PROBABILITY SYLLABUS, Fall Semester 2008
Budapest Semesters in Mathematics
T 10:30 - 11:15am, W 10:15am - 12:00, Room 104

Márton Balázs

Tel:	+36 30 386 6827
E-mail:	`balazs@math.bme.hu`
Office Hours:	Tuesday 11:15am - 12:00, Room 104
Office outside BSM:	Budapest University of Technology and Economics (BME)
(just in case)	Office 3a, fifth floor, Building H, 1 Egry József u., 1111 Budapest

Text is:	*A First Course in Probability*, Seventh Edition by S. Ross
Final Exam:	**10:15am, Tuesday December 16.**
Course homepage:	`www.math.bme.hu/~balazs/pro/`

Course Description: This is a first course on the mathematical phenomenon of uncertainty and techniques used to handle them. Not only being challenging itself, this field is of increasing interest in many areas of engineering, economical, physical, biological and sociological sciences as well. In this course we cover the basic notions and methods of probability theory, also giving emphasize on examples, applications and problem solving. Briefly, the topics include probability in discrete sample spaces, methods of enumeration (combinatorics), conditional probability and independence, random variables, properties of expectations, the Weak Law of Large Numbers, and the Central Limit Theorem.

Probability is a conceptually difficult field, although it might seem easy and straightforward at first. One has to distinguish between very different mathematical objects, and find their connection to real-life situations within the same problem. Therefore it is very important to follow classes and deeply understand the material during the semester.

Grading and assignments: There will be two in-class exams, weekly homeworks to be handed in during the semester, and a final exam.

- **The in-class exams** are on **Tuesdays October 7 and November 18, at 10:30am. Please let me know immediately if later you will have a conflict with these dates.** The first exam will be on Chapters 1, 2 and 3, the second one will be on Chapters 4, 5 and 6. **Each** worth **160 points** (each 20% of the total possible points).

- **The homeworks** are to be handed in on the due dates seen on page 3. **Each** worth **20 points** The worst of all homeworks will be dropped. In this way, a **total** of **240 points** (30% of the total possible points) can be earned from these assignments. Solving the homework problems by no means guarantees that you have the necessary level of practice. Please do other exercises (and check the answers in the back of the book *after* solving them) until you feel safe with problems on the topics in question. It is a good idea to simulate exam-like situations: solve exercises in limited time, without the use of the book or your notes (or your classmates).

- **The final exam** is at **10:15am, Tuesday December 16**. Half of it will cover Chapters 1 to 6, the other half is on Chapters 7 and 8 of the book. It worth **240 points** (30 % of the total possible points).

- **Bonus questions** are also to be found below. While a total of 800 points can be earned by the exams and homeworks, an additional **4 points** can be given for a solution of each bonus problem.

Grades will be based on the total of 800 points approximating the following standards:

Grade	**Points**
A^+	≥ 775
A	$\in [745, 775)$
A^-	$\in [720, 745)$
B^+	$\in [695, 720)$
B	$\in [665, 695)$
B^-	$\in [640, 665)$
C^+	$\in [615, 640)$
C	$\in [585, 615)$
C^-	$\in [560, 585)$
D	$\in [480, 560)$
F	< 480

Because of this standard, you are not in competition with your classmates nor does their performance influence positively or negatively your performance. You are encouraged to form study/problem groups with your classmates; things not clear to you may become obvious when you try to explain them to others or when you hear other points of view. Sometimes just verbalizing your mathematical thoughts can deepen your understanding. However, if you discuss with others the exercises, each person should write up her/his own version of the solution. Please note that much less can be learned by just understanding and writing up someone else's solution than by coming up (or even just trying to come up) with original ideas and solving the problem.

Please feel free to contact me any time outside class via e-mail, phone, or in person if you have questions or suggestions about this course.

Schedule for BSM Probability, Fall 2008

We aim to follow the next schedule. However, we might deviate from it during the semester. Please turn to the next pages to see the homework problem sets.

Date	Chapter of the book	Homework due on the date to the left
Sep 9T	1.1, 1.2, 1.3, 1.4	-
Sep 10W	1.4, 1.5, 1.6, 2.2, 2.3	-
Sep 16T	2.4, 2.5	-
Sep 17W	2.5, 3.2	Homework #1
Sep 23T	3.3	-
Sep 24W	3.3, 3.4	Homework #2
Sep 30T	3.4, 3.5	-
Oct 1W	4.1, 4.2, 4.3, 4.4,	Homework #3
Oct 7T	**Midterm 1.**	-
Oct 8W	4.5, 4.6	Homework #4
Oct 14T	4.7	-
Oct 15W	4.8, 4.9, 5.1	Homework #5
Oct 21T	5.2	-
Oct 22W	5.3, 5.4	Homework #6
Oct 28T	5.4	-
Oct 29W	5.5, 5.7	Homework #7
Nov 4T	6.1	-
Nov 5W	6.1, 6.2, 6.3	Homework #8
Nov 11T	6.3	-
Nov 12W	6.4, 6.5, 7.1, 7.2	Homework #9
Nov 18T	**Midterm 2.**	-
Nov 19W	7.2, 7.4	Homework #10
Nov 25T	7.4	-
Nov 26W	7.4, 7.5	Homework #11
Dec 2T	7.5, 7.6	-
Dec 3W	7.6, 7.7, 7.8	Homework #12
Dec 9T	8.2	-
Dec 10W	8.3, (8.4)	Homework #13
Dec 16T	**Final Exam at 10:15am**	**-**

Homework problem sets for BSM Probability, Fall 2008

Group work is encouraged, but write up your own solution. Please show your work leading to the result, not only the result. Problems are from the book and written here explicitly. Please make sure you solve the problem indicated here, and not another one (the one below or above it, or the problem with the same number but from another chapter or from the other edition, etc.). Each problem worth the number of $^{\bullet}$'s you see right next to it. Hence, for example, Problem 4 of Chapter 1 worth two points, while Theoretical Exercise 10 of Chapter 1 worth three points. Bonus problems are also to be found here, each worth 4 points. Note that they are also due on the due date of the corresponding homework.

Homework #1 (Due on September 17)

Chapter 1, Problem $4^{\bullet\bullet}$, $7^{\bullet\bullet\bullet}$, $15^{\bullet\bullet\bullet}$, $19^{\bullet\bullet\bullet}$, $21^{\bullet\bullet\bullet}$, $31^{\bullet\bullet\bullet}$,
Chapter 1, Theoretical Exercise $10^{\bullet\bullet\bullet}$

Homework #2 (Due on September 24)

Chapter 2, Problem $13^{\bullet\bullet}$, $17^{\bullet\bullet}$, $21^{\bullet\bullet}$, $23^{\bullet\bullet}$, $27^{\bullet\bullet}$, $33^{\bullet\bullet}$, $44^{\bullet\bullet}$,
Problem #2A$^{\bullet\bullet\bullet\bullet}$: We roll a die ten times. What is the probability that each of the results $1, 2, \dots, 6$ shows up at least once? HINT: Define the events $A_i := \{$number i doesn't show up at all during the ten rolls$\}$, $i = 1 \dots 6$. Note that these events are *not* mutually exclusive.
Problem #2B$^{\bullet\bullet}$: For the events A and B, we know that $P(A) \geq 0.8$ and $P(B) \geq 0.5$. Show that $P(A \cap B) \geq 0.3$.
Bonus problem #2C: A closet contains n pair of shoes. If $2r$ shoes are randomly selected ($2r \leq n$), what is the probability that there will be (a) no complete pair, (b) exactly one complete pair, (c) exactly two complete pairs?

Homework #3 (Due on October 1)

Chapter 3, Problem $7^{\bullet\bullet\bullet}$ (HINT: it is not $\frac{1}{2}$.), $12^{\bullet\bullet\bullet\bullet}$, $42^{\bullet\bullet\bullet}$, $44^{\bullet\bullet\bullet\bullet}$, $50^{\bullet\bullet\bullet}$
Problem #3A$^{\bullet\bullet\bullet}$: We repeatedly roll two dice at the same time, and only stop when at least one of them shows a six. What is the probability that the other also shows a six? (HINT: it is not $\frac{1}{6}$).

Homework #4 (Due on October 8)

Chapter 3, Problem $27^{\bullet\bullet\bullet}$, $74^{\bullet\bullet\bullet\bullet}$, $86^{\bullet\bullet\bullet}$,
Chapter 3, Theoretical Exercise $9^{\bullet\bullet\bullet}$, $22^{\bullet\bullet\bullet}$
Problem #4A$^{\bullet\bullet\bullet\bullet}$: Cities A, B, C, D are located (in this order) on the four corners of a square. Between them, we have the following roads: $A \leftrightarrow B$, $B \leftrightarrow C$, $C \leftrightarrow D$, $D \leftrightarrow A$, $B \leftrightarrow D$. One night each of these roads gets blocked by the snow independently with probability 1/2. Show that the next morning city C is accessible from city A with probability 1/2.
Bonus Problem #4B, note: this one is for 6 points, BUT no points for both Problem #4A and #4B, I'll give the highest of the scores earned by these two problems: Solve Problem #4A without computations, using a dual map and probabilistic coupling: instead of snow blocks, use, in a clever way, crossroads with crossings open in one direction but closed in the traverser direction. Now imagine four new cities where these crossroads meet, in a way that the map of the new cities and the crossroads be similar to the original road map. Then think about the question in terms of cities A, C and the corresponding two new cities. (Write up each step of your argument in details.)

Homework #5 (Due on October 15)

Chapter 4, Problem $4^{\bullet\bullet}$, $21^{\bullet\bullet}$, $25^{\bullet\bullet}$, $26^{\bullet\bullet}$, $30^{\bullet\bullet}$, $37^{\bullet\bullet}$, $38^{\bullet\bullet}$,
Chapter 4, Theoretical Exercise $6^{\bullet\bullet\bullet}$, $7^{\bullet\bullet\bullet}$

Homework #6 (Due on October 22)

Chapter 4, Problem $42^{\bullet\bullet}$, $46^{\bullet\bullet}$, $50^{\bullet\bullet\bullet}$, $51^{\bullet\bullet}$, $56^{\bullet\bullet}$, $64^{\bullet\bullet\bullet}$,
Chapter 4, Theoretical Exercise $16^{\bullet\bullet\bullet}$, $25^{\bullet\bullet\bullet}$,
Bonus problem #6A: The average density of a forest is 16 trees on every 100 square yards. The tree trunks can be considered as cylinders of a diameter of 0.2 yards. We are standing inside the forest, 120 yards from its edge. If we shoot a gun bullet out of the forest without aiming, what is the probability that it will hit a tree trunk? (Ignore the marginal fact that centers of the tree trunks cannot be closer than 0.2 yards to each other.)

Homework #7 (Due on October 29)

Chapter 4, Problem $19^{\bullet\bullet}$, $75^{\bullet\bullet\bullet}$, $78^{\bullet\bullet\bullet}$
Chapter 5, Problem $2^{\bullet\bullet}$, $3^{\bullet\bullet}$, $5^{\bullet\bullet}$, $6^{\bullet\bullet\bullet}$,
Problem #7A$^{\bullet\bullet\bullet}$: For which values of α and c will the function $F(x) = \exp(-ce^{-\alpha x})$ be a distribution function? For such values, what is the corresponding density?
Bonus Problem #7B: Two players target shoot, player one begins. They shoot by turns, and hit the target with probability p_1 and p_2, respectively. The winner is the one who hits the target first. What is the probability that the first player wins? What is the expected number of shots in this game? (The expected number of shots is easy to compute in case $p_1 = p_2 = p$ (why?). Compute it, and verify if it agrees with your result when plugging in $p_1 = p,\ p_2 = p$.)

Homework #8 (Due on November 5)

Chapter 5, Problem $10^{\bullet\bullet}$, $12^{\bullet\bullet\bullet}$ (If you are interested, try to find the optimal solution (no points for this, but it's fun).), $18^{\bullet\bullet}$, $21^{\bullet\bullet}$, $26^{\bullet\bullet\bullet}$, $33^{\bullet\bullet}$,
Problem #8A$^{\bullet\bullet\bullet}$: A stick of length l is broken randomly. What is the distribution function of the length of the shorter piece?
Problem #8B$^{\bullet\bullet\bullet}$: A fraction p of citizens in a city smoke. We are to determine the value of p by making a survey involving n citizens who we select randomly. If k of these n people smoke, then $p' = k/n$ will be our result. How large should we choose n if we want our result p' to be closer to the real value p than 0.005 with probability at least 0.95? In other words: determine the smallest number n_0 such that $P(|p' - p| \le 0.005) \ge 0.95$ for any $p \in (0,\, 1)$ and $n \ge n_0$.
Bonus Problem #8C: Compute the "absolute moments"

$$\mathbb{E}(|Y|^k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} |y|^k e^{-y^2/2} dy$$

of the standard normal distribution.
HINT: for even k's, compute and use

$$\frac{d^l}{d\lambda^l} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\lambda y^2/2} dy \bigg|_{\lambda=1}.$$

For odd k's, introduce the new variable $z = y^2$ in the integral.

Homework #9 (Due on November 12)

Chapter 5, Problem $41^{\bullet\bullet\bullet}$,
Chapter 6, Problem $1^{\bullet\bullet\bullet}$, $16^{\bullet\bullet\bullet}$,
Chapter 5, Theoretical Exercise $14^{\bullet\bullet}$, $29^{\bullet\bullet\bullet}$, $30^{\bullet\bullet}$ and also$^{\bullet}$: show here (without much computation) that CY^{α} has also lognormal distribution with parameters $\mu' = \alpha\mu + \log C$ and $\sigma'^2 = \alpha^2\sigma^2$. HINT: As we have $Y = e^X$ with X normal, write CY^{α} as e^Z, and figure out the connection of Z to X. What kind of variable is Z? With which parameters?
Problem #9A$^{\bullet\bullet\bullet}$: We randomly select a point on the $[0, 1]$ interval of the x axis. Let ξ denote the distance between this point and the point at coordinate $(0, 1)$ of the plane. Determine the density function of the distribution of the random variable ξ.
Bonus Problem #9B: Are the functions

$$F(x, y) = \exp(-e^{-(x+y)}), \qquad G(x, y) = \exp(-e^{-x} - e^{-y})$$

joint distribution functions?

Homework #10 (Due on November 19)

Chapter 6, Problem $12^{\bullet\bullet}$, $13^{\bullet\bullet}$, $20^{\bullet\bullet}$, $40^{\bullet\bullet}$, $43^{\bullet\bullet}$, $51^{\bullet\bullet}$ HINT: Write up the definition of the joint distribution function of R and Θ, which is in fact the probability of some set of points. Make a good picture of that set to determine its probability, i.e. the joint distribution function. Then find the density. It's easy!
Chapter 6, Theoretical Exercise $6^{\bullet\bullet}$, $14^{\bullet\bullet}$
Problem #10A$^{\bullet\bullet}$: Choose a random point in the unit square of the plane. Let ξ be its distance from the closest edge of the square. Determine the distribution function of ξ. HINT: Draw!
Problem #10B$^{\bullet\bullet}$: We flip a fair coin three times. Denote the number of heads and tails by X and Y, respectively. Compute the expectation and variance of the variable $Z := XY$.
Bonus Problem #10C: Let X and Y be i.i.d. geometric random variables, both with parameter p. Define $U := \min\{X, Y\}$ and $V := X - Y$. Show that U and V are independent.

Homework #11 (Due on November 26)

Chapter 7, Problem $5^{\bullet\bullet}$, $8^{\bullet\bullet\bullet}$, $12^{\bullet\bullet\bullet\bullet}$, $18^{\bullet\bullet}$, $21^{\bullet\bullet\bullet}$ (Number of distinct birthdays = number of days with at least one birthday.),
Chapter 7, Theoretical Exercise $9^{\bullet\bullet\bullet}$, $13^{\bullet\bullet\bullet}$ HINT: Define the indicator variable I_i to be one if X_i happens to be a record value, and use the fact that each of $X_1, X_2, \ldots, X_i$ is equally likely to be the largest among themselves.

Homework #12 (Due on December 3)

Chapter 7, Problem $26^{\bullet\bullet}$ HINT: Find the distribution function of max or $\min(X_1, \ldots, X_n)$. $30^{\bullet\bullet}$, $40^{\bullet\bullet}$, $47^{\bullet\bullet\bullet}$ HINT: Use indicator variables for the edges. $50^{\bullet\bullet}$, $54^{\bullet\bullet\bullet}$, $68^{\bullet\bullet}$
Chapter 7, Theoretical Exercise $29^{\bullet\bullet}$ HINT: Use symmetry and the fact that (finite) summation and conditional expectation (with the same condition) are interchangeable.
Problem #12A$^{\bullet\bullet}$: In my purse the number of pennies, nickels, dimes and quarters are i.i.d. Poisson random variables with parameter λ. Use linearity of expectations to compute the expectation and variance of the joint value of my coins.

Homework #13 (Due on December 10)

Chapter 7, Problem $73^{\bullet\bullet\bullet}$, $76^{\bullet\bullet\bullet}$, $78^{\bullet\bullet\bullet}$
Chapter 7, Theoretical Exercise $41^{\bullet\bullet\bullet}$, $50^{\bullet\bullet\bullet}$,
Problem #13A$^{\bullet\bullet}$: A man takes the train and then transfers to the bus when commuting to

work each day. It takes two minutes for him to walk from the train to the bus at the transfer station. *In principle* the train arrives at 7:30am, and the bus leaves at 7:37am. The fact is that the train arrives at a normally distributed random time, having mean 7:30am and standard deviation 4 minutes. Independently, the bus leaves at a normally distributed random time with mean 7:37am and standard deviation 3 minutes. What is the probability that our man misses his bus at most once on the five working days of the week? HINT: Use what we know about the sum of normal variables.

Problem #13B•••: This problem shows how to decompose a bivariate normal distribution to independent normals. Fix $\sigma_x,\, \sigma_y > 0$, $\mu_x,\, \mu_y \in \mathbb{R}$, and $-1 < \varrho < 1$ parameters. Let X be a normal random variable with mean μ_x, variance σ_x^2. Let Z be another normal random variable, independent of X, with mean $\mu_y - \varrho \cdot \mu_x \cdot \frac{\sigma_y}{\sigma_x}$ and variance $(1-\varrho^2)\cdot\sigma_y^2$. Define $Y := \varrho \cdot \frac{\sigma_y}{\sigma_x} \cdot X + Z$. Show that the pair $(X,\, Y)$ has bivariate normal distribution with respective means μ_x and μ_y, variances σ_x^2 and σ_y^2, and correlation ϱ.

Bonus Problem #13C: Let X and Y be independent standard normal variables. Define $U := (X+Y)/\sqrt{2}$ and $V := (X-Y)/\sqrt{2}$. Show that U and V are also independent standard normal variables. HINT: Find the joint moment generating function of U and V, using what we know about the one of X and Y.