Generalized law of iterated logarithm for the infinite horizon Lorentz gas

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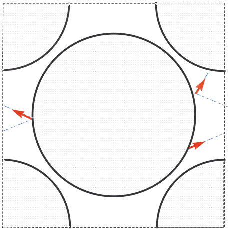
Stochastics and Influences

Happy Birthday, Bálint!

Sinai billiard on $\mathbb{T}^2$ and Lorentz gas on $\mathbb{R}^2$



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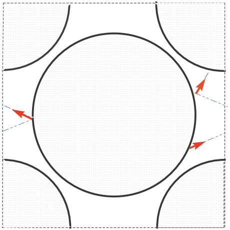

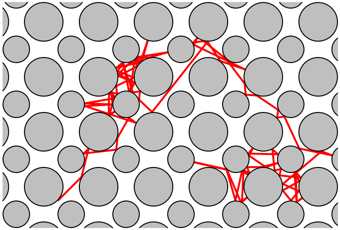

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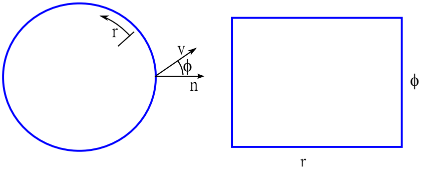
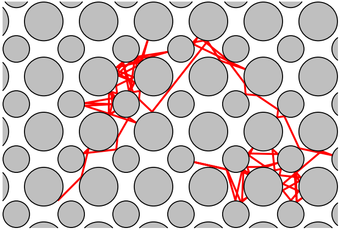
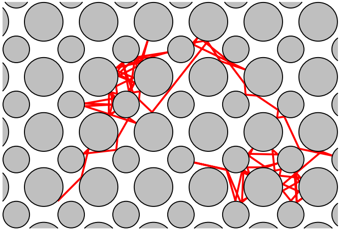
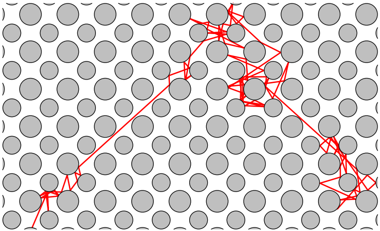




- Sinai billiard on $\mathbb{T}^2 \Rightarrow$ unfolding: periodic Lorentz gas on $\mathbb{R}^2$
- H. Lorentz, 1905 to model motion of electrons in metals
  - $T : M \to M$ billiard map, μ acip, $d\mu = c \cdot \cos\phi \, dr \, d\phi$
  - $\Delta : M \to \mathbb{R}^2$ free flight displacement
  - $\Delta_n = S_n\Delta(x) = \Delta(x) + \cdots + \Delta(T^{n-1}x)$ position on the plane

Finite vs. infinite horizon





- $\Delta(x)$ bounded
- $\Delta(x)$ unbounded

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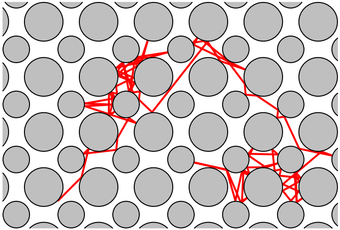
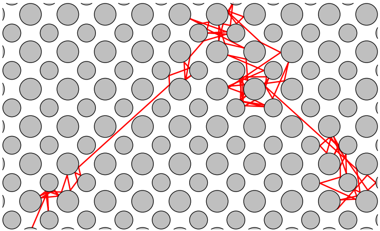
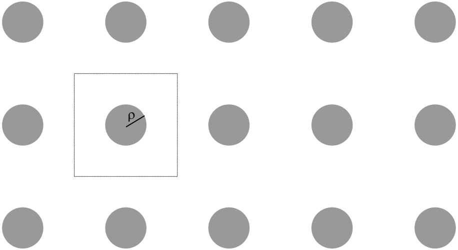




- $\Delta(x)$ bounded
- normal diffusion

- $\Delta(x)$ unbounded
- anomalous diffusion

Single circular scatterer of radius $\rho < \frac{1}{2}$ on $\mathbb{T}^2$



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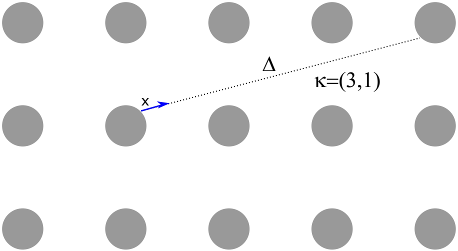

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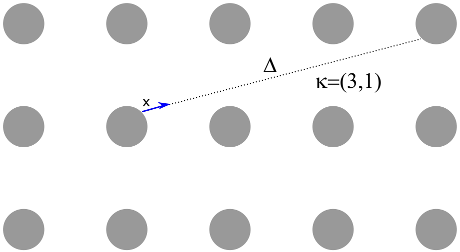

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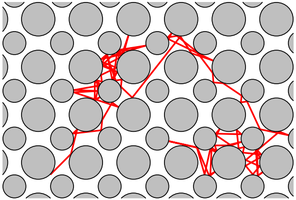
- $\kappa_n (= \kappa_{n,\rho}) = S_n(\kappa) = \sum_{i=0}^{n-1} \kappa \circ T^i$
- $\kappa(x) = \Delta(x) + d(x) - d(Tx) \;\Rightarrow\; \Delta_n - \kappa_n = O(1)$
- $\mathbb{E}_\mu \Delta = \mathbb{E}_\mu \kappa = 0 \;\Rightarrow$ study fluctuations

Limit laws



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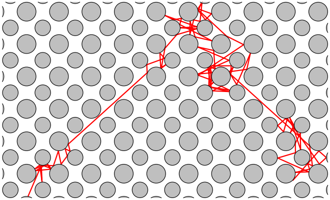
- $\frac{\Delta_n}{\sqrt{n}} \xrightarrow{\mathcal{D}} \mathcal{N}(0, \sigma^2)$



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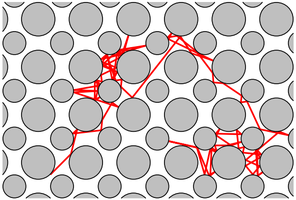

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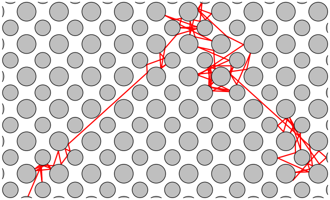
- σ^2: Green-Kubo



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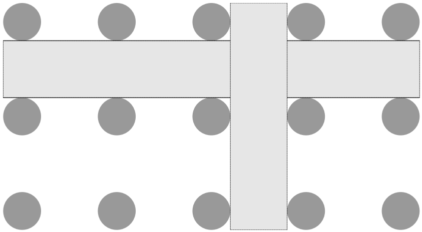
- Σ^2: corridor sum

Covariance matrix – corridor sum



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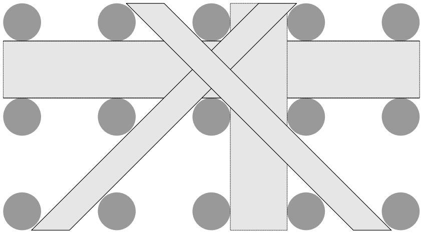

- parallel to $\xi = (p, q) \in \mathbb{Z}^2$
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- width: $d_\rho(\xi) = |\xi|^{-1} - 2\rho$

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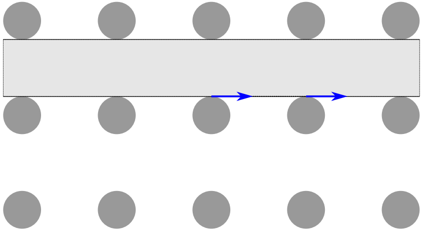

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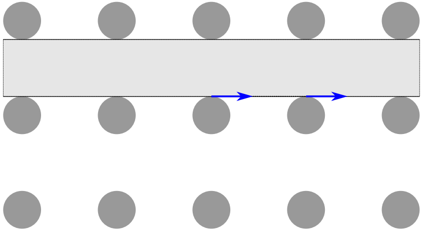

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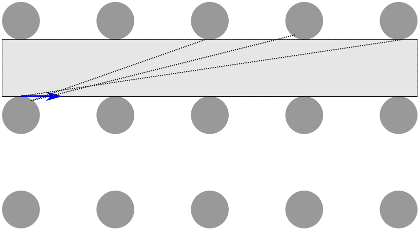
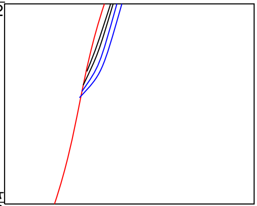
- For such points $\kappa(x) = \xi$ and we have

$$\Sigma(\rho)^2 = \frac{1}{8\rho\pi} \sum_{x \in X_\rho} \frac{d^2(\kappa(x))}{|\kappa(x)|} \kappa(x) \otimes \kappa(x)$$

Tail of κ





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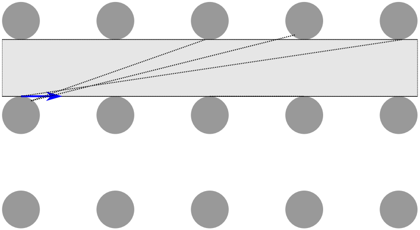

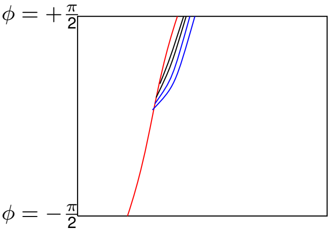

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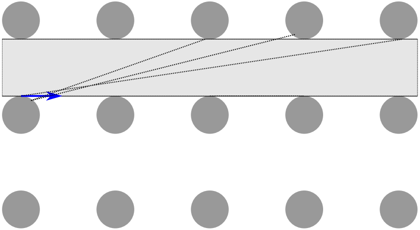

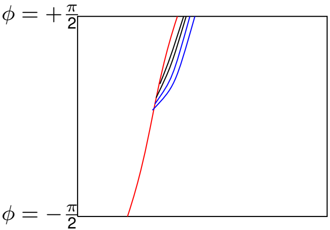

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- $\kappa, \kappa \circ T, \ldots$ behaves as i.i.d. sequence:

$$\frac{\kappa_n}{\sqrt{n \cdot \log n}} \xrightarrow{\mathcal{D}} \mathcal{N}(0, \Sigma(\rho)^2); \qquad \Sigma(\rho)^2 = A_\rho \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

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Almost sure invariance principle and LIL

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  $$\limsup_{n\to\infty} \frac{|S_n|}{\hat{c}_n} = a, \text{ almost surely, for some } a \in (0, \infty).$$

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  - still implies standard LIL
- extensions to $\eta \in \mathbb{R}^d$, $d \geq 2$, and/or dependent, ongoing

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- Melbourne–Nicol 2006,2009; ASIP for systems with Young towers, martingale approximation

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- Cuny–Dedecker–Korepanov– Merlevède, 2018– (ongoing) Young towers, regain KMT

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  - for generic good c_n, let $A(c_n) = \mathbb{E}(\eta^2 \cdot \mathbf{1}_{|\eta|<c_n})$, then
  - $a = \sup\left\{ b \geq 0 \,\middle|\, \sum_{n=1}^{\infty} \frac{1}{n} \exp\left(-\frac{b^2\, c_n^2}{2nA(c_n)}\right) = \infty \right\}$

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  - *with the sequence* $\hat{c}_n$ *specified above.*

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- c_n is good if and $\sum_{n=1}^{\infty} \mu(|\Delta| > c_n) \asymp \sum_{n=1}^{\infty} c_n^{-2} < \infty$;
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- $\hat{c}_n^2 = n\, L(n)\, LL(n)\, (1 + LL(n) \sin^2(LLL(n)))$ is very good:
- for $\hat{c}_n$, $\ell_1(2^n) = L(n)\, (1 + L(n) \sin^2(LL(n)))$, and the sumability condition follows from
- $\int_{x=1000}^{\infty} \frac{x}{1+e^x \sin^2 x}\, dx < \infty$. $(LL(n) \to x)$

Comments on the proof

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