW}$

Conjecture (Convergence and -1:1:2 scaling)

$$n^{1/2}T(nt, n^{1/2}x; ns, n^{1/2}y) \to D^{\mathrm{BN}}(t,x;s,y)$$

as $n \to \infty$ where $D^{\mathrm{BN}}(0,0;1,[h,\infty)) \stackrel{\mathrm{d}}{=} T_h$ and

$$n^{1/2}D^{\mathrm{BN}}(nt, n^{1/2}x; ns, n^{1/2}y) \stackrel{\mathrm{d}}{=} D^{\mathrm{BN}}(t,x;s,y).$$

The end

Happy birthday, Bálint!

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