Sum of independent exponentials

Lemma 1. Let \((X_i)_{i=1}^{n}, \ n \geq 2\), be independent exponential random variables with pairwise distinct respective parameters \(\lambda_i\). Then the density of their sum is

\[
f_{X_1 + X_2 + \cdots + X_n}(x) = \prod_{i=1}^{n} \lambda_i \sum_{j=1}^{n} \frac{e^{-\lambda_j x}}{\prod_{k \neq j}^{n} (\lambda_k - \lambda_j)}, \quad x > 0.
\]

Remark. I once (in 2005, to be more precise) thought this stuff would be part of some research-related arguments, but I ended up not using it. Later on I realized it’s actually Problem 12 of Chapter I in Feller: An Introduction to Probability Theory and its Applications, Volume II. And recently I have read about it, together with further references, in “Notes on the sum and maximum of independent exponentially distributed random variables with different scale parameters” by Markus Bibinger under http://arxiv.org/abs/1307.3945. Moreover, I now know that this distribution is known as the Hypoexponential distribution (thanks János!).

Proof. First we compute the convolutions needed in the proof.

\[
e^{-ax} * e^{-bx} = \int_{0}^{x} e^{-a(u-x)} e^{-bu} \, du = e^{-ax} \left( e^{(a-b)x} - 1 \right) / (a-b).
\]

For \(n = 2\),

\[
f_{X_1 + X_2}(x) = f_{X_1}(x) * f_{X_2}(x) = \lambda_1 \lambda_2 e^{-\lambda_2 x} / (\lambda_2 - \lambda_1) = \lambda_1 \lambda_2 \left[ \frac{e^{-\lambda_1 x}}{\lambda_2 - \lambda_1} + \frac{e^{-\lambda_2 x}}{\lambda_1 - \lambda_2} \right],
\]

in accordance to (1). Now inductively, fix \(n \geq 3\), and assume the statement is true for \(n - 1\). Then

\[
f_{X_1 + X_2 + \cdots + X_n}(x) = f_{X_1 + X_2 + \cdots + X_{n-1}}(x) * f_{X_n}(x) = \prod_{i=1}^{n} \lambda_i \sum_{j=1}^{n-1} \frac{e^{-\lambda_j x}}{\prod_{k \neq j}^{n} (\lambda_k - \lambda_j)} * f_{X_n}(x)
\]

\[
= \prod_{j=1}^{n} \lambda_j \sum_{j=1}^{n-1} \frac{e^{-\lambda_j x}}{(\lambda_j - \lambda_n) \prod_{k \neq j}^{n} (\lambda_k - \lambda_j)} = \prod_{j=1}^{n} \lambda_j \sum_{j=1}^{n-1} \frac{e^{-\lambda_j x}}{\prod_{k \neq j}^{n} (\lambda_k - \lambda_j)} = \prod_{j=1}^{n} \lambda_j \sum_{j=1}^{n-1} \frac{e^{-\lambda_n x}}{\prod_{k \neq j}^{n} (\lambda_k - \lambda_j)}.
\]

The proof is done as soon as we show that the coefficient of \(e^{-\lambda_n x}\) fits the coefficients seen in the sum of (1), i.e.

\[
- \sum_{j=1}^{n-1} \frac{1}{\prod_{k \neq j}^{n} (\lambda_k - \lambda_j)} = \frac{1}{\prod_{k=1}^{n} (\lambda_k - \lambda_n)}.
\]
or, equivalently,
\[
\sum_{j=1}^{n} \frac{1}{\prod_{k \neq j}^{n} (\lambda_k - \lambda_j)} = 0.
\]

To this order, we write
\[
\sum_{j=1}^{n} \frac{1}{\prod_{k \neq j}^{n} (\lambda_k - \lambda_j)} = \sum_{j=1}^{n} \frac{\prod_{k \neq j}^{n} (\lambda_k - \lambda_l)}{\prod_{k \neq j}^{n} (\lambda_k - \lambda_l)}
\]
which is zero if and only if
\[
\sum_{j=1}^{n} \prod_{k \neq j}^{n} (\lambda_k - \lambda_l)
\]
is zero. We transform the latter in the following display. The nontrivial steps are changing orders of \(\lambda\)'s and thus signs in the factors of the products.

\[
\sum_{j=1}^{n} \prod_{k \neq j}^{n} (\lambda_k - \lambda_l) = \sum_{j=1}^{n} \prod_{k \neq j}^{n} (\lambda_k - \lambda_l) \prod_{k = j}^{n} (\lambda_k - \lambda_l)
\]
\[
= \pm \sum_{j=1}^{n} \prod_{k \neq j}^{n} (\lambda_k - \lambda_l)^2 \prod_{k = j}^{n} (\lambda_k - \lambda_l) \prod_{k = j}^{n} (\lambda_k - \lambda_l) (-1)^{n-j} =
\]
\[
= \pm \prod_{k > l}^{n} (\lambda_k - \lambda_l) \sum_{j=1}^{n} \prod_{k \neq j}^{n} (\lambda_k - \lambda_l) (-1)^{n-j},
\]
which is zero if and only if
\[
(3)
\sum_{j=1}^{n} \prod_{k \neq j}^{n} (\lambda_k - \lambda_l) (-1)^j
\]
is zero. Notice that the product here is a Vandermonde determinant of the form
\[
\begin{vmatrix}
1 & \lambda_1 & \lambda_1^2 & \cdots & \lambda_1^{n-2} \\
1 & \lambda_2 & \lambda_2^2 & \cdots & \lambda_2^{n-2} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
1 & \lambda_{j-1} & \lambda_{j-1}^2 & \cdots & \lambda_{j-1}^{n-2} \\
1 & \lambda_{j+1} & \lambda_{j+1}^2 & \cdots & \lambda_{j+1}^{n-2} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
1 & \lambda_n & \lambda_n^2 & \cdots & \lambda_n^{n-2}
\end{vmatrix}
\]
and hence (3) is nothing but the expansion of the determinant

\[
\begin{vmatrix}
1 & 1 & \lambda_1 & \lambda_1^2 & \cdots & \lambda_1^{n-2} \\
1 & 1 & \lambda_2 & \lambda_2^2 & \cdots & \lambda_2^{n-2} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
1 & 1 & \lambda_n & \lambda_n^2 & \cdots & \lambda_n^{n-2}
\end{vmatrix}
\]

w.r.t. its second column. As this determinant is zero, so is (3) and thus (2) is proven. \( \square \)