Problems for the thirteenth week

Mathematics A3 for Civil Engineering students

1. If $\mathbb{E}(X) = 1$ and $\mathbf{Var}(X) = 5$ find
   (a) $\mathbb{E}[(2 + X)^2]$,
   (b) $\mathbf{Var}(4 + 3X)$.

2. Let $X_1, X_2, \ldots$ be independent and identically distributed random variables with expectation μ and variance σ^2. Compute the expectation and variance of
$$Y_n := \frac{X_1 + X_2 + \cdots + X_n - n\mu}{\sqrt{n} \cdot \sigma}.$$

3. The county hospital is located at the center of a square whose sides are 3 kilometers wide. If an accident occurs within this square, then the hospital sends out an ambulance. The road network is rectangular, so the travel distance from the hospital, whose coordinates are $(0, 0)$, to the point (x, y) is $|x| + |y|$. If an accident occurs at a point that is uniformly distributed in the square, find the expected travel distance of the ambulance.

4. If X and Y are independent and identically distributed with mean μ and variance σ^2, find $\mathbb{E}[(X - Y)^2]$.

5. The numbers of 1, 2, 5, 10, 20, 50 and 100-forint coins in my pocket are independent Poisson(λ) distributed random variables. Find the expectation and variance of the joint value of my coins.

6. Suppose that X and Y are independent random variables having a common mean μ. Suppose also that $\mathbf{Var}(X) = \sigma_X^2$ and $\mathbf{Var}(Y) = \sigma_Y^2$. The value of μ is unknown and it is proposed to estimate μ by a weighted average of X and Y. That is, $\lambda X + (1 - \lambda)Y$ will be used as an estimate of μ, for some appropriate value of λ. Which value of λ yields the estimate having the lowest possible variance? Explain why it is desirable to use this value of λ.

7. We flip a fair coin three times. Denote the number of heads and tails by X and Y, respectively. Compute the expectation and variance of the variable $Z := XY$.

8. Compute the standardized version of a Binomial(n, p) distribution as $n \to \infty$ in the cases $p = 0.4$, $p = 0.02$, and $p = 0.96$.

9. What is the probability that the number of outcomes six is between 970 and 1050 if we roll a die 6 000 times?

10. Each item produced by a certain manufacturer is, independently, of acceptable quality with probability 0.95. Approximate the probability that at most 10 of the next 150 items produced are unacceptable.

11. Two types of coins are produced in a factory: a fair coin and a biased one that comes up heads 55 percent of the time. We have one of these coins but do not know whether it is a fair coin or a biased one. In order to ascertain which type of coin we have, we shall perform the following statistical test: We shall toss the coin 1000 times. If the coin lands on heads 525 or more times, then we shall conclude that it is a biased coin, whereas, if it lands heads less than 525 times, then we shall conclude that it is the fair coin. If the coin is actually fair, what is the probability that we shall reach a false conclusion? What would it be if the coin were biased?

12. Determine the integer k for which the probability that the number of heads is between 195 and k is approximately 0.5 when a coin is flipped 400 times.

13. How many times should a coin be tossed so as to having the number of heads between 47% and 53% of the number of all tosses with probability at least 0.95?

14. Dömötör plays roulette in the casino. Every round he bets 10 tokens on 'red'. After 100 rounds he has lost 300 tokens. Is he reasonable when he thinks that the croupier is cheating? (On the game roulette one has 37 fields, numbered from 0 to 36. Out of these, '0' has color green, and 18 fields are red, and 18 are black.)

15. Let X be the number of times that a fair coin, flipped 40 times, lands heads. Find the probability that $X = 20$. Use the normal approximation and then compare it to the exact solution. (Hint for the normal approximation: $\mathbb{P}\{X = 20\} = \mathbb{P}\{19.5 \leq X < 20.5\}$, which of course does not matter while X is considered a discrete binomial variable, but becomes important when we apply the DeMoivre-Laplace Theorem.

16. A fraction p of citizens in a city smoke. We are to determine the value of p by making a survey that involves n citizens whom we select randomly. If k of these n people smoke, then $p' = k/n$ will be our result. How large should we choose n if we want our result p' to be closer to the real value p than 0.005 with probability at least 0.95? In other words: determine the smallest number n_0 such that

$$P(|p' - p| \leq 0.005) \geq 0.95$$

for any $p \in (0,\, 1)$ and $n \geq n_0$.

17. What is the probability that the sum of 50 independent and identically distributed random variables falls in the interval $[0,\, 30]$ if the distribution of these variables
    (a) is uniform;
    (b) has density function $f(x) = 2x$

    on the interval $[0,\, 1]$?

18. Approximate the probability that the sum of 10 000 rolls of a fair die falls between 34 800 and 35 200.

19. A die is continually rolled until the total sum of all rolls exceeds 300. Approximate the probability that at least 80 rolls are necessary.

20. One has 100 lightbulbs whose lifetimes are independent exponentials with mean 5 hours. If the bulbs are used one at a time, with a failed bulb being replaced immediately by a new one, approximate the probability that there is still a working bulb after 525 hours.

21. In Problem 20 suppose that it takes a random time, uniformly distributed over $(0,\, 0.5)$, to replace a failed bulb. Approximate the probability that all bulbs have failed by time 550.