Additional problems for BSM PRO Theory

Fall, 2009

Here are additional problems which I find interesting and worth thinking about. You can hand these, or some of these, in if you wish, then I mark them. *Handing or not handing in these problems will not, in any way, affect your score in the course.*

First set, posted on October 11:

1.1 In a small town n TV repairmen are available, and one day k households call for TV repairmen. What is the probability that precisely i of the n repairmen will be called, $i = 1, 2, \ldots, \min(n, k)$?

1.2
- **Fermi-Dirac distribution.** k balls are randomly distributed into $n \geq k$ urns in such a way that each urn contains at most one ball. (This will be the distribution after a long time if in every second a random ball is chosen, and that ball is moved to the clockwise neighboring urn, provided that that urn is empty. If the neighboring urn is occupied then nothing happens.) Find the probability that the first urn is occupied.
- **Maxwell-Boltzmann distribution.** k distinguishable balls are randomly distributed into n urns. (This will be the distribution after a long time if in every second a random ball is chosen, and that ball is moved into the clockwise neighboring urn.) Find the probability that the first urn contains i balls, $i = 1 \ldots k$.
- **Bose-Einstein distribution.** k indistinguishable balls are randomly distributed into n urns. (This will be the distribution after a long time if in every second a random urn is chosen, and a ball, if any, from that urn is moved into the clockwise neighboring urn.) Find the probability that the first urn contains i balls, $i = 1 \ldots k$.
- Now in each of these cases consider the limit $n, k \to \infty$ such that $k/n \to \lambda$. Show that the answers for the above questions converge:
  * for Fermi-Dirac: $\mathbf{P}\{\text{the first urn contains a ball}\} \to \lambda$, this is the Bernoulli distribution;
  * for Maxwell-Boltzmann: $\mathbf{P}\{\text{the first urn contains } i \text{ balls}\} \to \frac{\lambda^i}{i!} \cdot \mathrm{e}^{-\lambda}$, this is called the Poisson distribution;
  * for Bose-Einstein: $\mathbf{P}\{\text{the first urn contains } i \text{ balls}\} \to \left(\frac{\lambda}{\lambda+1}\right)^i \cdot \frac{1}{\lambda+1}$, this is called the Geometric distribution.

1.3 **Pólya urn.** Initially an urn contains 1 red and 1 blue balls. In every step we choose a random ball which we replace plus we also insert another ball of the same color. (For example, if the first choice was red, then in the second step we choose from 2 red balls and one blue ball.) Show by induction that, after the n^{th} step, the urn will contain i red balls with probability $\frac{1}{n+1}$ for each $i = 1, 2, \ldots, n+1$.

1.4 Urn 1 contains n blue balls, and Urn 2 contains n red balls. At each step we randomly choose a ball from Urn 1, throw it away, and move a red ball from Urn 2 into Urn 1. After the n^{th} step, Urn 2 is empty. What is the proportion of the blue balls in Urn 1 at this time? (Assume that n is large.)

1.5 A hunter discovers a fox in 30 yards distance, and shoots it. If the fox survives the first shot, then it tries to escape at a velocity of 10 yards/second. The hunter loads the gun and shoots again in every 3 seconds until he either kills the fox, or it disappears in the horizon. The probability that the hunter hits the fox is proportional to the inverse square of the distance: $\mathbf{P}\{\text{The hunter hits the fox from a distance } x\} = 675x^{-2}$ $(x \geq 30)$. Even when the fox is hit, it is not necessary fatal: the fox survives each hit, independently of previous hits, with probability 1/4. Is the probability that the fox survives this unpleasant adventure positive or zero?

1.6 Given are an urn of infinite volume and infinitely many balls numbered 1, 2, 3, …. Initially the urn is empty. One minute before midnight we place balls no. 1, 2, …, 10 into the urn, shake the urn well, choose a random ball which we throw away. Half a minute before midnight we place balls no. 11, 12, …, 20 into the urn, shake the urn well, choose a random ball which we throw away. … 2^{-n} before midnight we place balls no. $10n+1$, $10n+2$, …, $10n+10$ into the urn, shake the urn well, choose a random ball which we throw away. We keep on this until midnight. Prove that the urn is empty at midnight with probability one.

1.7 As a result of a big survey it turned out that there are, on average, 2.4 children in a family, however, children have, on average, 1.55 siblings. What is the standard deviation of the number of children in a family?

1.8 A hundred passengers are in queue for boarding a hundred-seat plane. The first passenger lost his boarding pass, so he sits on a randomly chosen seat. Other passengers board one by one, and sit on their seats if they can, and sit to randomly chosen empty seats if their own is already occupied. What is the probability that the last passenger will occupy the seat that was assigned to him?

1.9 An $\mathbb{R} \mapsto \mathbb{R}$ function is *convex*, if for every $y \in \operatorname{dom} f$ there is a real v such that for every $x \in \operatorname{dom} f$

$$f(x) \ge f(y) + v \cdot (x - y).$$

Suppose that f is a convex function, X is a random variable with values in $\operatorname{dom} f$, and both X and $f(X)$ have expectations. Show *Jensen's inequality*:

$$\mathbf{E}(f(X)) \ge f(\mathbf{E}X).$$

1.10 Let $x_1, x_2, \ldots, x_n$ be nonnegative reals, and X a uniform choice of these numbers: $\mathbf{P}\{X = x_i\} = 1/n$, $i = 1 \ldots n$. Apply Jensen's inequality for X and

- the r^{th} power function, $r > 1$;
- the reciprocal function: connection of the arithmetic and the harmonic mean;
- the logarithm: connection of the arithmetic and the geometric mean (how does Jensen's inequality work for *concave* functions?).
- The value at time t of a deposit with continuously compounded interest is

$$a(t) = a(0) \cdot \mathrm{e}^{\int_0^t r(s)\,\mathrm{d}s},$$

where $r(s)$ is the interest rate at time s. Alex always plans in advance, so he deposits an amount in the bank for exactly 10 years. Bob on the other hand only knows that he will withdraw his money from the bank at a random time T which has mean $\mathbf{E}(T) = 10$ years. Which of the two fellows can expect more money at the time of withdrawal, if
  * $r(s)$ is constant 0.1, or
  * $r(s)$ is of the form $0.5/(s + 1)$?

- Ross: Chapter 2, Theoretical Exercise 18.
- Ross: Chapter 3, Problem 55.
- Ross: Chapter 3, Problem 58.
- Ross: Chapter 3, Theoretical Exercise 7.
- Ross: Chapter 3, Theoretical Exercise 23.
- Ross: Chapter 4, Theoretical Exercise 13.
- Ross: Chapter 4, Theoretical Exercise 35 (**Pólya urn again**).

Second set, posted on October 22:

- Ross: Chapter 4, Problem 14.
- Ross: Chapter 4, Problem 24.
- Ross: Chapter 4, Problem 70.
- Ross: Chapter 4, Theoretical Exercises 13 and 18.
- Ross: Chapter 4, Theoretical Exercise 17/(b).
- Ross: Chapter 7 (yes, Seven), Theoretical Exercise 4, and Self-test Problem 10. Pretty surprising at first.

2.1 The rules of ditch-running competition are the following: the track is straight and contains infinitely many identical ditches. Two contestant of equal strength compete and run shoulder to shoulder. Each of the contestants can spring over each ditch independently with probability 2^{-L}, where L is the length of the ditches. If one of them falls in a ditch then the competition ends.

a) Show that probability that the game ends in a draw is smaller than ε if and only if

$$L < \log_2\Big(\frac{1+\varepsilon}{1-\varepsilon}\Big).$$

b) Whenever a game ends in a draw they start a new game until a winner is found. For which value of L will the expected number of jumps of contestants be minimal?

2.2 Find expectation of $(1+X)^{-1}$ when

a) $X \sim \text{Binom}(n,\, p)$,

b) $X \sim \text{Poi}(\lambda)$.

2.3 After a foot-race, 300 of the participants found one tick and 75 found two ticks in their bodies. Estimate the total number of participants.

2.4 In a country families bear children until the first boy is born. What is the distribution of sexes in this country? Are the sexes of children within a family independent?

2.5 **Surveys.** In a population of size N, m people smoke. A survey is made by randomly selecting and asking n different people whether they smoke or not. Let X be the smokers among those asked. Show that the distribution of X converges to $\text{Binom}(n,\, p)$ if $N \to \infty$, $m \to \infty$ such that $m/N \to p$ while n is kept fixed.

2.6 **Geometric is memoryless.** Show that, if $X \sim \text{Geom(p)}$, then

$$\mathbf{P}\{X \geq n+k \,|\, X > n\} = \mathbf{P}\{X \geq k\},$$

and the Geometric is the only positive integer variable with this property. The above display interprets as the *memoryless property*: given that more than n trials are needed for the first success, the probability that an additional k or more trials are needed is the same as if we just started the experiments.

Third set, posted on October 30:

- Ross: Chapter 5, Problem 31.
- Ross: Chapter 5, Theoretical Exercise 5.
- Ross: Chapter 5, Theoretical Exercise 20. You don't need to know about Gamma distributions, nor Gamma functions, all you need is the hint. (The Gamma function is defined through

$$\Gamma(z) = \int_0^\infty t^{z-1}\mathrm{e}^{-t}\,\mathrm{d}t, \qquad z \in \mathbb{R}, z \notin -\mathbb{N}.$$

Integration by parts gives $\Gamma(z+1) = z\Gamma(z)$, in particular, $\Gamma(n+1) = n!$ for $n \in \mathbb{N}$.)

- Ross: Chapter 5, Theoretical Exercise 28.
- Ross: Chapter 7 (yes, Seven), Theoretical Exercise 7 and 8.

3.1 Let F be a continuous distribution function with $F(0) = 0$. Show that

$$G(x) := \begin{cases} 0, & \text{if } x \leq 1, \\ F(x) - F(x^{-1}), & \text{if } x > 1 \end{cases}$$

is also a distribution function. Give a probabilistic interpretation to this formula.

3.2 Is there a continuous $[0,\, 1] \to [0,\, \infty)$ function for which $\int_0^1 g(x)\,\mathrm{d}x = 1$, $\int_0^1 x \cdot g(x)\,\mathrm{d}x = a$, $\int_0^1 x^2 \cdot g(x)\,\mathrm{d}x = a^2$?

3.3 Let X be normally distributed with mean zero, variance σ^2. Prove that for any $x > 0$ we have

$$\frac{1}{\sqrt{2\pi}} e^{-(x^2/2\sigma^2)} \left(\frac{\sigma}{x} - \frac{\sigma^3}{x^3}\right) \quad < \quad \mathbf{P}\{X > x\} \quad < \quad \frac{1}{\sqrt{2\pi}} e^{-(x^2/2\sigma^2)} \frac{\sigma}{x}.$$

Hint: differentiate all three terms of the inequality and compare the derivatives.

3.4 There are two identical insurance companies, both having ten thousand customers. At the beginning of the year every customer pays \$500 to his/her insurance company. During the year each customer, independently, establishes a claim to \$1500 for damages. Both companies also have \$50 000 in reserve from the previous year. A company goes bankrupt, if it cannot settle the claims. Does it make sense for the two companies to join? Let p_1 be the probability that at least one of the two companies goes bankrupt, and let p_2 be the probability that the joint companies would go bankrupt. Compute p_1 and p_2 (approximately) and draw the conclusion.

3.5 A random variable is said to have a *symmetric distribution*, if there is a $\mu \in \mathbb{R}$ for which $X - \mu$ and $\mu - X$ have the same distribution. Denote by F the distribution function of X.

a) Compute $F(\mu)$. (*Careful, tricky question.*)

b) Compute $\int\limits_{\mu-a}^{\mu+a} F(x)\,\mathrm{d}x$.

3.6 For which $c > 0$ value will $F(x) = \frac{\mathrm{e}^x}{\mathrm{e}^x + c}$ be the distribution function of a symmetric random variable?

3.7 Let X be a random variable such that $\mathbf{P}\{X = 0\} = 0$, and $Y := X^{-1}$. Under what conditions will X and Y have the same distribution?

3.8 Prove that if ξ is a Cauchy random variable with density $f(x) = \frac{1}{\pi}\frac{1}{1+x^2}$, then $X := 1/\xi$, $Y := 2\xi/(1-\xi^2)$, and $Z := (3\xi - \xi^3)/(1 - 3\xi^2)$ are also Cauchy distributed. *Hint: use the trigonometric identity: if* $\xi = \tan(\alpha)$*, then* $1/\xi = \tan(\frac{\pi}{2} - \alpha)$*,* $2\xi/(1-\xi^2) = \tan(2\alpha)$*, and* $(3\xi - \xi^3)/(1 - 3\xi^2) = \tan(3\alpha)$.

3.9 Let X be a Cauchy random variable with density $f(x) = \frac{1}{\pi}\frac{1}{1+x^2}$. We know that $\mathbf{E}|X| = \infty$, but $\mathbf{E}(|X|^{1-\varepsilon}) < \infty$ for any $\varepsilon > 0$. Determine the limit $\lim\limits_{\varepsilon \searrow 0} \varepsilon \mathbf{E}(|X|^{1-\varepsilon})$.

3.10 Let X be uniformly distributed on the interval $[-3, 4]$, and $\Psi(x) = |x-1| + |x+1|$. Determine the distribution function $G(y)$ of the random variable $Y = \Psi(X)$. Is Y a continuous random variable? Give the decomposition of G to the sum of a discrete, an absolutely continuous, and a continuous but singular, nondecreasing function.

3.11 Benford's distribution describes the fact that many real-world quantity (e.g., physical constants, salaries, stocks) behave in a scale-invariant way within some boundaries. Let $0 < a < b$ be real numbers, then the precise definition is that X *has Benford distribution*, if it is continuous, almost surely $a \le X < b$, and for all $a < z < b$ we have

$$\mathbf{P}\{z \le X < \alpha z\} \text{ does not depend on } z \text{ whenever } 1 < \alpha < b/z.$$

(As an example, consider the case when $a \le 1$ and $b \ge 4$. Then X has the same probability of falling between 1 and 2 as falling between 2 and 4.)

a) Formulate the above statement in terms of the distribution function of X.

b) Differentiate the equation of part a) with respect to z, and draw the conclusion: the density of X is of the form C/x ($a \le x < b$). Find the value of C.

c) Find the distribution function of X.

d) From now on we fix the value $a = 1$. Let $b = 10^n$ ($n > 1$ integer), and show that the distribution of the first digit of X does not depend on n.

e) Therefore, we also fix the value of b now: let $a = 1$ and $b = 10$. Compute the probabilities

$$\mathbf{P}\{\text{the first digit of } X \text{ is } i\}, \qquad i = 1, 2, \ldots, 9.$$

Fourth set, posted on November 6:

- Ross: Chapter 6, Problem 49.
- Ross: Chapter 6, Problem 53.
- Ross: Chapter 6, Theoretical Exercise 17.
- Ross: Chapter 6, Theoretical Exercise 18.
- Ross: Chapter 6, Theoretical Exercise 22.

4.1 We break a stick at two random, uniformly and independently chosen points.

a) What is the probability that, considering the three parts obtained as edges, a triangle can be assembled?

b) What is the probability that each of the three parts is shorter than a (a is a fixed number, more than the third of the length of the stick)?

4.2 Let X_1, X_2, X_3 be iid. $\mathcal{N}(0, 1)$ random variables. Define

$$\varrho := \sqrt{X_1^2 + X_2^2 + X_3^2}, \qquad \xi_i := X_i/\varrho, \qquad i = 1, 2, 3.$$

a) Compute the density of ϱ.

b) Define that ϱ and the vector $\underline{\xi} = (\xi_1, \xi_2, \xi_3)$ are independent, and $\underline{\xi}$ is uniformly distributed on the surface of the unit sphere.

4.3 Let X and Y be iid. Exp(λ) random variables. Prove that $U := X + Y$ and $V := X/(X + Y)$ are independent.

4.4 The *Monte Carlo integration* is basically the following. Suppose that f is a $[0, 1] \to [0, 1]$ continuous function, and we want to approximate the integral

$$I = \int_0^1 f(x)\,\mathrm{d}x.$$

We generate a uniform random point on the unit square $[0, 1]^2$ n times independently (for example by taking independent Uniform(0, 1) variables U_i and V_i, $i = 1, \ldots, n$), and we denote by X_n the number of such points that fall below the graph of the function f (that is, the number of indices i with $f(U_i) < V_i$). Our estimate for I is then X_n/n.

How large should we choose n for the case of the function $f(x) = x^2$, if we want 0.1 accuracy with at least 98% probability?

4.5 a) We take three uniform independent points in $[0, 1]$. Find the distribution function of the middle one.

b) We take n uniform independent points in $[0, 1]$. Find the distribution function of the k^{th} one.

Fifth set, posted on November 27:

- Ross: Chapter 7, Problem 25.
- Ross: Chapter 7, Problem 56.
- Ross: Chapter 7, Theoretical Exercise 17.
- Ross: Chapter 7, Theoretical Exercise 20.
- Ross: Chapter 7, Theoretical Exercise 22.
- Ross: Chapter 7, Self-test Problem 25.

5.1 Example 3c of Chapter 6 on page 284 in Ross is certainly true, and the DeMoivre-Laplace Theorem has generalizations in that direction, however, some parts of the argument like "...it follows that X_A and X_B will approximately have the same distribution as would independent normal random variables with expected values and variances as given in the preceding." are not precise and do not follow from the Theorem. Give a correct reasoning for the answers of parts (a) and (b) by only using what DeMoivre-Laplace Theorem says:

$$\frac{X_A - \mathbf{E}X_A}{\sqrt{\mathbf{Var}X_A}} \approx \mathcal{N}(0, 1) \qquad \text{and, independently,} \qquad \frac{X_B - \mathbf{E}X_B}{\sqrt{\mathbf{Var}X_B}} \approx \mathcal{N}(0, 1).$$

5.2 With every step while hiking, independently, Johnny might fall on his face and hurt his knee, or he also might fall back and hurt his elbow. Every ten miles, on average, he hurts his knee three times and his elbow two times. What is the longest hike his mother can let him go if she wants no injuries with probability at least 2/3?

5.3 The number of accidents in a city is Poisson(10) distributed. After each accident, with probability 0.6 the parties can come to an agreement themselves, but with probability 0.4 the police is called. Yesterday the police was called 5 times. What is the probability that 10 accidents occurred?

5.4 Let $X_1, X_2, \ldots, X_n$ be

- independent geometrically distributed with respective parameters $p_1, p_2, \ldots, p_n$, or
- independent exponentially distributed with respective parameters $\lambda_1, \lambda_2, \ldots, \lambda_n$.

In both cases find the distribution of $\min\{X_1, X_2, \ldots, X_n\}$. For the exponential case also find the probability that the minimum is achieved by X_i $(i = 1 \ldots n)$.

5.5 What is the probability that the sum of n independent Geometric(p) random variables is odd?

5.6 A fair coin is flipped repeatedly. Find the expected number of flips needed to first see the sequence HHH. How about HTH?

5.7 A type of sand consists of sphere-shaped grains with lognormally distributed diameters of parameter $\mu = -0.5$ and $\sigma = 0.3$. What *percentage of the weight* of the sand is in grains with diameter less than 0.5?

5.8 Let X and Y be independent and identically distributed, non negative random variables. Find $\mathbf{E}\big(\frac{X}{X+Y}\big)$.

5.9 Let $X_1, \ldots, X_n$ be iid. integer valued random variables, $m \le n$, and $S_m = \sum_{i=1}^{m} X_i$. Show that for any k with $\mathbf{P}\{S_n = k\} > 0$,

$$\mathbf{E}(S_m \,|\, S_n = k) = \frac{m \cdot k}{n}.$$

5.10 Let X and Y be both random variables that can only take on two values: $X = x_1, x_2$ and $Y = y_1, y_2$. Show that, as opposed to the general case, they are independent if and only if they are uncorrelated.

5.11 Let X and Y be the two coordinates of a point that is uniformly distributed within the circle of radius one, centered at $(1, 1)$. Compute $\mathbf{Cov}(X, Y)$.

5.12 Solve Buffon's needle problem without integrating, by the following argument.

- Let X be the number of times the needle intersects a line. By linearity of expectations, it is clear that $\mathbf{E}X$ is proportional to ℓ/d, where ℓ is the length of the needle and d is the distance between the parallel lines on the sheet.
- Let us now consider a smooth curve. By "dividing it into infinitesimal pieces" conclude that the same proportionality holds.
- The circle of diameter d is a nice curve where the constant of proportionality is easy to find out. Conclude that for any smooth curve we have

$$\mathbf{E}X = \frac{2}{\pi} \cdot \frac{\ell}{d}.$$

In particular, for a needle of length $\ell < d$ we have $X = 0$ or 1, hence this also agrees with the probability that the curve intersects a line.

5.13 A closed loop of thread, shorter than $2r\pi$, is dropped on the surface of a sphere of radius r. Show that the loop is entirely contained in the surface of a half-sphere. *Of course, this is a probability problem.*

5.14 We are given a set $\underline{v}^1, \underline{v}^2, \ldots, \underline{v}^n$ of unit vectors in $\mathbb{R}^d$. Show that there exist $\varepsilon_i = \pm 1$ numbers $(i = 1 \ldots n)$ such that

$$\Big|\sum_{i=1}^{n} \varepsilon_i \cdot \underline{v}^i\Big| \le \sqrt{n}.$$

Of course, this is also a probability problem.

5.15 Let $A_1 \ldots A_n$ be events. Their indicator variables will be denoted by $\mathbf{1}\{A_i\}$ $(i = 1 \ldots n)$ for this problem. Which event does

$$\prod_{i=1}^{n} \big(1 - \mathbf{1}\{A_i\}\big)$$

indicate? On one hand, taking expectation will give the probability of this event. On the other hand, expand the above product according to a 1, singletons of $\mathbf{1}\{A_i\}$'s, products of pairs $\mathbf{1}\{A_i\}\mathbf{1}\{A_j\}$, products of triplets $\mathbf{1}\{A_i\}\mathbf{1}\{A_j\}\mathbf{1}\{A_k\}$, etc. Apply an expectation on each of these terms and conclude the inclusion-exclusion formula.

5.16 Twelve people enter the elevator on the ground floor of a ten-storey building. Compute the expectation and variance of the number of times the elevator stops.

Sixth set, posted on December 7:

- Ross: Chapter 7, Problem 16.
- Ross: Chapter 7, Problem 46.
- Ross: Chapter 7, Theoretical Exercise 7.
- Ross: Chapter 7, Theoretical Exercise 8.
- Ross: Chapter 7, Theoretical Exercise 10.
- Ross: Chapter 7, Theoretical Exercise 52.
- Ross: Chapter 7, Self-test Problem 20.
- Ross: Chapter 8, Theoretical Exercise 4, 5.
- Ross: Chapter 8, Self-test Problem 12.

6.1 A prison warden wants to make room in his prison and is considering liberating one hundred prisoners, thereby freeing one hundred cells. He therefore assembles one hundred prisoners and asks them to play the following game: he lines up one hundred urns in a row, each containing the name of one prisoner, where every prisoner's name occurs exactly once. The game is played as follows: every prisoner is allowed to look inside fifty urns. If he or she does not find his or her name in one of the fifty urns, all prisoners will immediately be executed, otherwise the game continues. The prisoners have a few moments to decide on a strategy, knowing that once the game has begun, they will not be able to communicate with each other, mark the urns in any way or move the urns or the names inside them. Choosing urns at random, their chances of survival are almost zero, but there is a strategy giving them a 30% chance of survival, assuming that the names are assigned to urns randomly – what is it?

Remark: First of all, the survival probability using random choices is $1/2^{100}$ so this is definitely not a practical strategy. 30% seems ridiculously out of reach – but yes, you read the problem correctly.

6.2 **Law of Large Numbers for renewal processes.** Let $\tau_1, \tau_2, \ldots$ be nonnegative iid. random variables (*waiting times between renewals*), and suppose that $\mathbf{E}\tau_i = \mu < \infty$. Let $T(n) := \sum_{i=1}^{n} \tau_i$ (*the time of the* n^{th} *renewal*), and $N(t) = \max\{n : T(n) \le t\}$ (*the number of renewals up to time* t). The Weak Law of Large Numbers states that $T(n)/n \to \mu$ in probability. Prove the following dual statement: for any $\delta > 0$,

$$\lim_{t\to\infty} \mathbf{P}\Big\{\Big|\frac{N(t)}{t} - \frac{1}{\mu}\Big| > \delta\Big\} = 0.$$

6.3 (a) Let $X_1, X_2, \ldots, X_n$ be iid. random variables, $\mathbf{E}X_i = 0$. Let also $S_n = \sum_{i=1}^{n} X_i$. Express the third and fourth moments of S_n in terms of those of the X_i's.

(b) Let $f : [0, 1] \to \mathbb{R}$ be a four times continuously differentiable function. Compute

$$\lim_{n\to\infty} n \int_0^1\!\!\int_0^1 \cdots \int_0^1 \Big\{f\Big(\frac{x_1 + x_2 + \cdots + x_n}{n}\Big) - f\Big(\frac{1}{2}\Big)\Big\}\,\mathrm{d}x_1\,\mathrm{d}x_2 \ldots \mathrm{d}x_n.$$

6.4 Prove that

$$\lim_{n\to\infty} \int_0^1\!\!\int_0^1 \cdots \int_0^1 \frac{x_1^2 + x_2^2 + \cdots + x_n^2}{x_1 + x_2 + \cdots + x_n}\,\mathrm{d}x_1\,\mathrm{d}x_2 \ldots \mathrm{d}x_n = \frac{2}{3}.$$

6.5 Let f be a $[0, 1] \to \mathbb{R}$ continuous function. Prove that

$$\lim_{n\to\infty} \int_0^1\!\!\int_0^1 \cdots \int_0^1 f\Big(\frac{x_1 + x_2 + \cdots + x_n}{n}\Big)\,\mathrm{d}x_1\,\mathrm{d}x_2 \ldots \mathrm{d}x_n = f\Big(\frac{1}{2}\Big),$$

$$\lim_{n\to\infty} \int_0^1\!\!\int_0^1 \cdots \int_0^1 f\big((x_1 x_2 \ldots x_n)^{1/n}\big)\,\mathrm{d}x_1\,\mathrm{d}x_2 \ldots \mathrm{d}x_n = f\Big(\frac{1}{\mathrm{e}}\Big).$$

6.6 **Conjecturing the true order of magnitude of normal fluctuations.** Let $X_1, X_2, \ldots$ be iid. random variables with mean $\mathbf{E}X_i = 0$ and variance $\mathbf{Var}X_i = \sigma^2 < \infty$. Let also $S_n = \sum_{i=1}^{n} X_i$. The WLLN states that for any fixed $\delta > 0$

$$\lim_{n\to\infty} \mathbf{P}\Big\{\frac{|S_n|}{n} > \delta\Big\} = 0.$$

Prove the following, much stronger, statement: for any sequence b_n that converges to ∞ we have, for any $\delta > 0$,

$$\lim_{n\to\infty} \mathbf{P}\Big\{\frac{|S_n|}{b_n \cdot \sqrt{n}} > \delta\Big\} = 0.$$