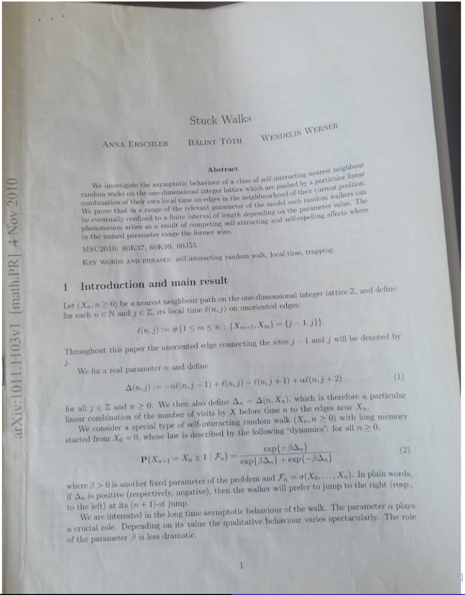
A story of self-interacting random walks getting stuck in atypical behaviours

Daniel Kious

University of Bath

Bálint's birthday conference

First encounter with Bálint - some time in 2012



arXiv:1011.1103v1 [math.PR] 4 Nov 2010

Stuck Walks

Anna Erschler   Bálint Tóth   Wendelin Werner

Abstract

We investigate the asymptotic behaviour of a class of self-interacting nearest neighbour random walks on the one-dimensional integer lattice which are pushed by a particular linear combination of their own local time on edges in the neighbourhood of their current position. We prove that in a range of the relevant parameter of the model such random walkers can be eventually confined to a finite interval of length depending on the parameter value. The phenomenon arises as a result of competing self-attracting and self-repelling effects where in the named parameter range the former wins.

MSC2010: 60K37, 60K99, 60J55

Key words and phrases: self-interacting random walk, local time, trapping

1 Introduction and main result

Let $(X_n, n \geq 0)$ be a nearest neighbour path on the one-dimensional integer lattice $\mathbb{Z}$, and define for each $n \in \mathbb{N}$ and $j \in \mathbb{Z}$, its local time $\ell(n, j)$ on unoriented edges:

$$\ell(n,j) := \#\{1 \leq m \leq n : \{X_{m-1}, X_m\} = \{j-1, j\}\}.$$

Throughout this paper the unoriented edge connecting the sites $j-1$ and j will be denoted by j.

We fix a real parameter α and define

$$\Delta(n,j) := -\alpha\ell(n,j-1) + \ell(n,j) - \ell(n,j+1) + \alpha\ell(n,j+2) \quad (1)$$

for all $j \in \mathbb{Z}$ and $n \geq 0$. We then also define $\Delta_n = \Delta(n, X_n)$, which is therefore a particular linear combination of the number of visits by X before time n to the edges near X_n.

We consider a special type of self-interacting random walk $(X_n, n \geq 0)$ with long memory started from $X_0 = 0$, whose law is described by the following "dynamics": for all $n \geq 0$,

$$\mathbf{P}(X_{n+1} = X_n \pm 1 \mid \mathcal{F}_n) = \frac{\exp\{\pm\beta\Delta_n\}}{\exp\{\beta\Delta_n\} + \exp\{-\beta\Delta_n\}} \quad (2)$$

where $\beta > 0$ is another fixed parameter of the problem and $\mathcal{F}_n = \sigma(X_0, \ldots, X_n)$. In plain words, if Δ_n is positive (respectively, negative), then the walker will prefer to jump to the right (resp., to the left) at its $(n+1)$-st jump.

We are interested in the long time asymptotic behaviour of the walk. The parameter α plays a crucial role. Depending on its value the qualitative behaviour varies spectacularly. The role of the parameter β is less dramatic.

1

First contact with Professor Tóth

From Daniel Kious • danielkious@gmail.com

To balint@math.bme.hu

Cc Pierre Tarres • tarres@maths.ox.ac.uk

Date 5 Sept 2013, 22:06

See security details

Dear Mister Tóth, dear Pierre in Cc,

I am a PhD student of Pierre Tarrès, in Toulouse. I have just finished writing an article about the conjecture introduced in Stuck Walks.

First contact with Professor Tóth

From Pierre Tarres • tarres@maths.ox.ac.uk

To Daniel Kious • danielkious@gmail.com

Date 5 Sept 2013, 22:41

See security details

C'est bien, mais il faut dire Professor plutôt que Mister (aussi en français).

First handshake with Bálint - 19 February 2014

PhD defense - 16 June 2014

Now the maths

Locally self-interacting walks in $\mathbb{Z}$ defined by Erschler, Tóth & Werner (2010/2012)

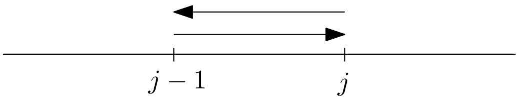
Transition probabilities are functions of the local time profile seen from the walker or of its gradient

Rich behaviour, with many open questions for future generations of Bálint's influencees

Definition

$X := (X_n)_{n\geq 0}$: nearest neighbor walk on the integer lattice $\mathbb{Z}$, starting at 0, i.e. $X_0 = 0$.
Local time, at time k, on the non-oriented **edge** $\{j-1, j\}$:

$$l_k(j) = \sum_{m=1}^{k} \mathbb{1}_{\{\{X_{m-1},X_m\}=\{j-1,j\}\}}.$$



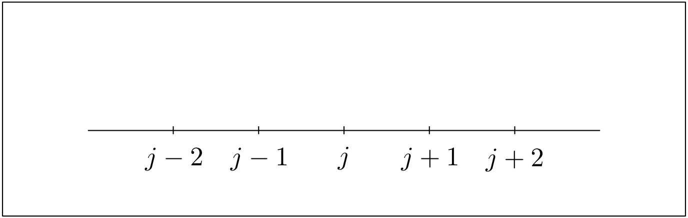
Definition

Fix $\alpha \in \mathbb{R}$. Define the **local stream**:

$$\Delta_k(j) = -\alpha l_k(j-1) + l_k(j) - l_k(j+1) + \alpha l_k(j+2).$$

Conditional transition probability:

$$\mathbb{P}(X_{k+1} = X_k \pm 1 | \mathcal{F}_k) \quad = \quad \frac{e^{\pm\beta\Delta_k(X_k)}}{e^{-\beta\Delta_k(X_k)} + e^{\beta\Delta_k(X_k)}}, \text{ where } \beta > 0.$$



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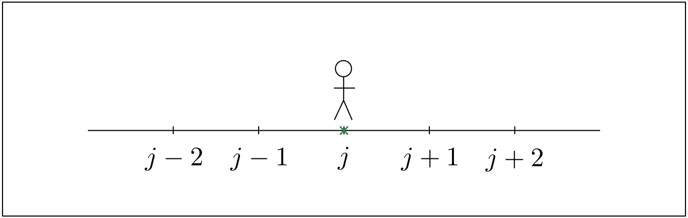

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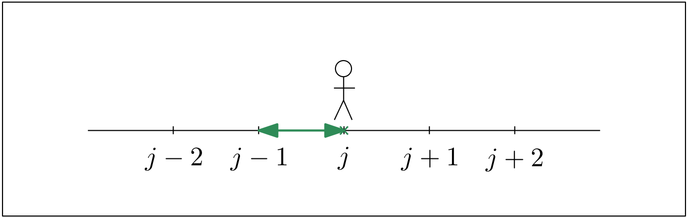

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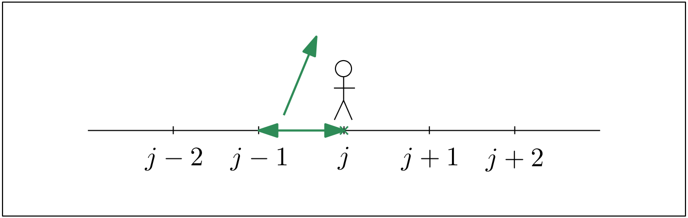

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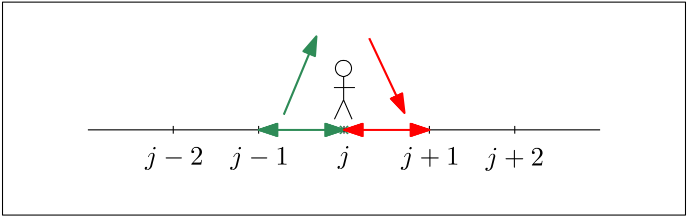

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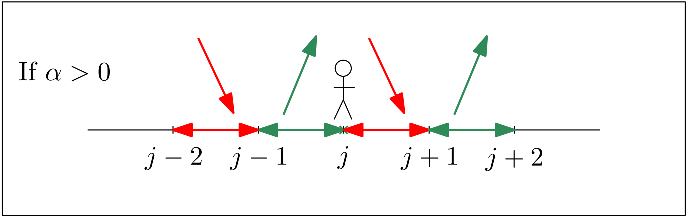

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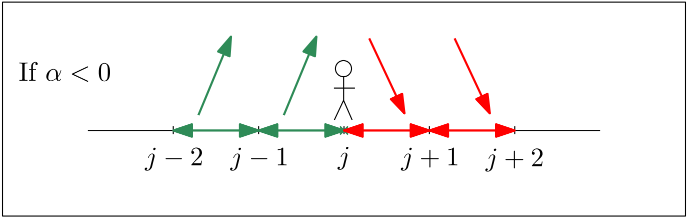


Generalisation of the True self repelling walk, corresponding to $\alpha = 0$

$$\mathbb{P}(X_{k+1} = X_k + 1 | \mathcal{F}_k) = \frac{e^{-2\beta\ell_k(X_k+1)}}{e^{-2\beta\ell_k(X_k)} + e^{-2\beta\ell_k(X_k+1)}}.$$

Non-degenerate scaling limits for $X_k / k^{2/3}$ proved by Tóth (1995)

The TSRW phase (Erschler, Tóth & Werner, 2012)

$$\Delta_k(j) = -\alpha l_k(j-1) + l_k(j) - l_k(j+1) + \alpha l_k(j+2).$$

$$\mathbb{P}(X_{k+1} = X_k + 1 | \mathcal{F}_k) = \frac{e^{\beta \Delta_k(X_k)}}{e^{-\beta \Delta_k(X_k)} + e^{\beta \Delta_k(X_k)}}.$$

Open question: for $\alpha \in [-1, 1/3[$, prove the scaling behaviour is similar to TSRW.

Open question: for $\alpha = 1/3$, the third derivative case, mysterious scaling. $k^{2/5}$? $k^{1/2}$?

The slow phase (Erschler, Tóth & Werner, 2012)

$$\Delta_k(j) = -\alpha l_k(j-1) + l_k(j) - l_k(j+1) + \alpha l_k(j+2).$$

$$\mathbb{P}(X_{k+1} = X_k + 1 | \mathcal{F}_k) = \frac{e^{\beta \Delta_k(X_k)}}{e^{-\beta \Delta_k(X_k)} + e^{\beta \Delta_k(X_k)}}.$$

Open question: $\alpha \in (-\infty, -1)$, prove that the walks builds traps in the environment and that its displacement is atypically slow.

More problems (Erschler, Tóth & Werner, 2012)

$$\mathbb{P}(X_{k+1} = X_k + 1 | \mathcal{F}_k) \quad = \quad \frac{e^{\beta\Delta_k(X_k)}}{e^{-\beta\Delta_k(X_k)} + e^{\beta\Delta_k(X_k)}}.$$

Even more questions remain open when the symmetry is dropped:

• Second derivative: transient deterministic behaviour $X_n \sim \sqrt{2n}$.

$$\Delta_k(j) = -l_k(j-1) + l_k(j) + l_k(j+1) - l_k(j+2).$$

• Logarithmic behaviour $X_n \sim \ln(n)/\ln(2)$.

$$\Delta_k(j) = -2l_k(j-1) + l_k(j) + l_k(j+1)$$

• Ballistic behaviour $X_n \sim cn$.

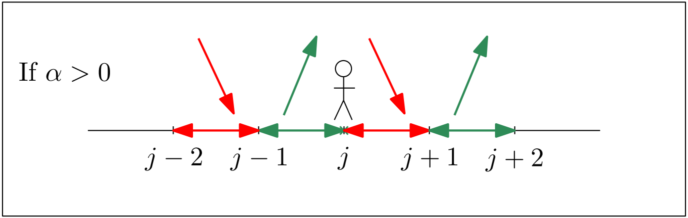
$$\Delta_k(j) = l_k(j-1) + l_k(j) - 2l_k(j+1)$$

The Stuck Walks

We will now focus on the case $\alpha > 1/3$, the Stuck Walks.

$$\Delta_k(j) = -\alpha l_k(j-1) + l_k(j) - l_k(j+1) + \alpha l_k(j+2).$$

$$\mathbb{P}(X_{k+1} = X_k + 1 | \mathcal{F}_k) = \frac{e^{\beta \Delta_k(X_k)}}{e^{-\beta \Delta_k(X_k)} + e^{\beta \Delta_k(X_k)}}.$$



Notations

Define $\alpha_1 = +\infty$ and for all $L \geq 2$:

$$\alpha_L = \frac{1}{1 + 2\cos(\frac{2\pi}{L+2})},$$

so that $(\alpha_L)_{L\geq 1}$ decreases from $+\infty$ to $1/3$.
Let R' be the set of points that are visited **infinitely often**, i.e.

$$R' = \{j \in \mathbb{Z} : l_\infty(j) + l_\infty(j+1) = \infty\}.$$

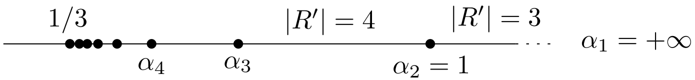
Stuck Walks (PTRF, 2012)

Conjecture (Erschler, Tóth and Werner, 2012)

If $\alpha \in (\alpha_{L+1}, \alpha_L)$, then $|R'| = L + 2$ almost surely.

Theorem (Erschler, Tóth and Werner, 2012)

Suppose that $L \geq 1$. We have:
- If $\alpha < \alpha_L$, then, almost surely, $|R'| \geq L + 2$, or $R' = \emptyset$;
- If $\alpha \in (\alpha_{L+1}, \alpha_L)$, then the probability that $|R'| = L + 2$ is positive.



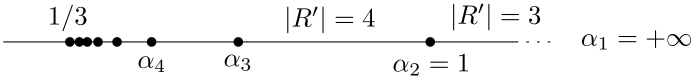
Results

Theorem (K, 2013/ AoP 2016)

Fix $L \geq 1$ and assume that $\alpha \in (\alpha_{L+1}, \alpha_L)$, then the walk localizes on $L+2$ or $L+3$ sites almost surely, i.e. $|R'| \in \{L+2, L+3\}$ a.s.

Theorem (K, 2013/ AoP 2016)

Assume that that $\alpha \in (1, +\infty) = (\alpha_2, \alpha_1)$, then the walk localizes on 3 sites almost surely, i.e. $|R'| = 3$ a.s.

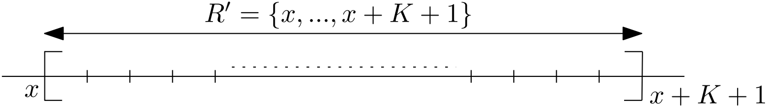


Sketch of proof

There is a **linear system** at the core of the longterm behaviour of this walk. When $X_k = j$, we have

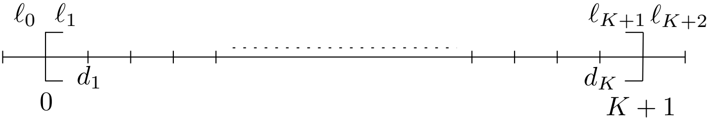
$$\mathbb{P}(X_{k+1} = j \pm 1 | \mathcal{F}_k) \quad \leq \quad e^{\pm 2\beta \frac{\Delta_k(j)}{k} \times k}.$$

This indicates:



$$\text{For any } j \in \{1, ..., K\} : d_j \sim \frac{\Delta_k(x+j)}{k} \to 0$$

$$\text{For any } j \in \{0, ..., K+2\} : l_j \sim \frac{l_k(x+j)}{k} \to ?$$



Solutions $(l_0, ..., l_{K+2})$ of the linear system defined by:

$$d_1 = d_2 = ... = d_K = 0,\ l_0 = l_{K+2} = 0 \text{ and } \sum_{j=1}^{K+1} l_j = 1,$$

where, for all $j \in \{1, ..., K\}$,

$$d_j = -\alpha l_{j-1} + l_j - l_{j+1} + \alpha l_{j+2}.$$

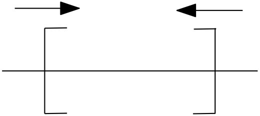
Define $d_0 = -l_1 + \alpha l_2$ and $d_{K+1} = -\alpha l_K + l_{K+1}$.

The linear system

- If $K < L$

$$\frac{\Delta_k(x)}{k} \sim d_0 < 0 \qquad 0 < d_{K+1} \sim \frac{\Delta_k(x+K+1)}{k}$$

- If $K = L$ or $L + 1$



$$\frac{\Delta_k(x)}{k} \sim d_0 > 0 \qquad 0 > d_{K+1} \sim \frac{\Delta_k(x+K+1)}{k}$$

- $K > L + 1$: Get more results in order to emphasize instability of some vertices through the **Rubin's construction**.

Rubin's construction

First used by Davis (1990) and Sellke (1994). Tarrès (2011) introduced a **variant** $\Rightarrow$ **powerful couplings** for VRRW.
Idea: define a continuous-time walk $(\widetilde{X}_t)_t$ on $\mathbb{Z}$ in order to couple to with $(X_k)_k$.

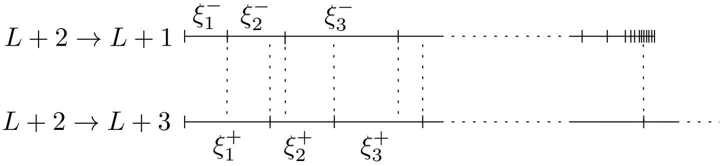
When $X_k = j$:

$$\begin{aligned}\mathbb{P}(X_{k+1} = j+1 | \mathcal{F}_k) &= \frac{e^{-2\beta(\ell_k(j+1)-\alpha\ell_k(j+2))}}{e^{-2\beta(\ell_k(j+1)-\alpha\ell_k(j+2))} + e^{-2\beta(-\alpha\ell_k(j-1)+\ell_k(j))}} \\ &= \mathbb{P}(\xi^{\ell}_{j,j+1} < \xi^{\ell}_{j,j-1})\end{aligned}$$

Rubin's construction

When $X_k = j$:

$$\mathbb{P}(X_{k+1} = j+1 | \mathcal{F}_k) = \frac{e^{-2\beta(\ell_k(j+1)-\alpha\ell_k(j+2))}}{e^{-2\beta(\ell_k(j+1)-\alpha\ell_k(j+2))} + e^{-2\beta(-\alpha\ell_k(j-1)+\ell_k(j))}}$$
$$= \mathbb{P}(\xi^\ell_{j,j+1} < \xi^\ell_{j,j-1})$$



The two papers by A. Erschler, B. Tóth, W. Werner:

- *Stuck Walks*, 2012
- *Some locally self-interacting walks on the integers*

Happy Birthday, Bálint!!