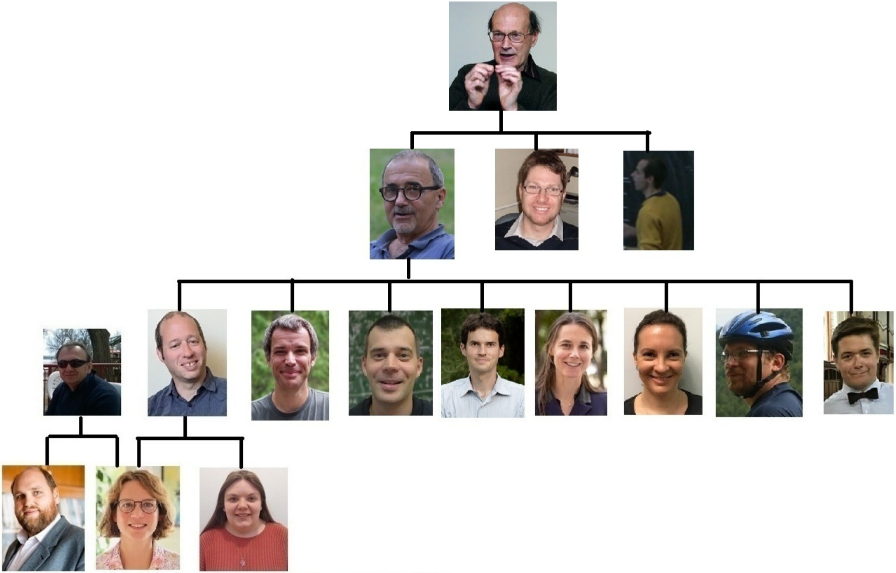
Connecting blocking measures of interacting particle systems with combinatorial objects

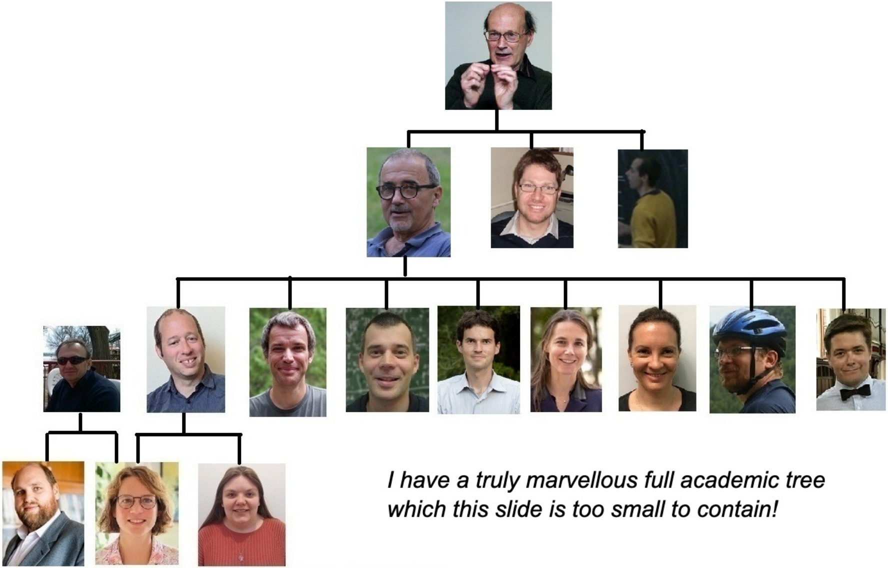
(Based on joint works with Daniel Adams, Márton Balázs, Dan Fretwell and Benjamin Lees)

Jessica Jay

Stochastics and Influences Workshop
On the ocassion of Bálint Tóth's 70^{th} *Birthday*

Lancaster University





I have a truly marvellous full academic tree which this slide is too small to contain!

Probability and Combinatorics

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- Recently Blocking Measures for Interacting Particle Systems have been shown to be linked to combinatorial objects.

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- Balázs and Bowen (2018) gave a purely probabilistic proof of a well-known combinatorial identity:

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$$\sum_{m\in\mathbb{Z}} q^{\frac{m(m+1)}{2}} z^m = \prod_{i\geq 1}(1-q^i)(1+q^i z)(1+q^{i-1}z^{-1})$$

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Jacobi Triple Product (JTP)

For $q \in (0,1)$ and $c \in \mathbb{R}$,

$$\frac{\sum_{m\in\mathbb{Z}} q^{\frac{m(m+1)}{2}-mc}}{\prod_{i\geq 1}(1+q^{i-c})(1+q^{i-1+c})} = \prod_{i\geq 1}(1-q^i)$$

- This paper constructs a **family** of blocking measures for Interacting Particle Systems on $\mathbb{Z}$.
- This proof of the Jacobi Triple Product identity follows from the Exclusion - Zero-range correspondence, by equating blocking measures.

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- Particle - Hole symmetry for ASEP leads to,

Durfee Rectangles Identity

For $q \in (0,1)$ and any fixed $n \in \mathbb{Z}$,

$$\frac{1}{\prod_{i\geq 1}(1-q^i)} = \sum_{k=\max\{-n,0\}}^{\infty} \frac{q^{k(n+k)}}{\prod_{i=1}^{n+k}(1-q^i) \cdot \prod_{j=1}^{k}(1-q^j)}.$$

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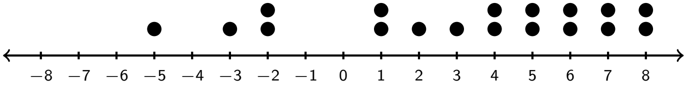
- Are ASEP/AZRP the **only** particle systems that have this connection to combinatorics via these identities?

- If this connection **is** deeper, does it reveal **new results** in Probability or Combinatorics?

0-1-2 Systems on $\mathbb{Z}$



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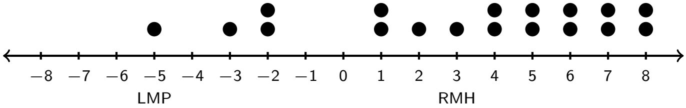

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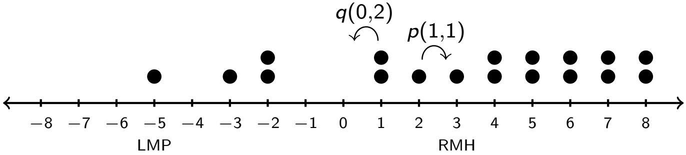

Jump rate functions: Right jump from site i to $i+1$ w/r $p(\eta_i, \eta_{i+1})$. Left jump from site i to $i-1$ w/r $q(\eta_{i-1}, \eta_i)$.

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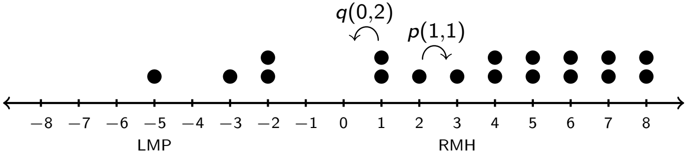

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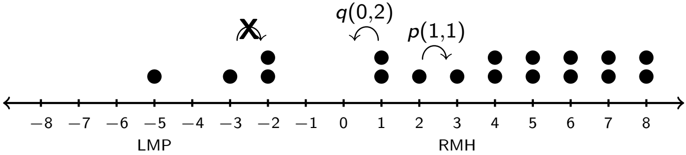

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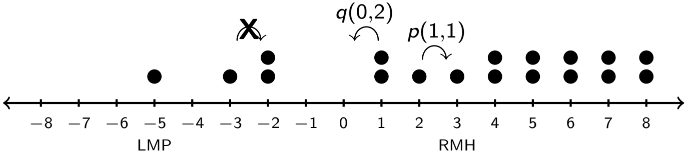

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- Exclusion: $p(0,\cdot) = p(\cdot,2) = q(2,\cdot) = q(\cdot,0) = 0$
- Attractivity: $p(y+1,z) \geq p(y,z)$, $p(y,z+1) \leq p(y,z)$, $q(y+1,z) \leq q(y,z)$, $q(y,z+1) \geq q(y,z)$

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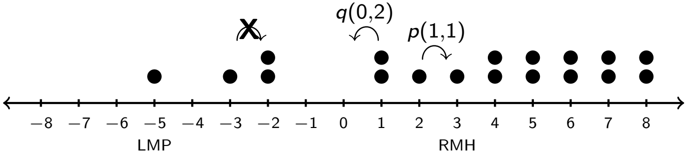

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- Algebraic recursive rates: an explicit condition. The rates can be parametrised by $q \in (0,1)$ and $t \geq 1$,

$$q := \frac{q(0,1)}{p(1,0)}, \qquad t := \left(\frac{p(1,0)q(0,2)}{q(0,1)p(1,1)}\right)^{\frac{1}{2}}.$$

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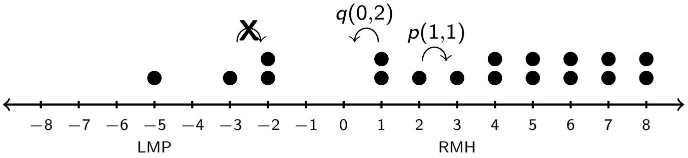

Family of reversible stationary measures

For any $c \in \mathbb{R}$ the following is reversible and stationary for the dynamics,

$$\underline{\mu}^c(\underline{\eta}) = \prod_{i=-\infty}^{0} \frac{t^{\mathbb{I}\{\eta_i=1\}} q^{(c-i)\eta_i}}{(1 + tq^{c-i} + q^{2(c-i)})} \prod_{i=1}^{\infty} \frac{t^{\mathbb{I}\{\eta_i=1\}} q^{(2-\eta_i)(i-c)}}{(1 + tq^{i-c} + q^{2(i-c)})}$$

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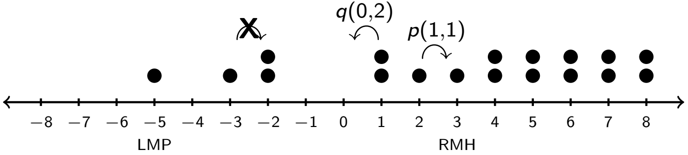

Conserved Quantity: $N(\underline{\eta}) = \sum_{i=1}^{\infty}(2-\eta_i) - \sum_{i=-\infty}^{0}\eta_i < \infty$

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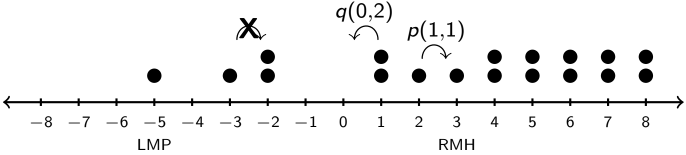

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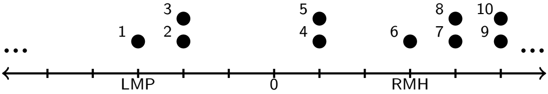
Unique stationary measures

The unique stationary measure on $\{\underline{\eta} : N(\underline{\eta}) = n\}$ (for n even) is given by,

$$\underline{\nu^n}(\underline{\eta}) = \frac{2\sum_{\ell=-\infty}^{\infty} q^{\ell(\ell+1)-2\ell c}\underline{\mu}^c(\underline{\eta})\mathbb{I}\{N(\underline{\eta}) = n\}}{q^{\frac{n(n+2)}{4}-nc}\left(1+\prod_{i=-\infty}^{\infty}(1-2\mu_i^c(1))\right)}$$

An equivalent family of particle systems on $\mathbb{Z}_{<0}$

For each $n \in \mathbb{Z}$ there is a bijection, $T^n : \Omega^n \to \mathcal{H}$, for some specific state space $\mathcal{H} \subset \mathbb{Z}_{\geq 0}^{\mathbb{Z}_{<0}}$.



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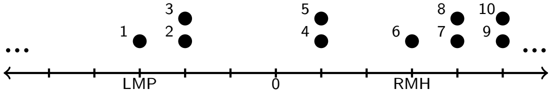

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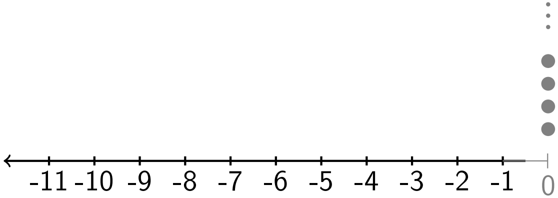

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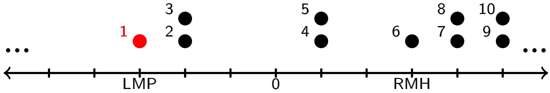

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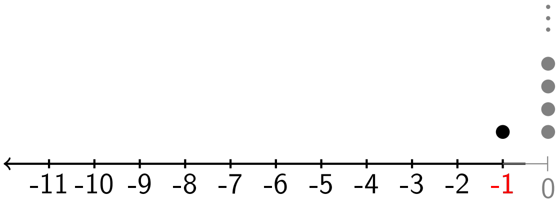

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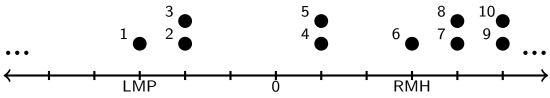

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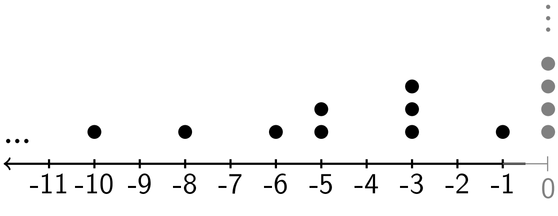

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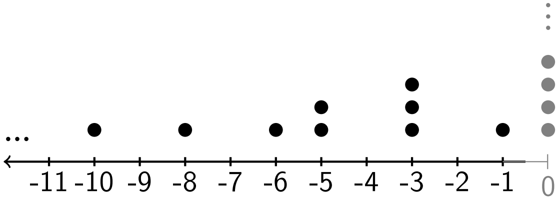
Family of Restricted Particle Systems



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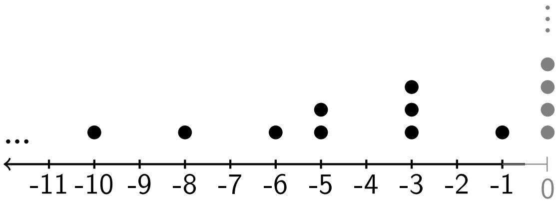

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$$\mathcal{H} \subset \mathcal{H}' := \{\underline{\omega} \in \mathbb{Z}_{\geq 0}^{\mathbb{Z}_{<0}} : \ \omega_{-i} = 0 \Rightarrow \omega_{-i-1} \neq 0, \forall i > 0\}$$

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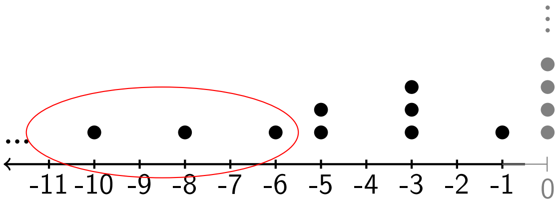

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- Far to the left agrees with 1,0,1,0,1... $\mathcal{H} = \mathcal{H}^e \cup \mathcal{H}^o$

$$\mathcal{H}^e := \{\underline{\omega} \in \mathcal{H}' : \exists N > 0 \ \ \text{s.t}\ \omega_{-i} = \mathbb{I}\{i \text{ even}\} \ \ \forall i \geq N\},$$

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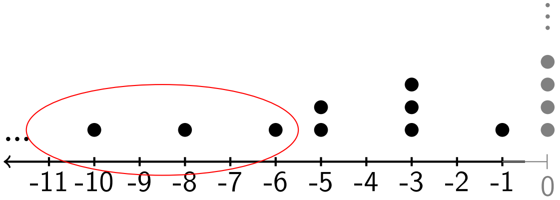

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Dynamics: inherited from family of 0-1-2 systems.

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Reversible stationary measure

On $\mathcal{H}^e$ the **unique reversible stationary** measure is,

$$\underline{\pi}^e(\underline{\omega}) = \frac{q^{\sum\limits_{i \text{ odd}} i\omega_{-i} + \sum\limits_{i \text{ even}} i(\omega_{-i}-1)} \, t^{2\left(\sum\limits_{i \text{ odd}} \mathbb{I}\{\omega_{-i}\geq 1\} - \sum\limits_{i \text{ even}} \mathbb{I}\{\omega_{-i}=0\}\right)}}{S_{\text{even}}(q,t)}$$

Probability Influencing Combinatorics

This equivalence proves **new** Jacobi style identities!

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Theorem (Balázs, Fretwell, J., (2022))

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$$2S_{\text{even}}(q,t)\sum_{\ell\in\mathbb{Z}} q^{\ell(\ell+1)}z^{2\ell}$$

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This identity was found from the probability and then we found the combinatorial interpretation.

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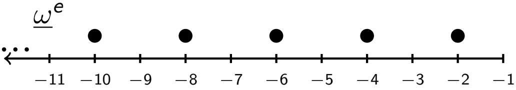

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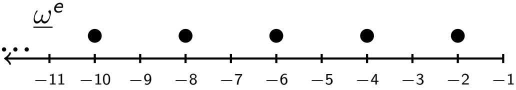

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Combinatorics gives:

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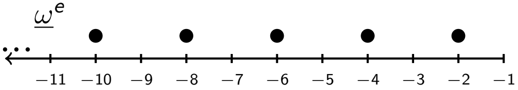
Combinatorics Influencing Probability

When we specialise to the 2-Exclusion process (all possible left jumps with rate q, right with rate 1) the identity specialises to a known one in the combinatorics literature.



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This is completely non-obvious from the form of the blocking measure!

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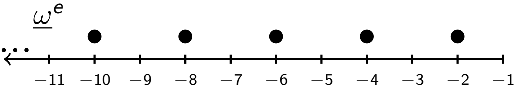

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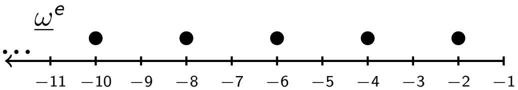

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Conjecture:

$$\sum_{\underline{\omega} \in \mathcal{H}^e} q^{\sum\limits_{i \text{ odd}} i\omega_{-i} + \sum\limits_{i \text{ even}} i(\omega_{-i}-1)} t^{2\left(\sum\limits_{i \text{ odd}} \mathbb{I}\{\omega_{-i} \geq 1\} - \sum\limits_{i \text{ even}} \mathbb{I}\{\omega_{-i}=0\}\right)}$$

$$= \frac{1}{\prod\limits_{m \geq 1} (1-q^{2m})} + \frac{\sum\limits_{i \geq 1} \sum\limits_{n \geq i} (-1)^{n-i} \binom{n+i-1}{2i-1} \frac{n}{i} q^{n^2} t^{2i}}{\prod\limits_{m \geq 1} (1-q^m)^2}$$

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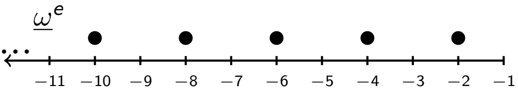

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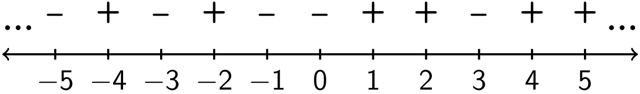
This is not clear from either the probability or combinatorics interpretation of our identities.

(Blocking) Ising process

State space: $\Omega^{\text{Is}} := \{-1, +1\}^{\mathbb{Z}}$



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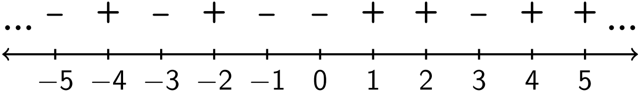

To any state we associate two quantities:

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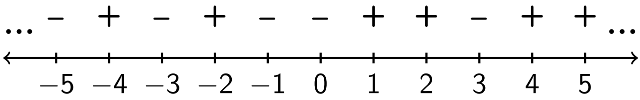

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$$H(\sigma) := \sum_{i\in\mathbb{Z}} \mathbb{I}\{\sigma_i \neq \sigma_{i+1}\}.$$

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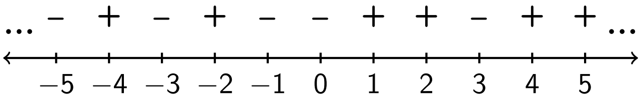

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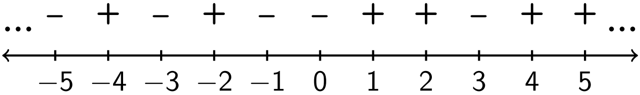

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We can also consider inhomogeneous interactions

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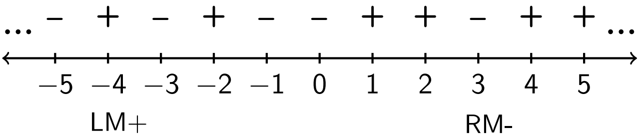
$$\mathcal{B} := \{\sigma \in \Omega^{\mathrm{Is}} : \exists a, b \in \mathbb{Z}\, s.t. \forall i \in \mathbb{N} \;\; \sigma_{a-i} = -1,\, \sigma_{b+i} = +1\}.$$



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We consider Kawasaki dynamics for the Ising process, where spins swap at sites $(i, i+1)$ with rates of the form,

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Family of Stationary Reversible Measures

For any $c \in \mathbb{R}$ the measure,

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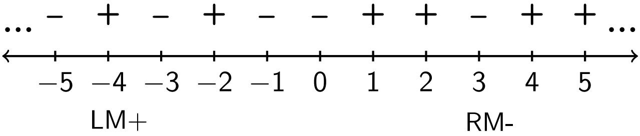

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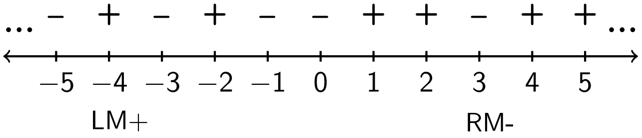

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The unique stationary measure on $\{\sigma : N(\sigma) = n\}$ is,

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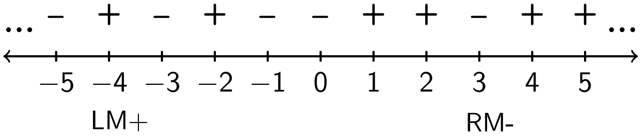

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Without the identity/ combinatorics the partition function is very complicated!

Combinatorics Influencing Probability

If we let $Q := q^2$, $z := q^{-2c}$ and $y := e^{-2\beta}$ the partition function of Blocking Ising is,

$$y + \sum_{\substack{L,R\geq 0\\ L+R>0}} y^{(L+R+1)} \sum_{\substack{\ell_1,\ldots,\ell_L\geq 1\\ m_1,\ldots,m_R\geq 1}} \prod_{j=1}^{L} \frac{Q^{\frac{1}{2}\ell_j(\ell_j-1)+(\ell_{L-j}+1)(\ell_L+\cdots+\ell_{L-j+1})} z^{-\ell_j}}{1-Q^{\ell_L+\cdots+\ell_{L-j+1}}}$$

$$\prod_{j=1}^{R} \frac{Q^{\frac{1}{2}m_j(m_j-1)+(m_{R-j}+1)(m_R+\cdots+m_{R-j+1})} z^{m_j}}{1-Q^{m_R+\cdots+m_{R-j+1}}} \left((1-y^{-1}) Q^{\ell_L+\cdots+\ell_1} Q^{m_R+\cdots+m_1} + y^{-1} \right).$$

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So the combinatorial identity gives us something highly non trivial!

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- We are seeing identities related to the ones from 0-1-2 systems.
- The parameter t in the 0-1-2 system can be seen as some sort of strength of gravity.
- We believe we can extend Balázs and Bowen's blocking family to 2 (potentially multiple) lanes.

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- Carinci, Giardiná, Redig and Sasamoto, also Kuan found certain particle systems can be built algebraically.
- **Question:** Can these particle system equivalences be seen algebraically?
- **Question:** Can the link with algebra give us more new results in probability?

Boldog Születésnapot Bálint!