Problems for the twelfth week

Mathematics A3 for Civil Engineering students

1. Which of the following functions can be a distribution function?

   (a) $F(x) = \begin{cases} 1 + \mathrm{e}^{1-x} & \text{, if } x > -1, \\ 0 & \text{, otherwise} \end{cases}$

   (b) $F(x) = \begin{cases} 2 - \dfrac{2}{x+1} & \text{, if } x \geq 0, \\ 0 & \text{, otherwise} \end{cases}$

   (c) $F(x) = \begin{cases} 1 - \mathrm{e}^{-x} & \text{, if } x \geq 0, \\ 0 & \text{, otherwise} \end{cases}$

   (d) $F(x) = \begin{cases} 0 & \text{, if } x \leq 0, \\ \dfrac{x}{4} \cdot (4 - x) & \text{, if } 0 < x \leq 2, \\ 1 & \text{, if } x > 2 \end{cases}$

2. Which of the following functions can be a probability density function?

   (a) $f(x) = \begin{cases} \dfrac{2}{x} & \text{, if } x > 1, \\ 0 & \text{, otherwise} \end{cases}$

   (b) $f(x) = \begin{cases} \dfrac{\sin(x)}{2} & \text{, if } 0 < x < 2, \\ 0 & \text{, otherwise} \end{cases}$

   (c) $f(x) = \begin{cases} 3^{x-1} \ln(3) & \text{, if } x \leq 0, \\ \dfrac{1}{3} \sin\left(\dfrac{x}{2}\right) & \text{, if } 0 < x < \pi, \\ 0 & \text{, otherwise} \end{cases}$

   (d) $f(x) = \begin{cases} 2\mathrm{e}^{-2x} & \text{, if } x > 0, \\ 0 & \text{, otherwise} \end{cases}$

3. Compute the expectation and variance of a random variable X with density

$$f(x) = \begin{cases} 2x & \text{, if } 0 < x < 1, \\ 0 & \text{, otherwise.} \end{cases}$$

4. Find the probabilities $\mathbb{P}\{m - \sigma < X < m + \sigma\}$ and $\mathbb{P}\{m - 2\sigma < X < m + 2\sigma\}$ in the previous question, where m stands for expectation and σ^2 for variance.

5. Consider the function

$$f(x) = \begin{cases} c(2x - x^3) & \text{, if } 0 < x < \dfrac{5}{2}, \\ 0 & \text{, otherwise.} \end{cases}$$

   Could f be a probability density function? If so, determine c.

Repeat if $f(x)$ were given by

$$f(x) = \begin{cases} c(2x - x^2) & \text{, if } 0 < x < \dfrac{5}{2}, \\ 0 & \text{, otherwise.} \end{cases}$$

6. A filling station is supplied with gasoline once a week. If its weekly volume of sales in thousands of liters is a random variable with probability density function

$$f(x) = \begin{cases} 5(1 - x)^4 & \text{, if } 0 < x < 1, \\ 0 & \text{, otherwise,} \end{cases}$$

what needs the capacity of the tank be so that the probability of the supply's being exhausted in a given week is 0.01?

7. Compute $\mathbb{E}(X)$ if X has a density function given by

   (a) $f(x) = \begin{cases} \dfrac{1}{4} x e^{-x/2} & \text{, if } x > 0, \\ 0 & \text{, otherwise;} \end{cases}$

   (b) $f(x) = \begin{cases} c(1 - x^2) & \text{, if } -1 < x < 1, \\ 0 & \text{, otherwise;} \end{cases}$

   (c) $f(x) = \begin{cases} \dfrac{5}{x^2} & \text{, if } x > 5, \\ 0 & \text{, otherwise?} \end{cases}$

8. The lifetime, in days, of a part of a machine is a random variable with density $f(x) = 2/x^3$ when $x > 1$. What is the probability that this part still works on February 1, if we bought it on January 26? Should we rather buy the part with density function $f(x) = 1/x^2$ when $x > 1$? What is the average lifetime for the two types of parts?

9. What is the probability that, if I wake with a start in the middle of the night, the minute hand of the clock is on the right hand-side of the vertical line that passes through the center of the clock? And what is the probability that the minute hand points inside the $\frac{1}{12}$ part of the circle that is located between the numbers 5 and 6?

10. What is the probability that, out of three independent points chosen in $(0, 1)$, precisely one falls in each of the intervals $(0, \frac{1}{3})$, $(\frac{1}{3}, \frac{2}{3})$, $(\frac{2}{3}, 1)$?

11. A long and high fence consists of columns of diameter D, placed in a distance of L from each other. I throw a ball of diameter d from a long distance, with my eyes closed to this fence. The ball either hits one of the columns, or it passes between them without touching. What is the probability that the ball passes without touching?

12. Trains headed for destination A arrive at the train station at 15-minute intervals starting at 7:00am, whereas trains headed for destination B arrive at 15-minute intervals starting at 7:05am.

    (a) If a certain passenger arrives at the station at a time uniformly distributed between 7:00 and 8:00am, and then gets on the first train that arrives, what proportion of time does he or she go to destination A?

(b) What if the passenger arrives at a time uniformly distributed between 7:10 and 8:10am?

13. You arrive at a bus stop at 10 o'clock, knowing that the bus will arrive at some time uniformly distributed between 10 and 10:30.

    (a) What is the probability that you will have to wait longer than 10 minutes?

    (b) If at 10:15 the bus has not yet arrived, what is the probability that you will have to wait at least an additional 10 minutes?

14. A bus travels between the two cities A and B, which are 100 km apart. If the bus has a breakdown, the distance from the breakdown to city A has a uniform distribution over $(0, 100)$. There is a bus service station in city A, in B, and in the center of the route between A and B. It is suggested that it would be more efficient to have the three stations located 25, 50, and 75 km, respectively, from A Do you agree? Why? What would be the optimal location of the three stations?

15. A rod of length l is broken into two pieces at a uniform random point. What is the distribution function of the shorter piece?

16. A public phone booth is occupied when I get there. The length of the phone call is random, and has an exponential distribution with parameter $1/3$ when measured in minutes. What is the probability that the call is still going on after 5 minutes? And if I know that the call has been going on for 2 minutes when I arrive?

17. The number of years a radio functions is exponentially distributed with parameter $\lambda = 1/8$. If Jones buys a used radio, what is the probability that it will be working after an additional 8 years?

18. 2% of electric components of a given type break down within 1000 hours of operation. Assume that time before breakdown has an exponential distribution. What is the probability that such a component will work for longer than the average?

19. The time (in hours) required to repair a machine is an exponentially distributed random variable with parameter $\lambda = 1/2$. What is

    (a) the probability that a repair time exceeds 2 hours;

    (b) the conditional probability that a repair takes at least 10 hours, given that its duration exceeds 9 hours?

20. For a memoryless light bulb, the probability that it operates for more than 2000 hours is $2/3$. In a city 200 of these light bulbs are installed. What is the probability that after 1000 hours exactly 150 bulbs are operational?

21. Compute the probabilities below, if X is a standard normal random variable:

    (a) $\mathbb{P}\{-1 < X < 1\}$

    (b) $\mathbb{P}\{-2 < X < 2\}$

    (c) $\mathbb{P}\{-3 < X < 3\}$

22. If X is a normal random variable with parameters $\mu = 10$ and $\sigma^2 = 36$, compute

    (a) $\mathbb{P}\{X > 5\}$;

(b) $\mathbb{P}\{4 < X < 16\}$;

(c) $\mathbb{P}\{X < 8\}$;

(d) $\mathbb{P}\{X < 20\}$;

(e) $\mathbb{P}\{X > 16\}$.

23. Suppose that X is a normal random variable with mean (that is, expected value) 5. If $\mathbb{P}\{X > 9\} = 0.2$, approximately what is $\mathbf{Var}(X)$?

24. Suppose that the height, in centimeters, of a 25-year-old man is a normal random variable with parameters $\mu = 180$ and $\sigma^2 = 169$. What percentage of 25-year-old men are over 2 meters tall? What percentage of men in the 2-meter club are over 2 meters and 10 centimeters?

25. A random variable X has expectation 0, variance 1. In which case is it more likely that $X > 1/2$; if X is normal, or if X is uniform?

26. Let $f(x)$ denote the probability density function of a normal random variable with mean μ and variance σ^2. Show that $\mu - \sigma$ and $\mu + \sigma$ are the points of inflection of this function.