Problems for the ninth week

Mathematics A3 for Civil Engineering students

1. Production line I of a factory works 60% of time, while production line II works 70% of time, independently of each other. What is the probability that
   (a) both lines operate,
   (b) at least one of the lines operates,
   (c) precisely one of the lines operates,
   (d) both lines are stopped?
2. Suppose that each child born to a couple is equally likely to be a boy or a girl independent of the sex distribution of the other children of the family. For a couple having 5 children, compute the probabilities of the following events:
   (a) All children are of the same sex.
   (b) The 3 eldest are boys and the others girls.
   (c) Exactly 3 are boys.
   (d) The 2 oldest are girls.
   (e) There is at least 1 girl.
3. We roll a red and a green die. Consider the following events: $A = \{$the sum of the numbers shown is 7$\}$, $B = \{$at least one of the dice shows a 6$\}$, $C = \{$both dice show odd numbers$\}$, $D = \{$the dice show different numbers$\}$, $E = \{$the green die shows a 4$\}$.
   (a) Are the events A and C independent?
   (b) Are the events A and C mutually exclusive?
   (c) What is the probability of B?
   (d) How are A and D related? What consequences follow for their probabilities? And for their independence?
   (e) Are the events A and E independent?
   (f) Based on the above, show examples of
      i. independent but not mutually exclusive events,
      ii. mutually exclusive but not independent events.
4. We roll a red and a green die. Consider the following events: $A = \{$the sum of the numbers shown is 7$\}$, $B = \{$the red die shows a 3$\}$, $C = \{$the green die shows a 4$\}$.
   (a) Are the events A and B independent? The events A and C? How about B and C?
   (b) Are the events A, B, C independent?
   (c) Are the events A and $B \cap C$ independent?
   (d) So, are A, B, C independent?

5*. Suppose that we want to generate the outcome of the flip of a fair coin but that all we have at our disposal is a biased coin which lands on heads with some unknown probability p that need not to be equal to 1/2. Consider the following procedure for accomplishing our task.

(1) Flip the coin.

(2) Flip the coin again.

(3) If both flips land heads or both land tails then return to step (1).

(4) Otherwise, if the two flips are different, then let the result of the last flip be the result of the experiment.

(a) Show that the result is equally likely to be either heads or tails.

(b) Could we use a simpler procedure that continues to flip the coin until the last two flips are different and then lets the result be the outcome of the final flip?

6. Independent flips of a coin that lands on heads with probability p are made. What is the probability that the first outcomes are

   (a) H, H, H, H,

   (b) T, H, H, H?

   (c*) What is the probability that the pattern (T, H, H, H) occurs before the pattern (H, H, H, H)? (Hint: How can the pattern (H, H, H, H) occur first?)

7. A true-false question is to be posed to a husband and wife team on a quiz show. Both the husband and the wife will, independently, give the correct answer with probability p. Which of the following is a better strategy for this couple?

   (a) Choose one of them and let that person answer the question; or

   (b) have them both consider the question and then either give the common answer if they agree or, if they disagree, flip a coin to determine which answer to give?

8. Between cities A, B, C, there are the following roads: $A - B$, $A - C$ and $B - C$. At a winter night, each of these road gets blocked by the snow independently with probability p. What is then the probability that on the next morning city C will be accessible from city A?

9. Between cities A, B, C, there are two independent roads between cities A and B, and another road between B and C. If each of these roads gets blocked independently with probability q, then what is the probability that city C will be accessible from city A?

10. Five men and 5 women are ranked according to their scores on an examination. Assume that no two scores are alike and all 10! possible rankings are equally likely. Let X denote the highest ranking achieved by a woman (for instance, X=1 if the top-ranked person is female). Find the probability mass function $\mathbb{P}\{X = i\}$, $i = 1, 2, \ldots, 10$ of X.

11. The mass function of X is given by $p(i) = \frac{i^2}{30}$, $i = 1, 2, 3, 4$. Compute the expectation of X.

12. Adam and Bob play the following game: they both roll a die, and Bob gives Adam an amount equal to the square of the difference of the numbers shown on the two dice. Adam gives Bob an amount equal to the sum of the two numbers shown on the dice. Who is favored by this game?

13. On a raffle they have one HUF1 000 000, 10 HUF50 000, and 100 HUF5 000 prizes. They produce 40 000 tickets for this raffle. What should the price of one ticket be, so that the expected return agrees to the half of that price?

14. Assume the fixed prizes of HUF700, HUF10 000, HUF789 thousand, and HUF535 million on the lottery (5 winner numbers out of 90). With a price HUF150 of a lottery ticket, what are our expected winnings on one lottery ticket?

15. Ann and Bob play with two dice. Ann pays Bob whenever both dice show odd numbers. Bob pays Ann whenever exactly one die shows an even number. In any other case, no payments are made. What amounts should they set up to make this game fair?

16. Let X be the outcome of rolling a die. Compute the expectation and standard deviation of X. What if the 'die' has n sides?

17. A total of 4 buses carrying 148 students from the same school arrive at a football stadium. The buses carry, respectively, 40, 33, 25, and 50 students. One of the students is randomly selected. Let X denote the number of students that were on the bus carrying this randomly selected student. One of the 4 bus drivers is also randomly selected. Let Y denote the number of students in her bus.

    (a) Which of $\mathbb{E}(X)$ or $\mathbb{E}(Y)$ do you think is larger? Why?

    (b) Compute $\mathbb{E}(X)$ and $\mathbb{E}(Y)$.

    (c) Find $\mathbf{Var}(X)$ and $\mathbf{Var}(Y)$.

18. We draw without replacement 3 balls from an urn that contains 4 red and 6 white balls. Denote by X the number of red balls drawn. Find the distribution of X, its expectation and standard deviation.

19. We randomly place a knight on an empty chessboard. What is the expected number of his possible moves? (A knight placed on the square $(i,\, j)$ of the chessboard can move to any of the squares $(i+2,\, j+1)$, $(i+1,\, j+2)$, $(i-1,\, j+2)$, $(i-2,\, j+1)$, $(i-2,\, j-1)$, $(i-1,\, j-2)$, $(i+1,\, j-2)$, $(i+2,\, j-1)$, provided they are on the chessboard.)

20. Rolling two dice, what is the expected value of the higher and of the smaller of the two numbers shown?

21. One of the numbers 1 through 10 is randomly chosen. You are to try to guess the number chosen by asking questions with „yes-no" answers. Compute the expected number of questions you will need to ask in each of the two cases:

    (a) Your ith question is to be „Is it i?", $i = 1,\, 2,\, \dots,\, 10$.

    (b) With each question you try to eliminate one-half of the remaining numbers, as nearly as possible. For example, your first question is „Is the number larger than 5?". If yes, then your second question is „Is the number greater than 7?", etc.

22. If $\mathbb{E}(X) = 1$ and $\mathbf{Var}(X) = 5$, find

    (a) $\mathbb{E}[(2+X)^2]$,

    (b) $\mathbf{Var}(4+3X)$.

23. Let X be a random variable having expected value μ and variance σ^2. Find the expected value and variance of
$$Y := \frac{X-\mu}{\sigma}.$$

24. How many people are needed so that the probability that at least one of them has the same birthday as you is greater than 1/2?