Order of current variance in the simple exclusion process

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Joint work with
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(University of Wisconsin - Madison)

Terschelling, September 2006

1. ASEP: Interacting particles
2. ASEP: Surface growth
3. Growth fluctuations
4. The second class particle
5. The upper bound
6. The lower bound

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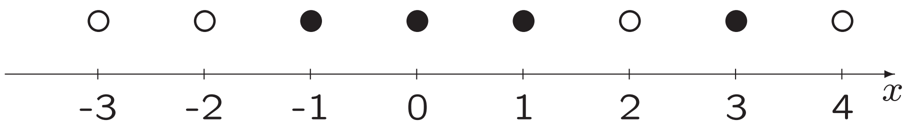

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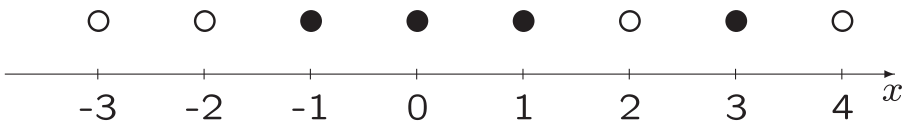

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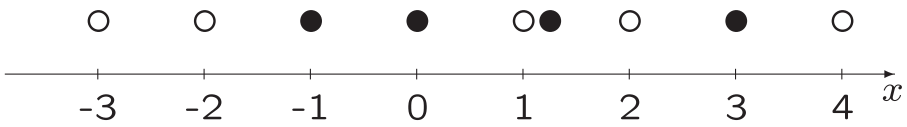

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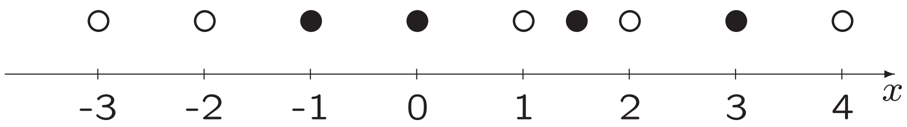

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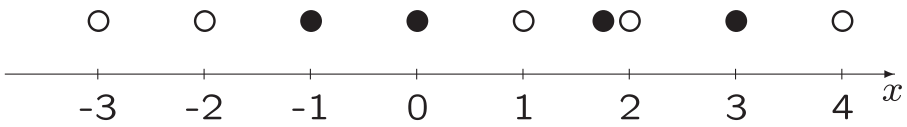

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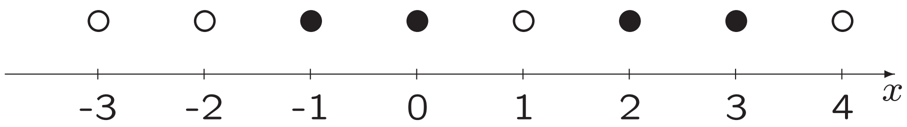

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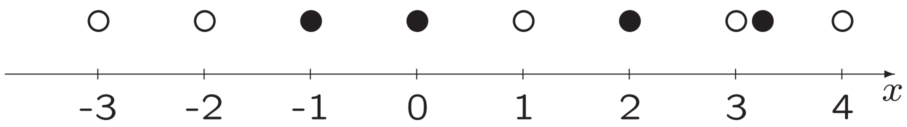

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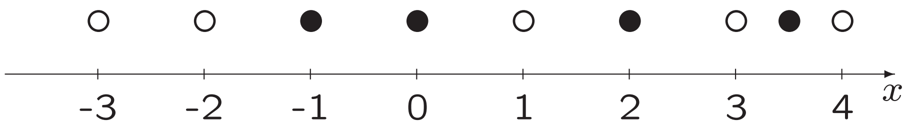

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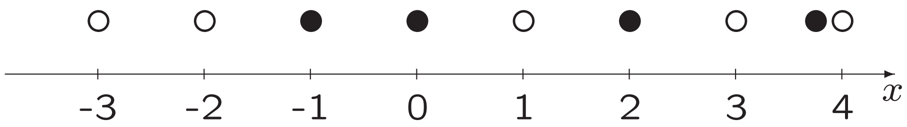

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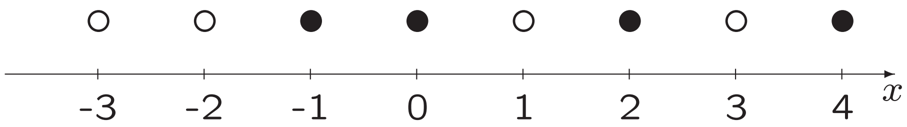

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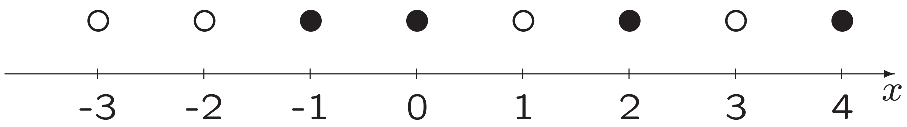

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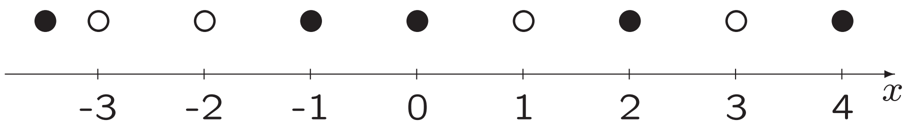

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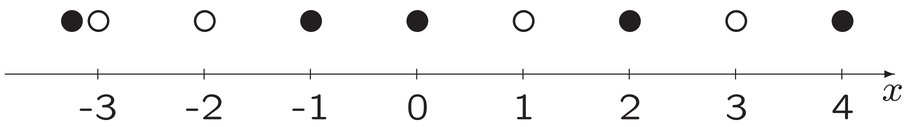

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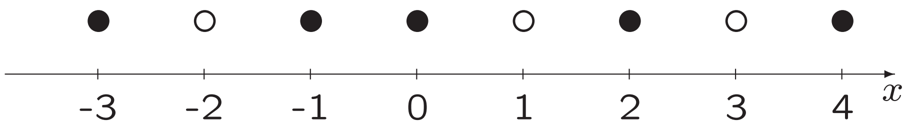

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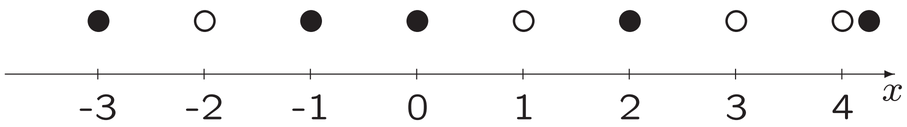

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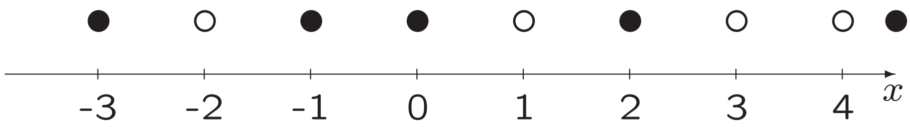

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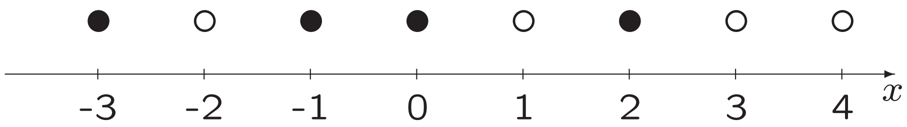

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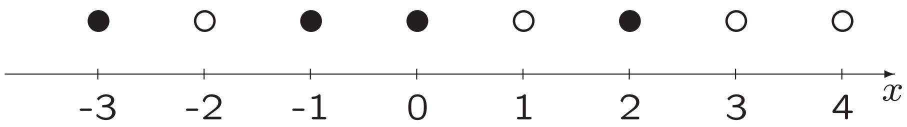

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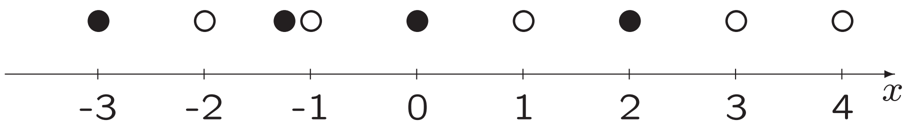

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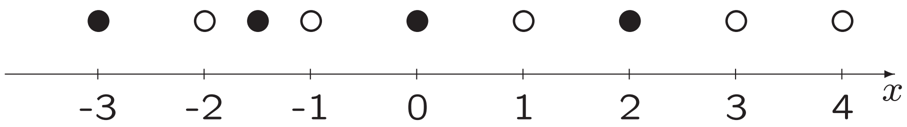

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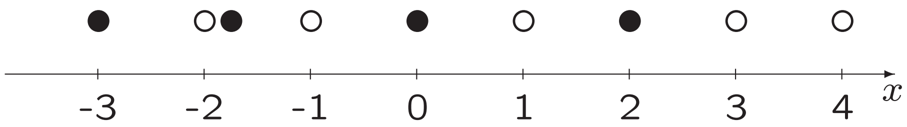

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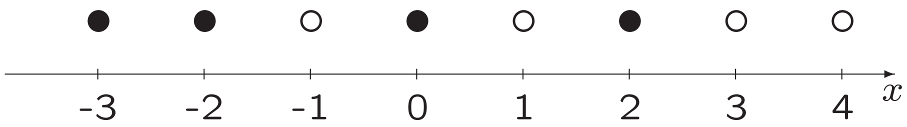

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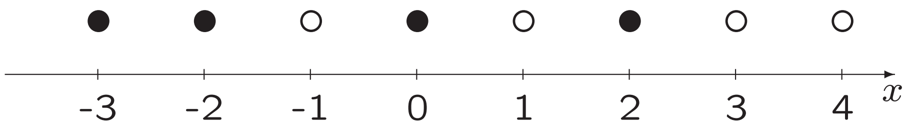

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The Bernoulli(ϱ) distribution is time-stationary for any $(0 \le \varrho \le 1)$. Any translation-invariant stationary distribution is a mixture of Bernoullis.

Hydrodynamics (briefly)

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$\rightsquigarrow$ The characteristic speed $C(\varrho) := a[1-2\varrho]$.
(ϱ is constant along $\dot{X}(T) = C(\varrho)$.)

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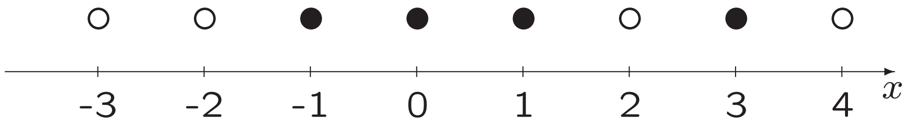

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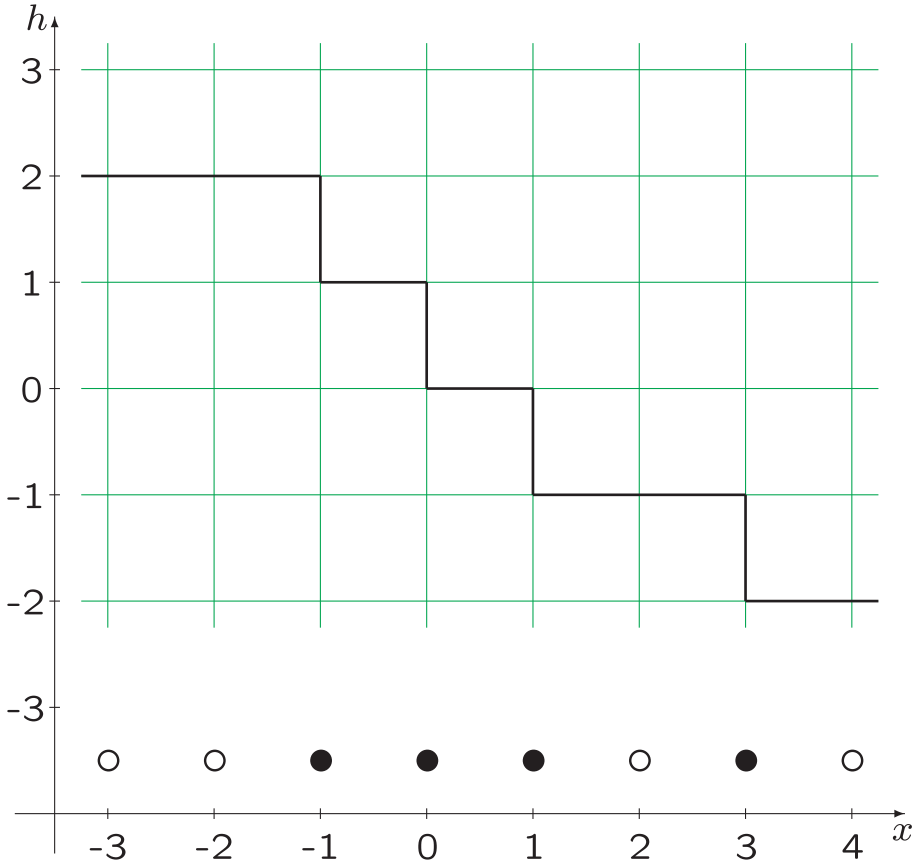

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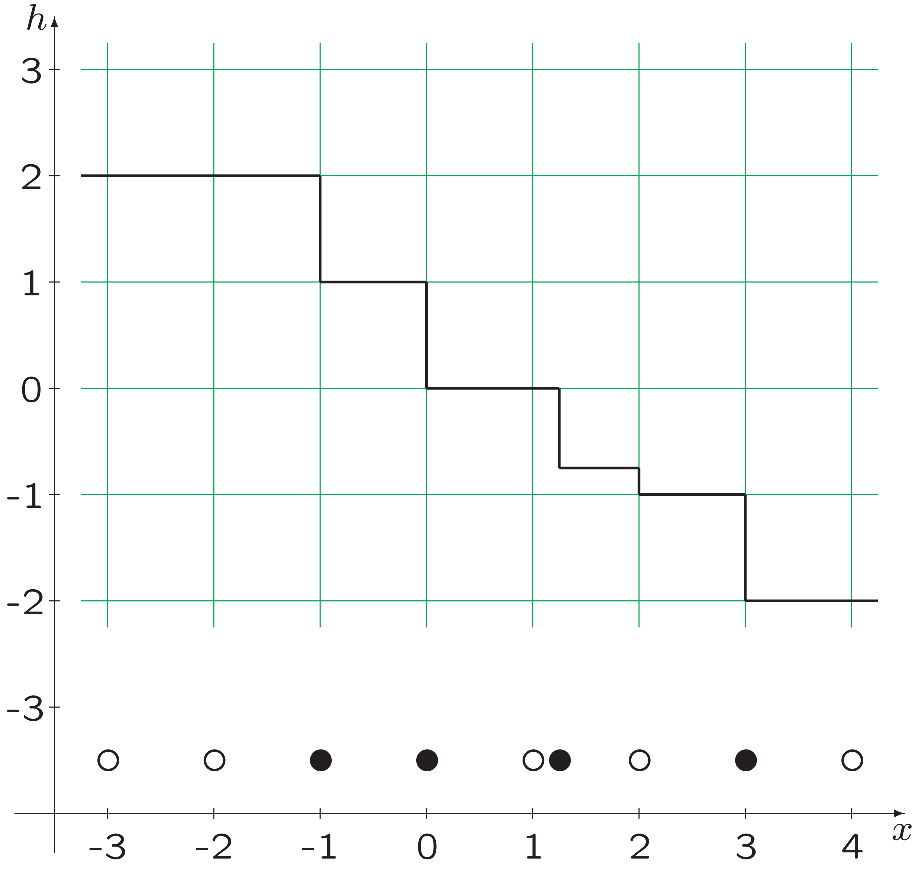

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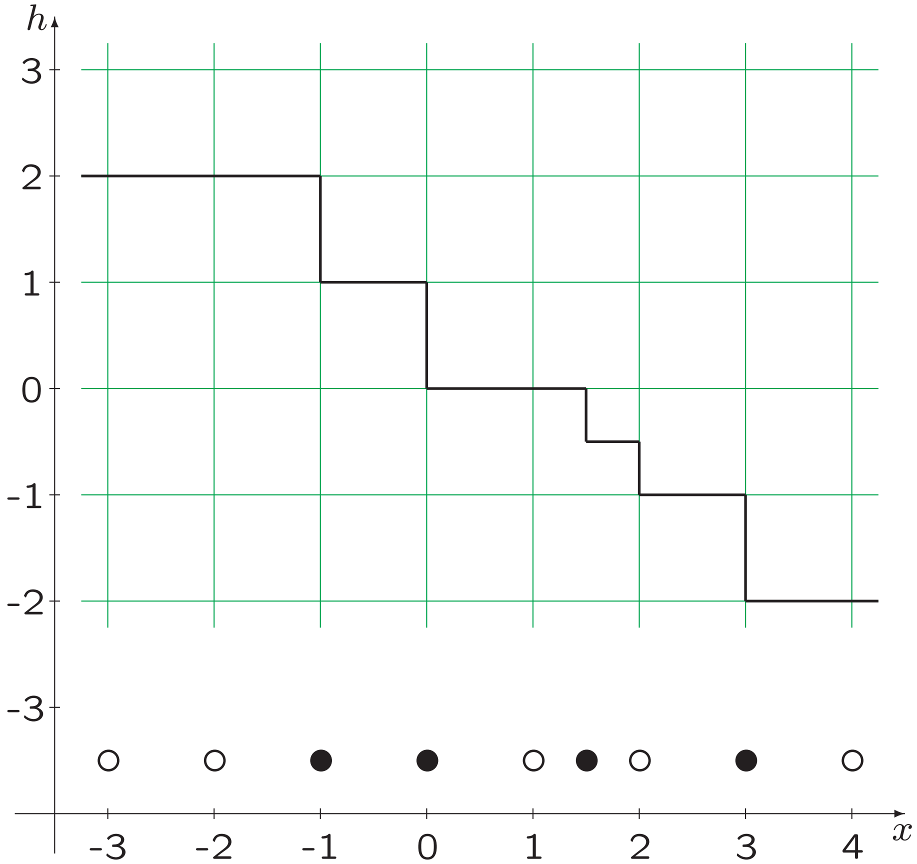

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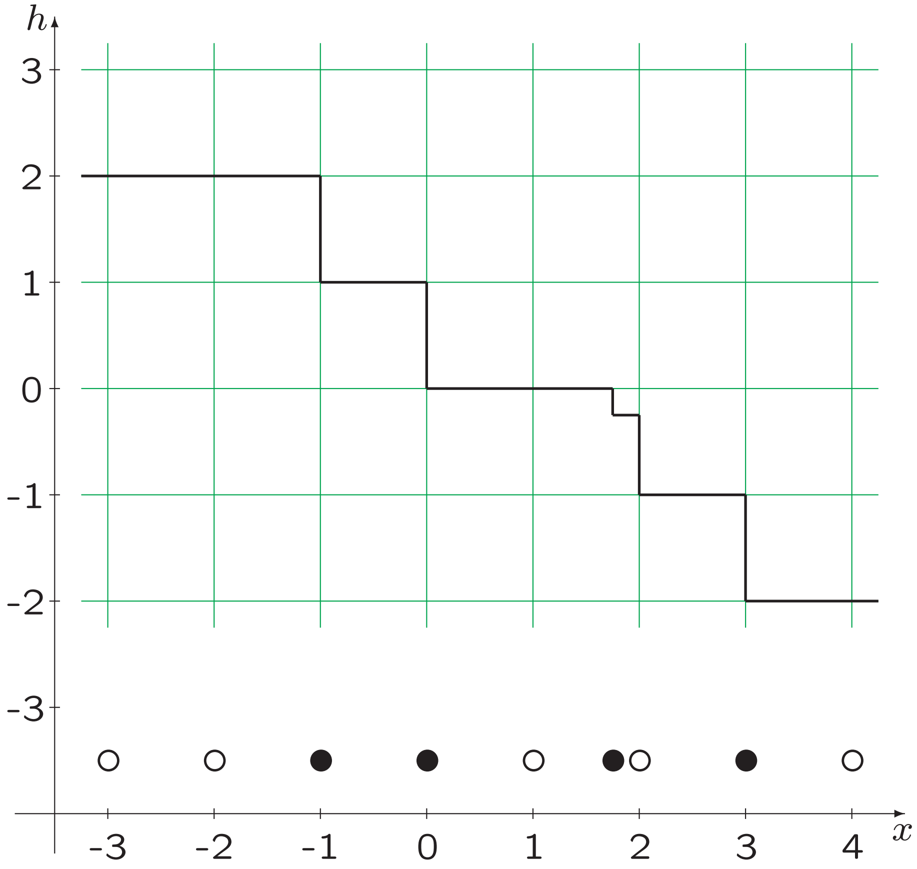

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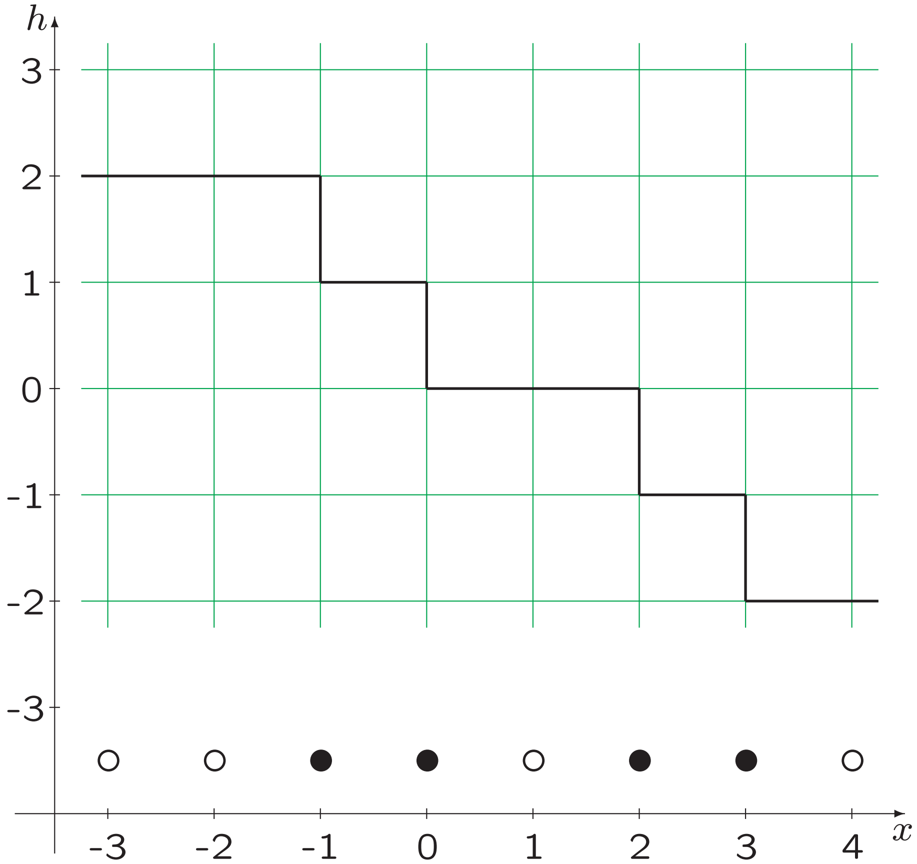

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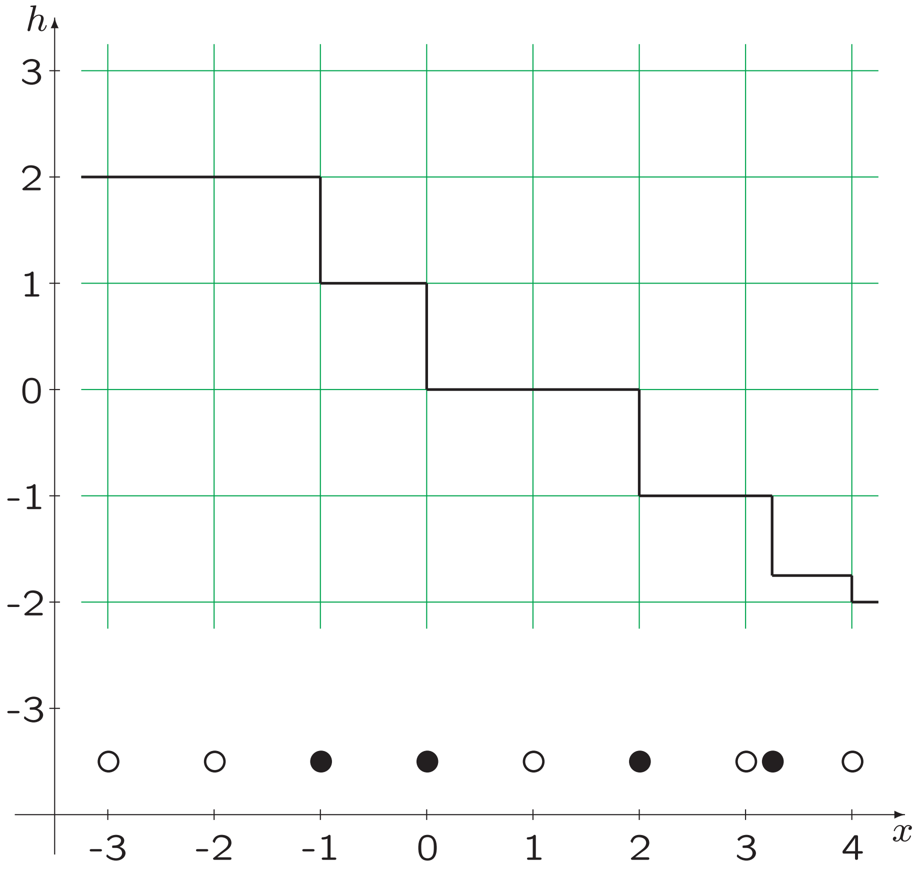

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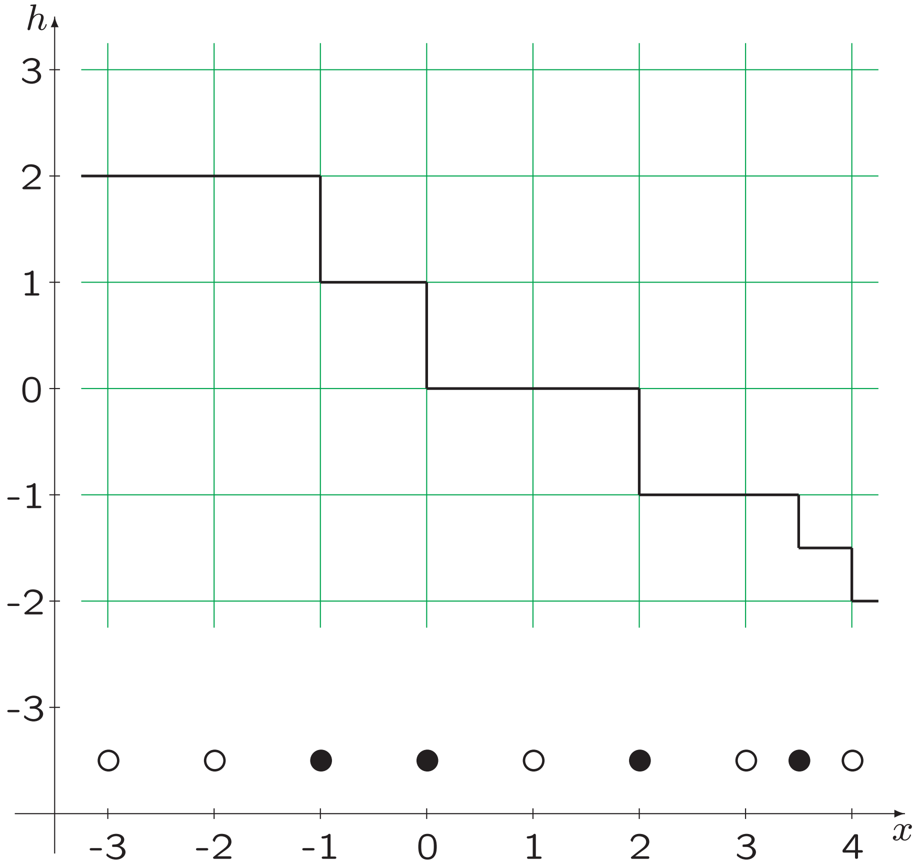

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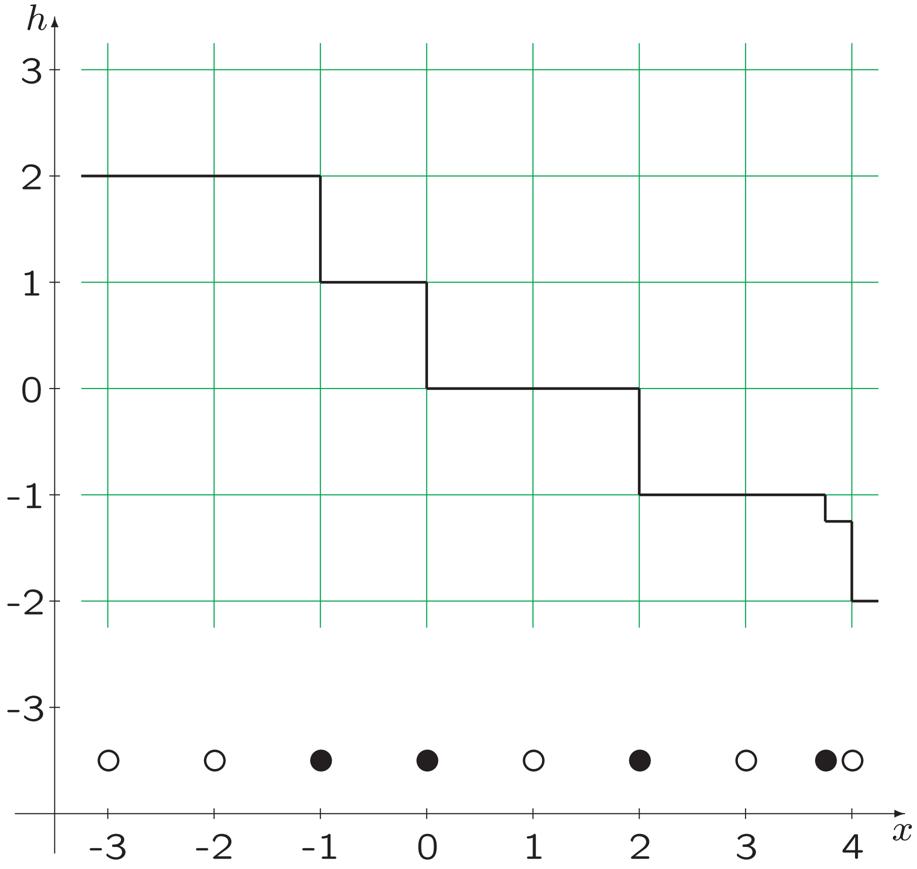

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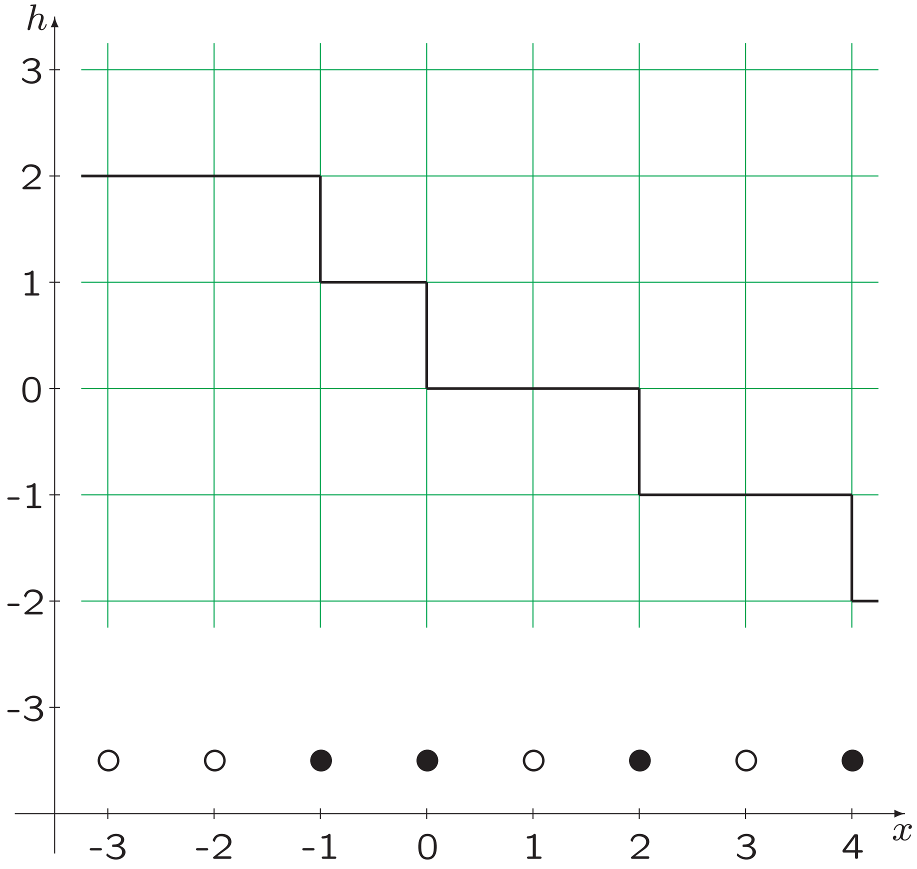

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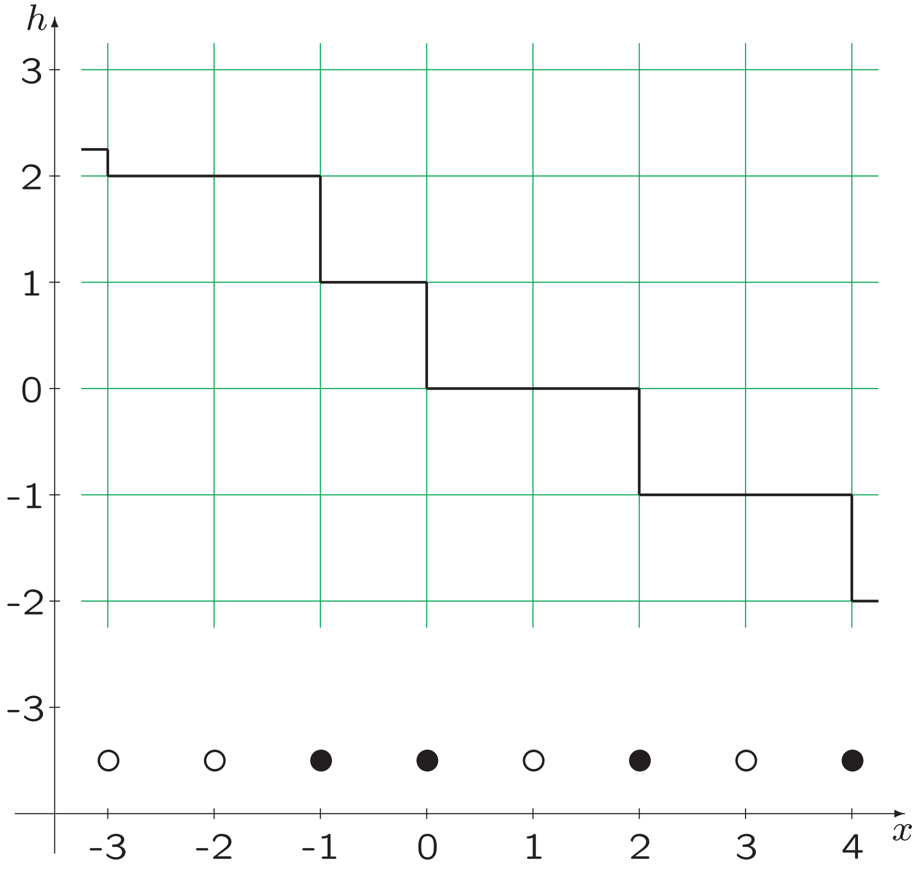

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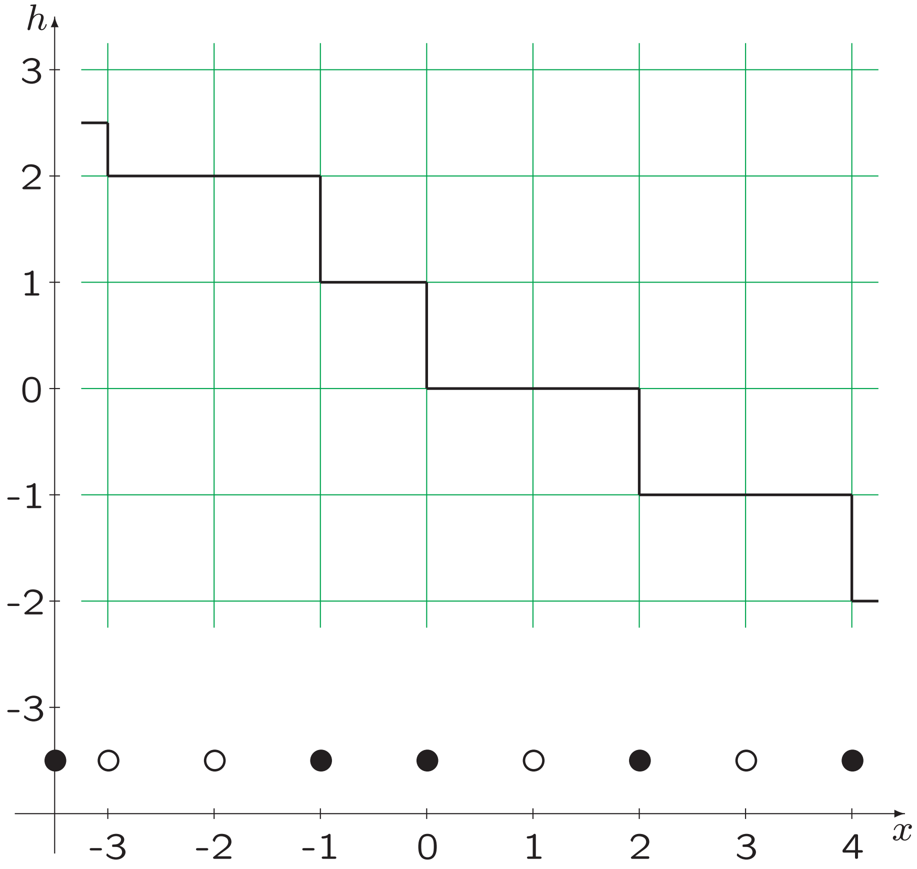

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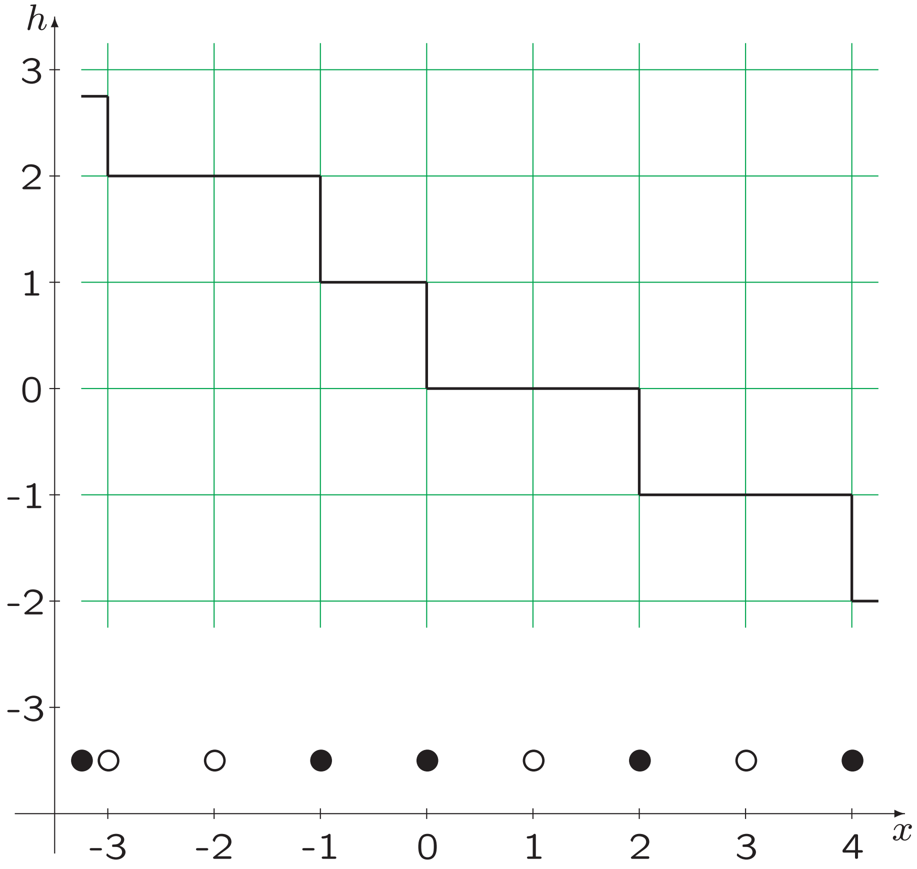

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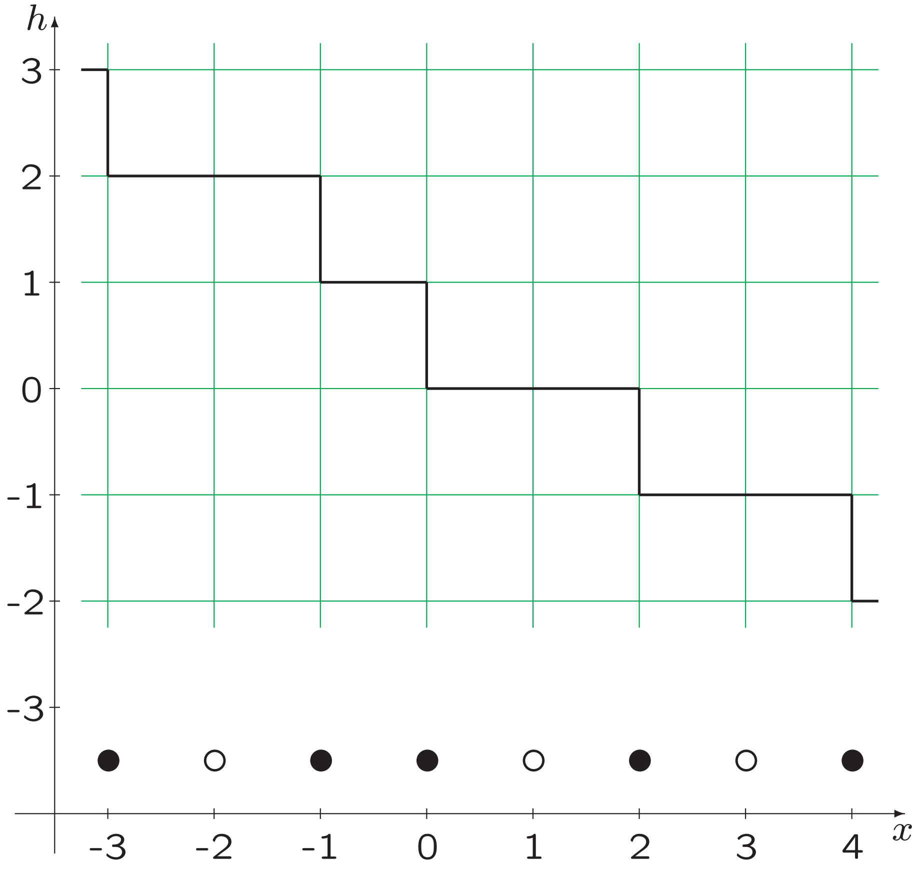

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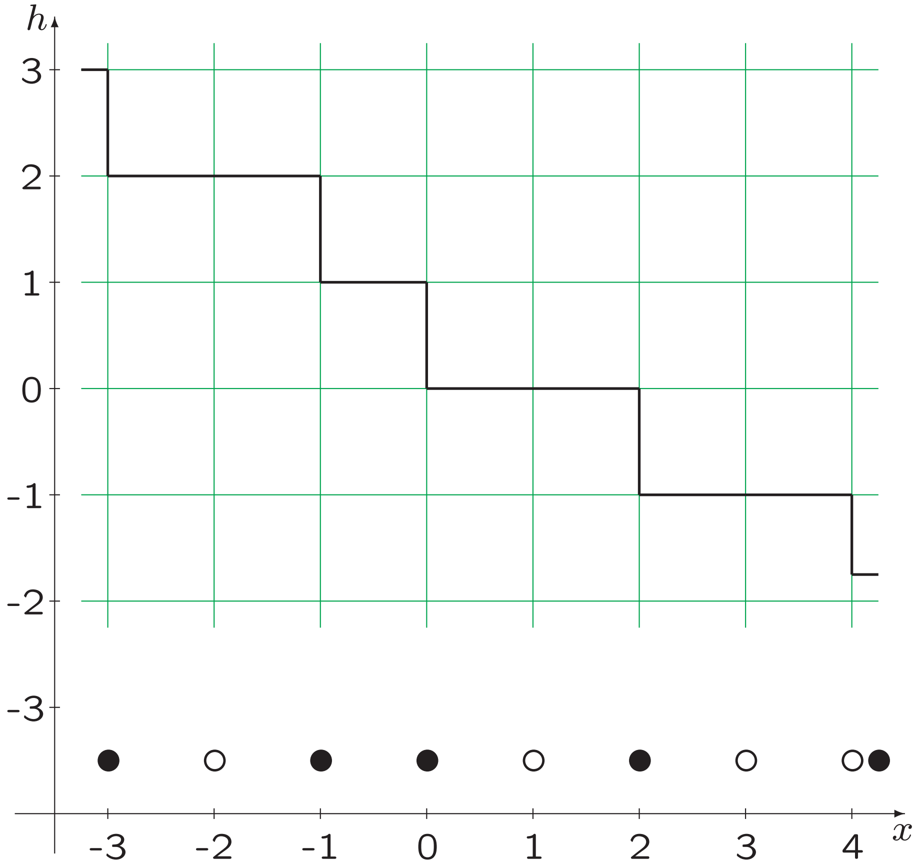

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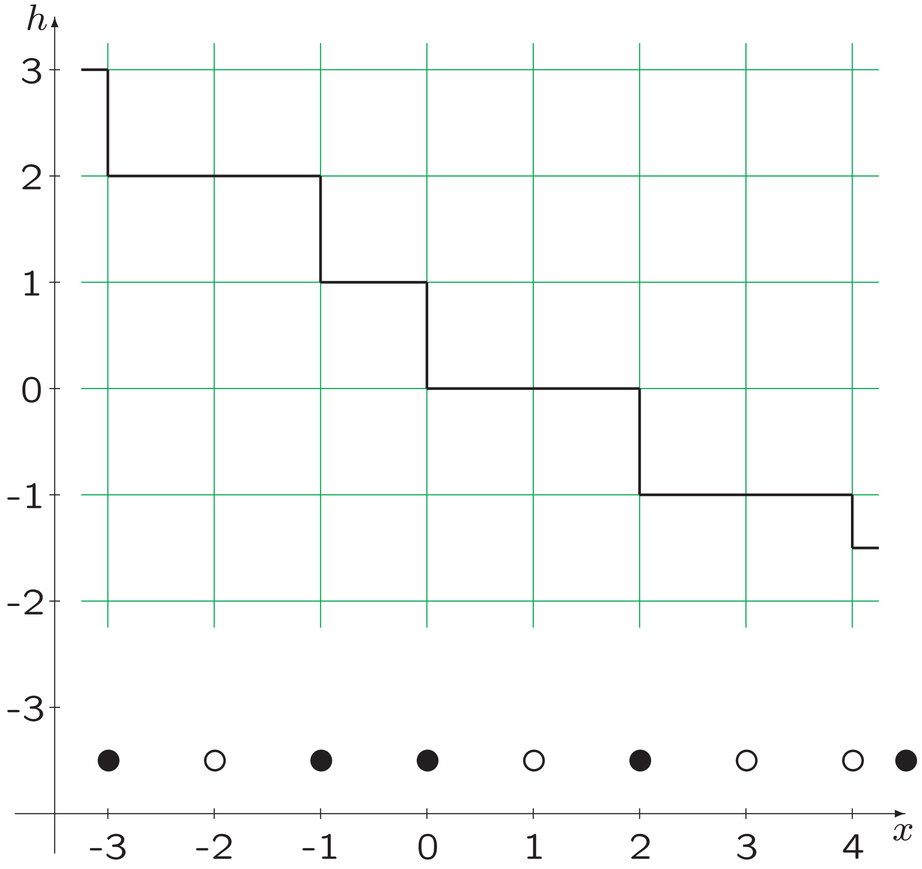

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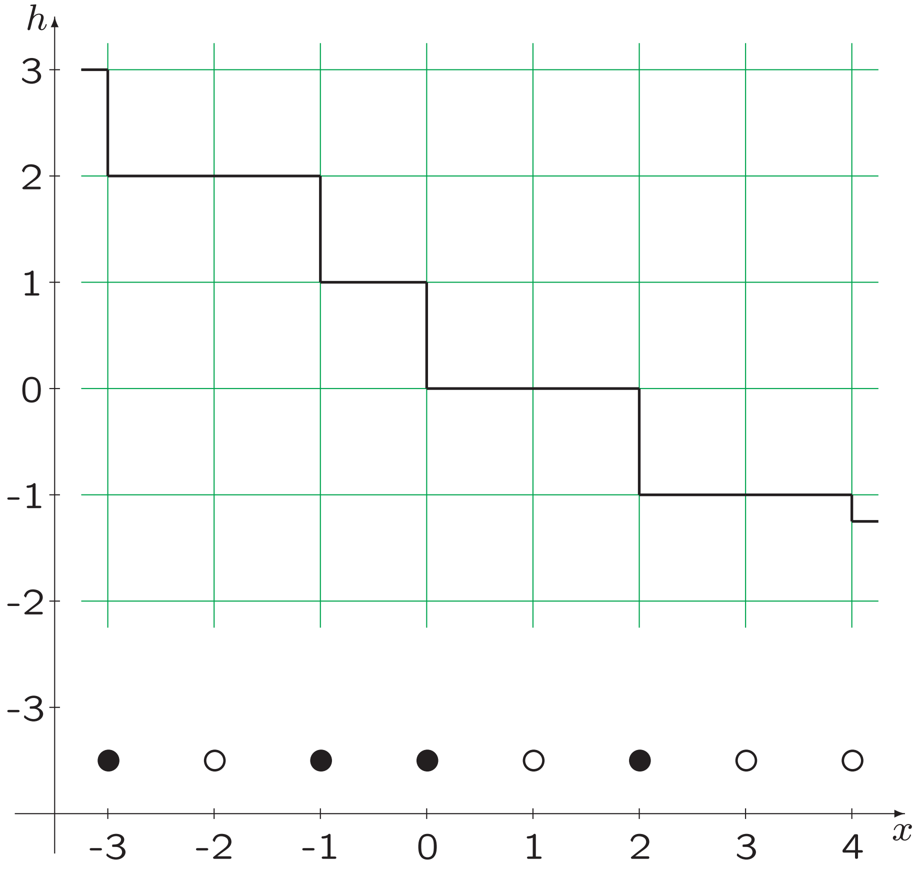

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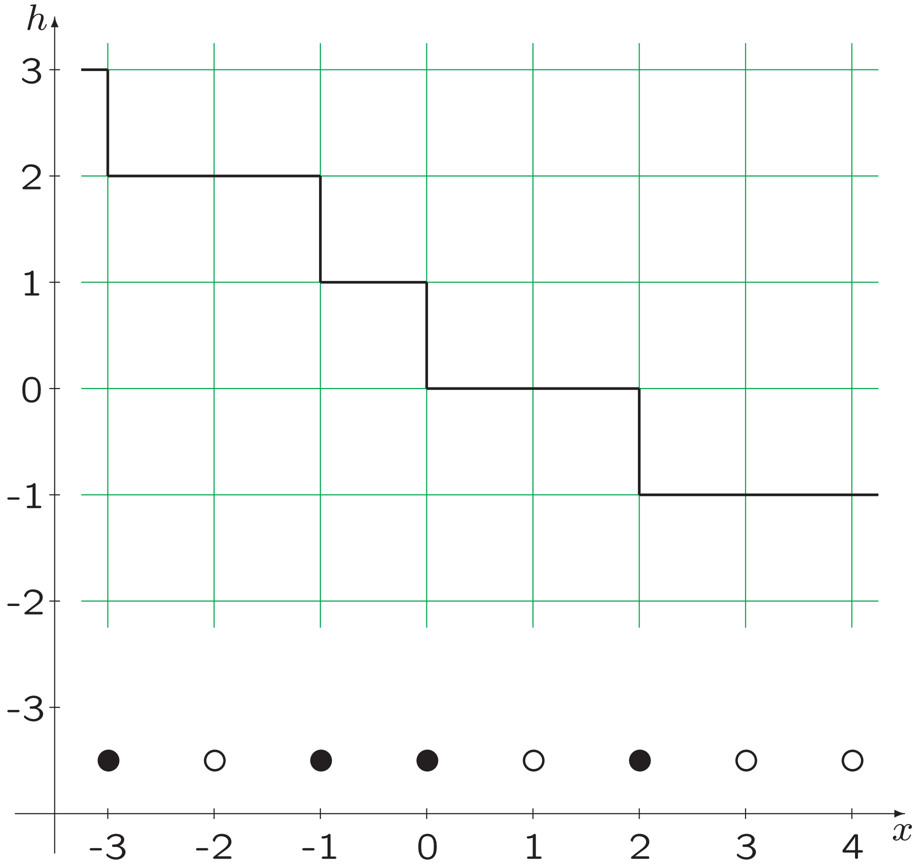

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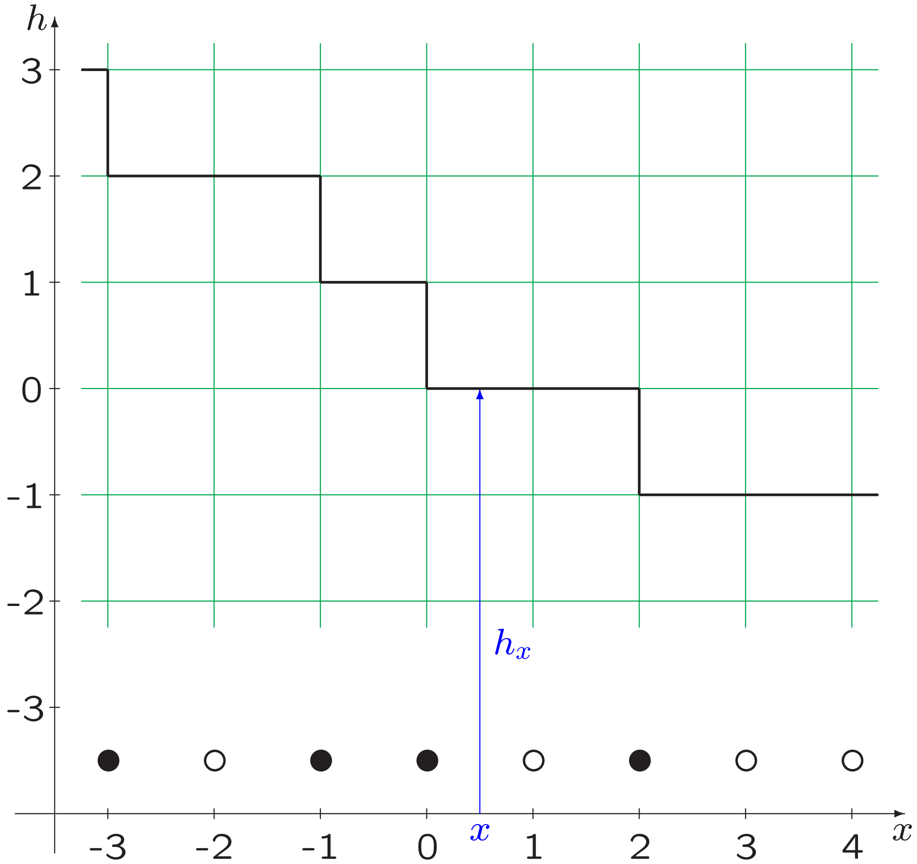

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$h_x(t)$ = height of the surface above x.

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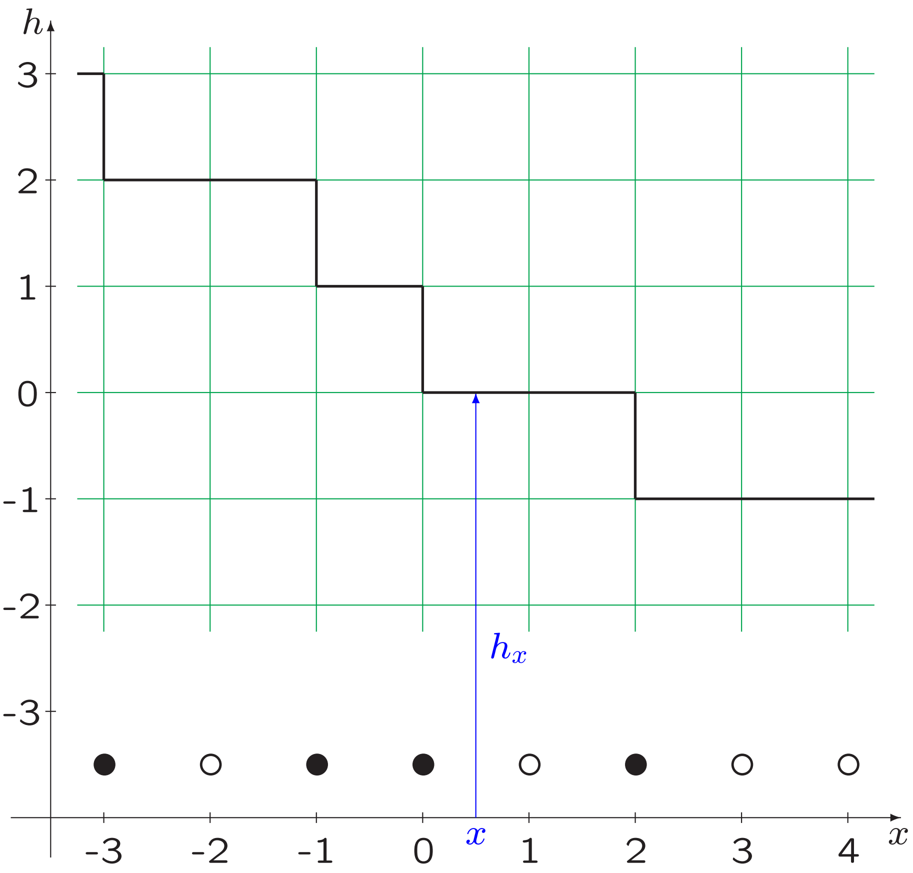

Bernoulli(ϱ) distribution

$h_x(t)$ = height of the surface above x.

$h_x(t) - h_x(0)$ = net number of particles passed above x.

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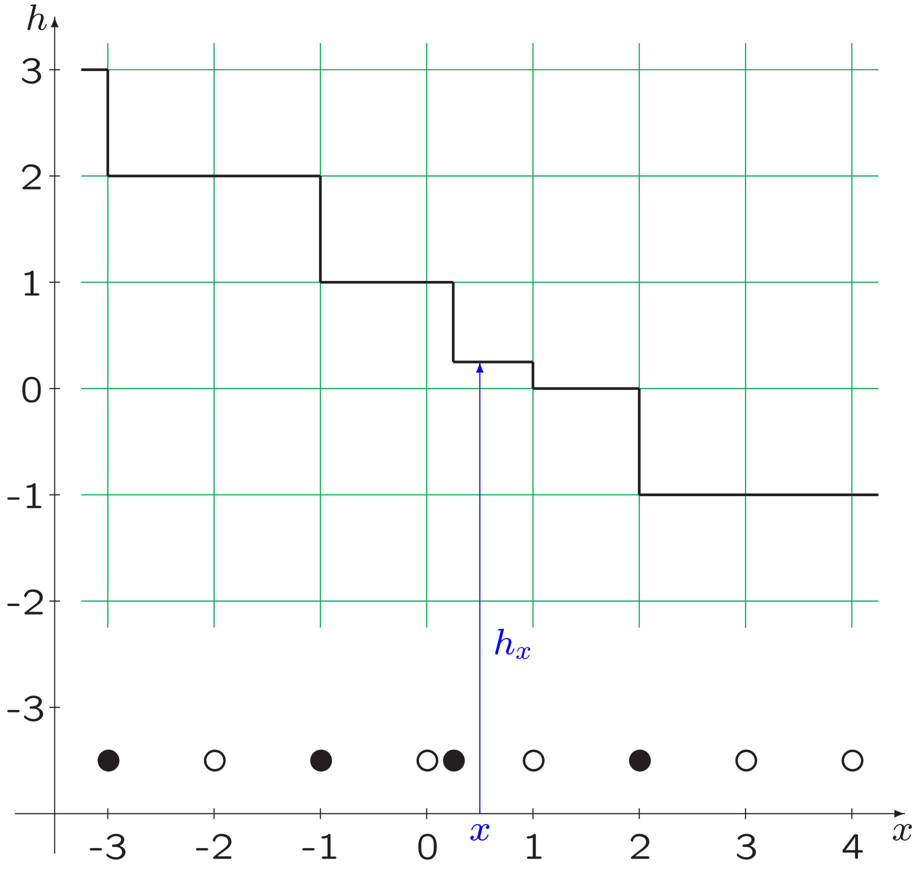

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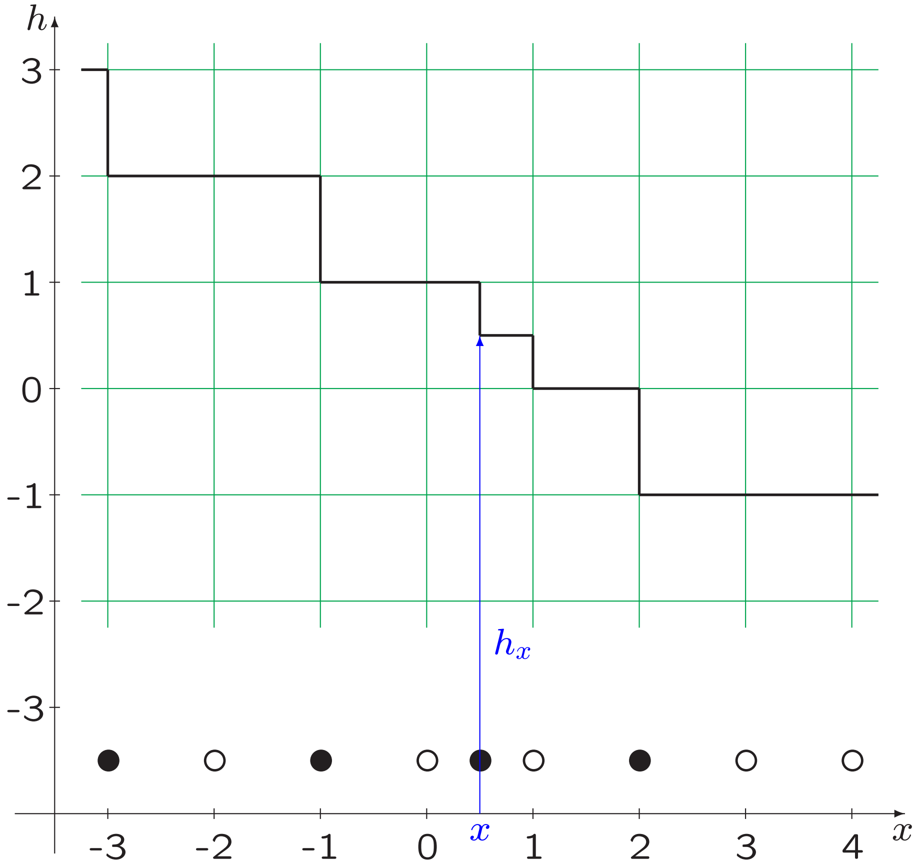

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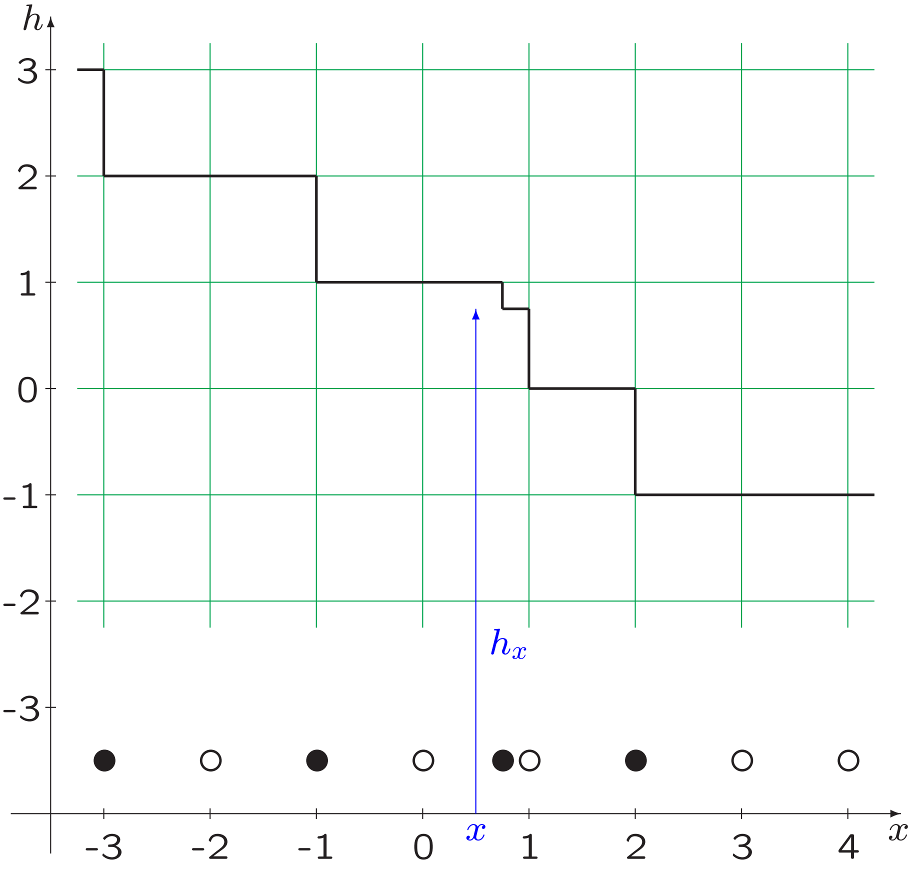

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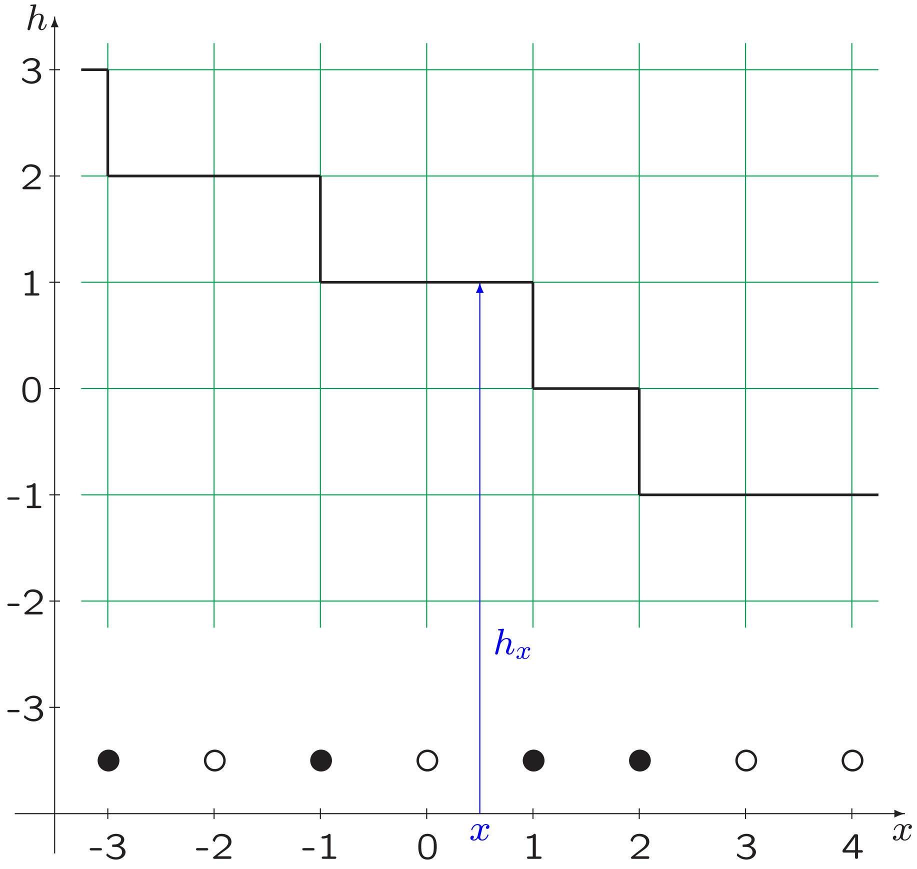

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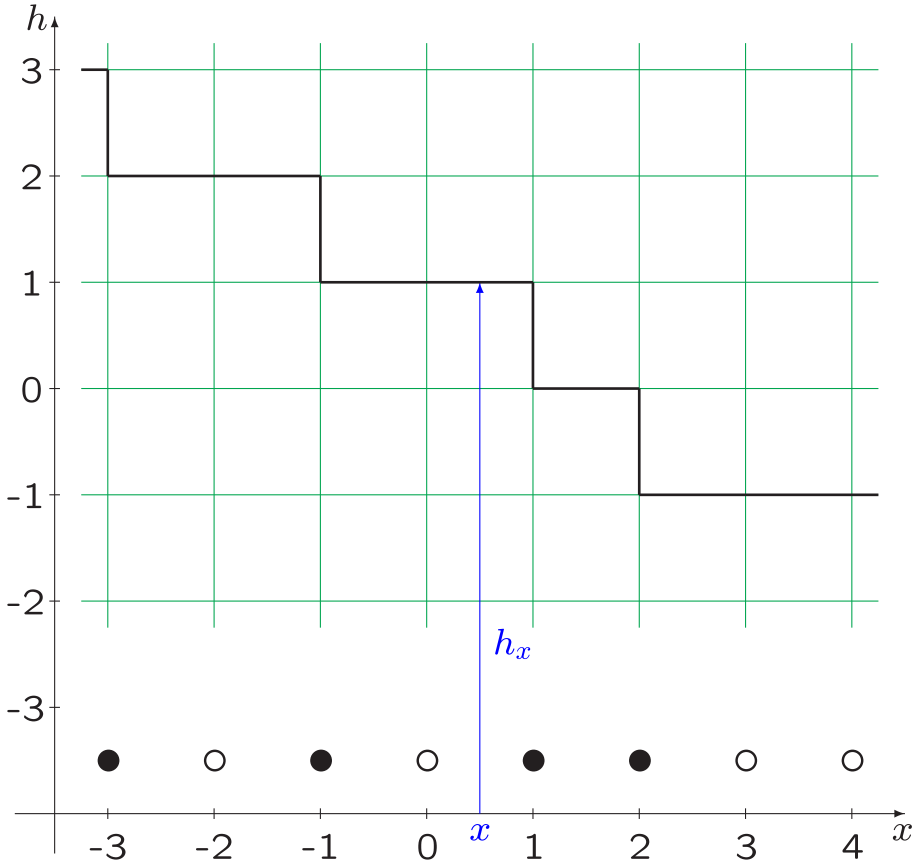

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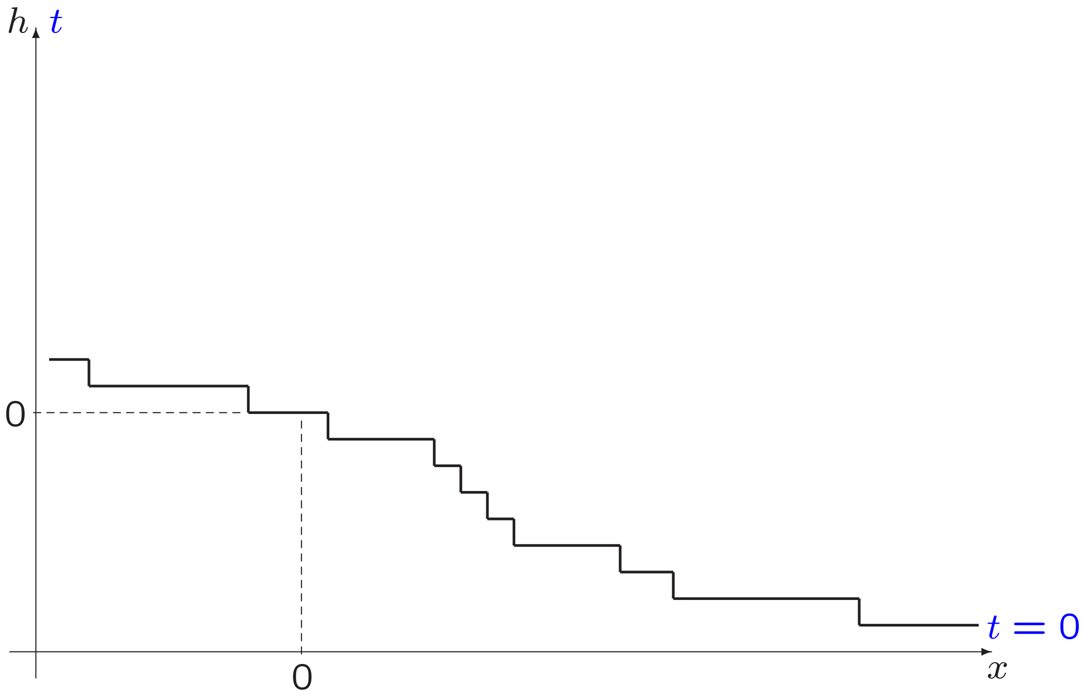
$h_{Vt}(t)$ = net number of particles passed through the moving window at Vt $(V \in \mathbb{R})$.

3. Growth fluctuations



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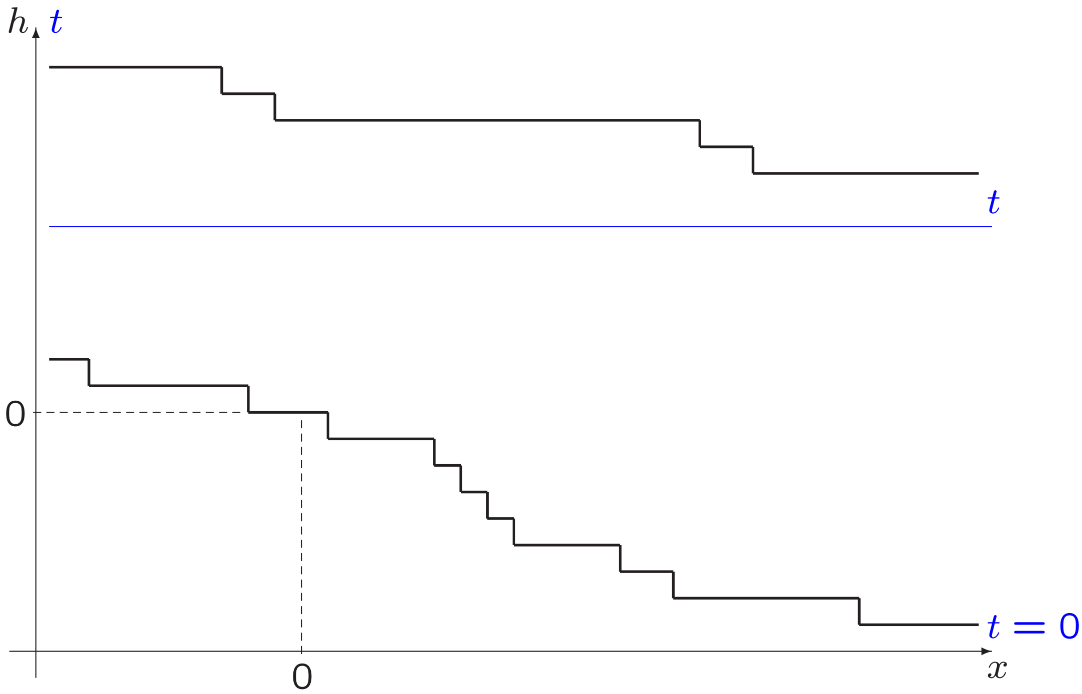

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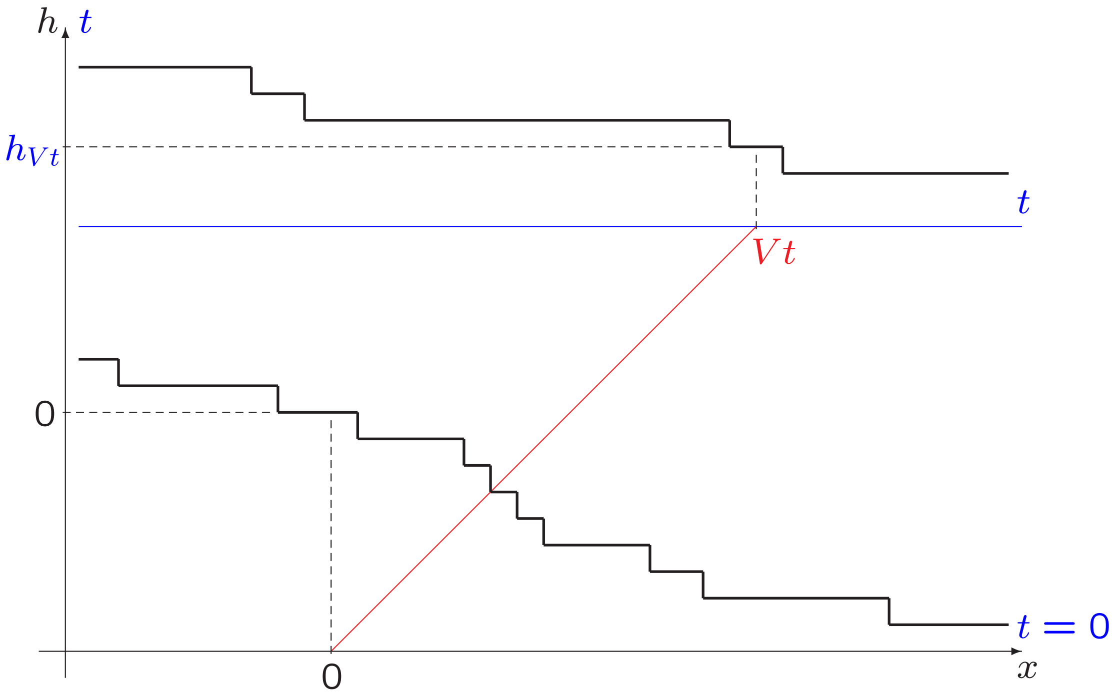

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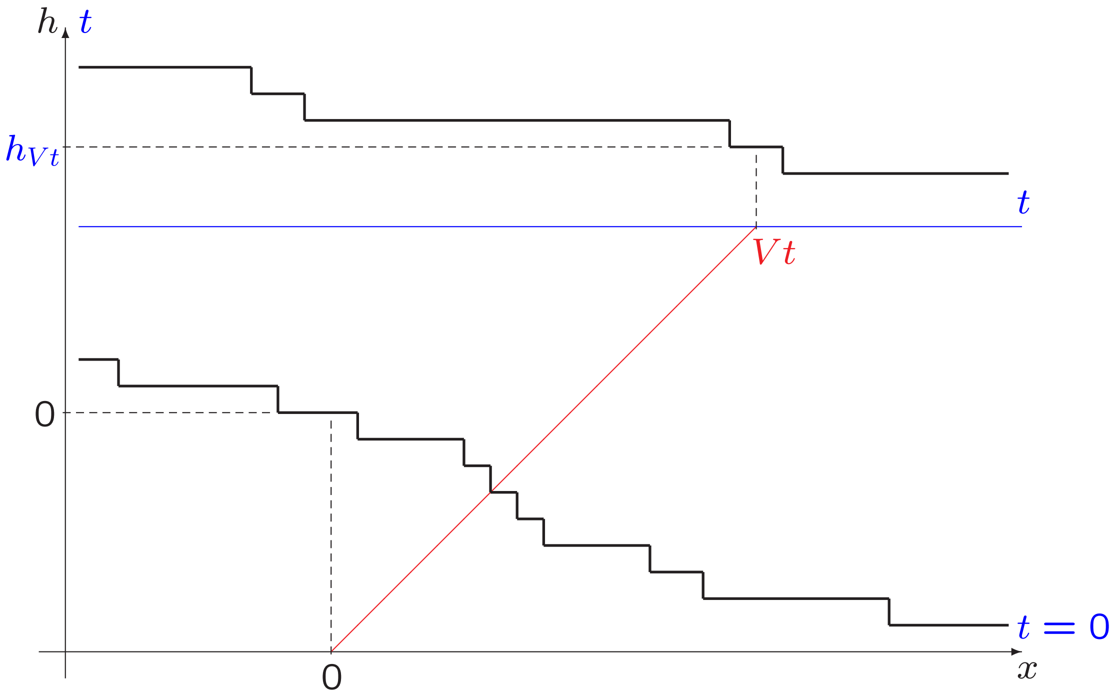

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$$\lim_{t\to\infty} \frac{\mathbf{Var}(h_{Vt}(t))}{t} = \text{const} \cdot |V - C(\varrho)|$$

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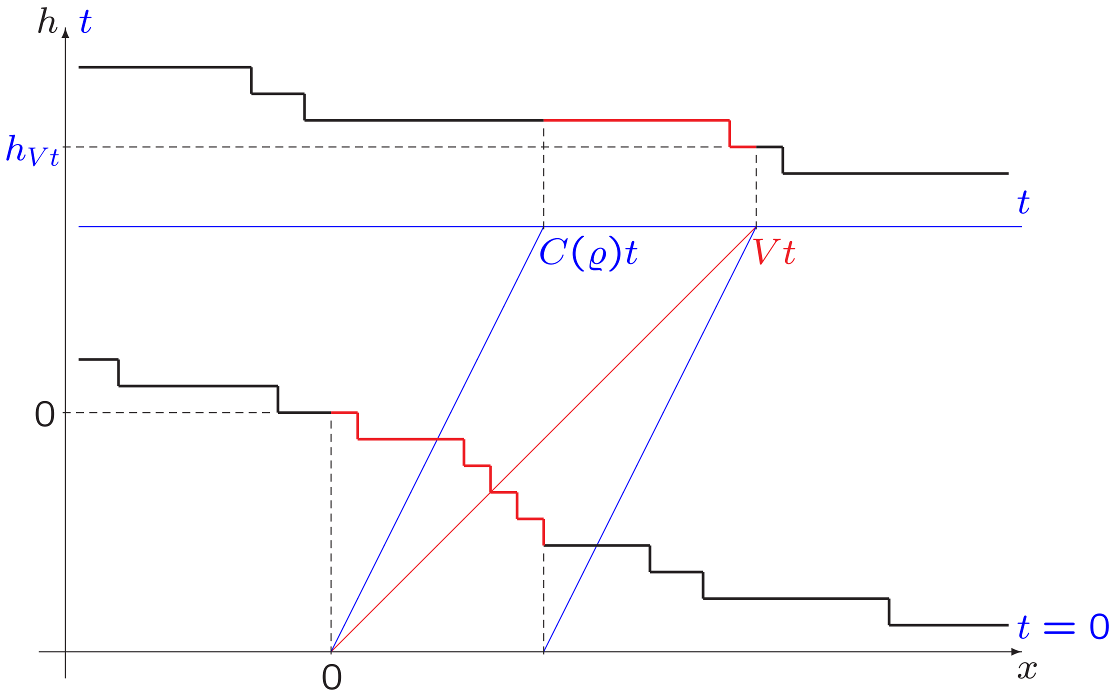

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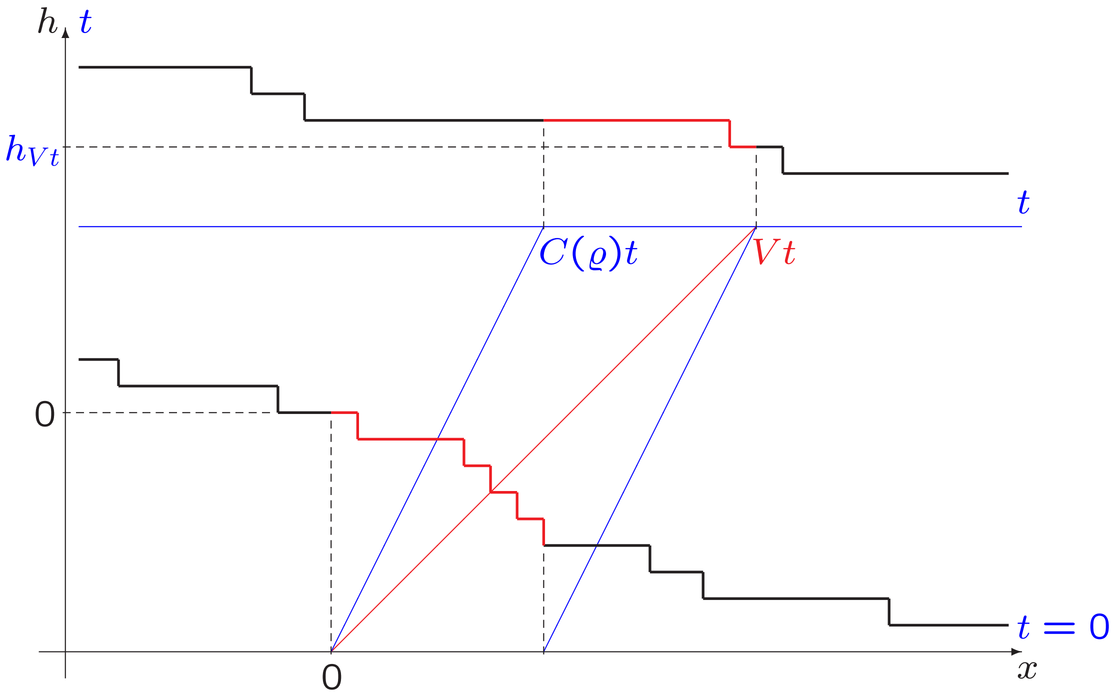

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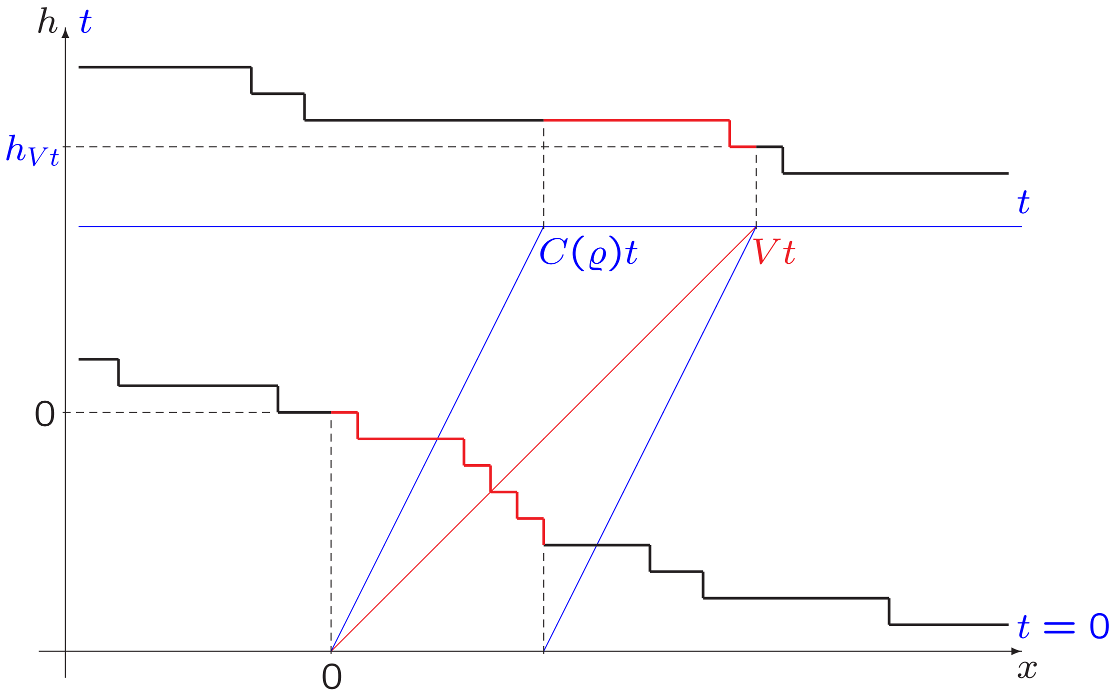

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Conjecture:

$$\lim_{t\to\infty} \frac{\mathbf{Var}(h_{C(\varrho)t}(t))}{t^{2/3}} = \text{[sg. non trivial]}.$$

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Theorem: For any $0 < \varrho < 1$, and any $q < p$,

$$0 < \liminf_{t\to\infty} \frac{\mathbf{Var}(h_{C(\varrho)t}(t))}{t^{2/3}} \leq \limsup_{t\to\infty} \frac{\mathbf{Var}(h_{C(\varrho)t}(t))}{t^{2/3}} \leq \infty.$$

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Limit distributions (not yet controlling the second moment) in terms of the Tracy-Widom distribution were found by Baik, Deift and Johansson 1999, Johansson 2000, and Ferrari and Spohn 2006 for the *totally* asymmetric exclusion (*TASEP*: $p = 1$, $q = 0$).

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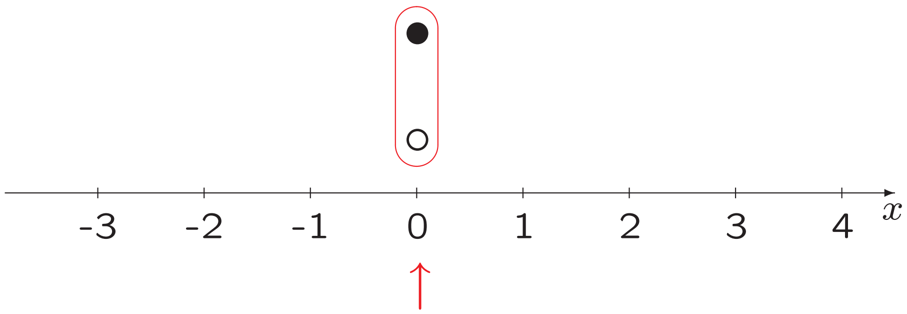
$\rightsquigarrow$ We needed to get rid of these tools. Premises: Cator and Groeneboom 2006 (Hammersley's process), B., Cator and Seppäläinen 2006 (TASEP, last passage).

4. The second class particle



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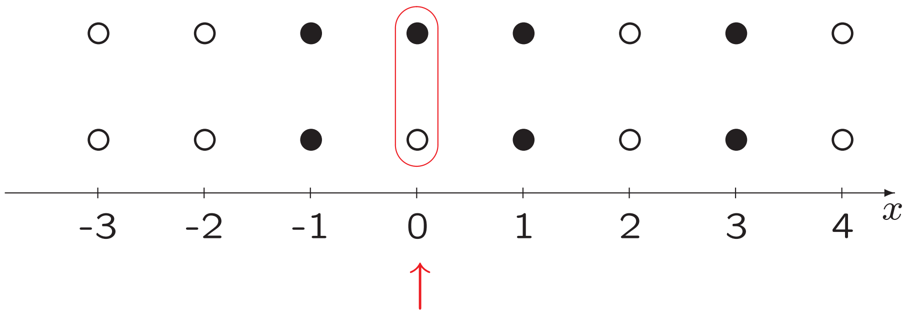

Bernoulli(ϱ) distribution except for 0

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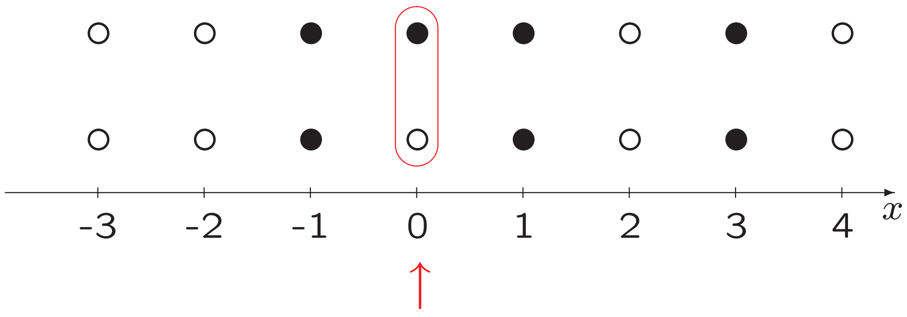

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Coupling: A single discrepancy is always conserved

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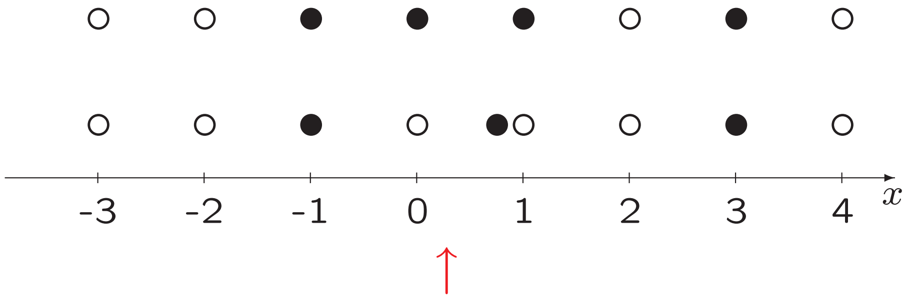

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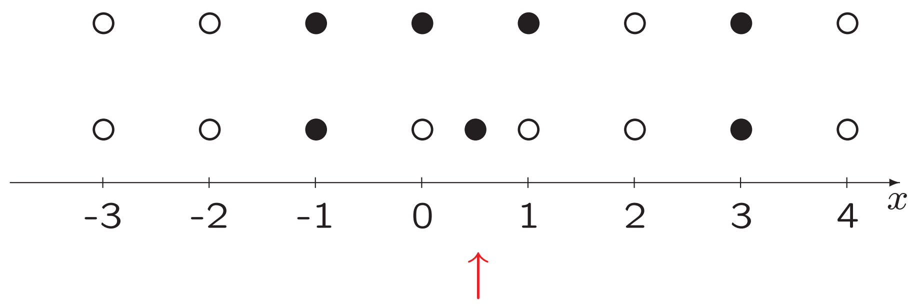

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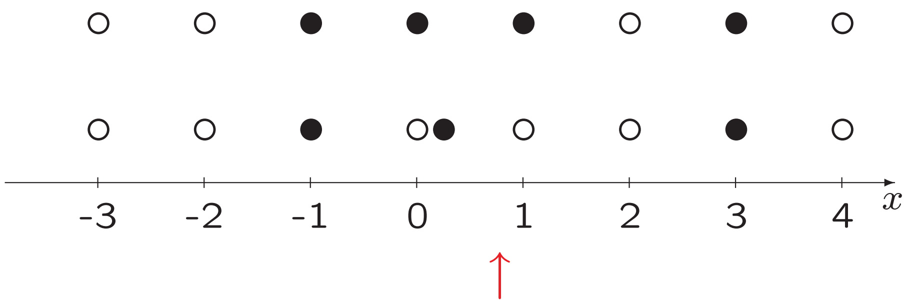

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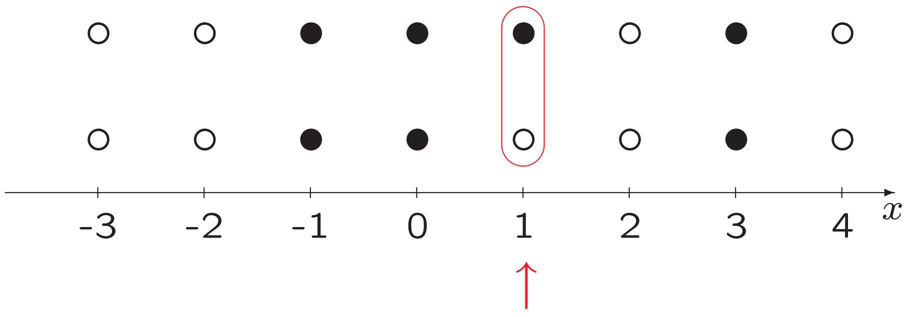

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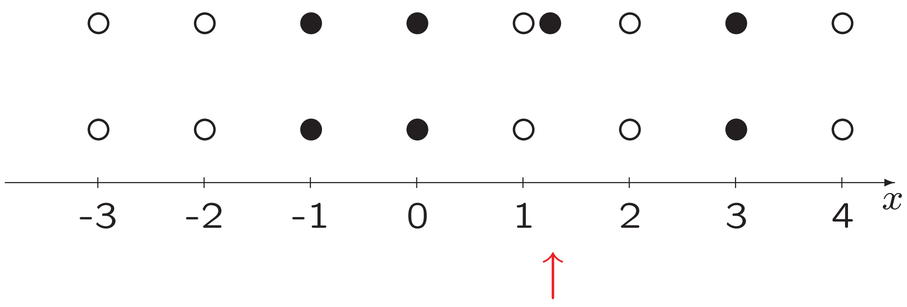

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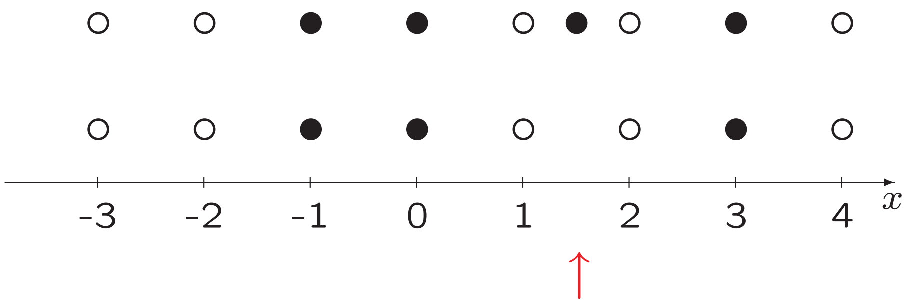

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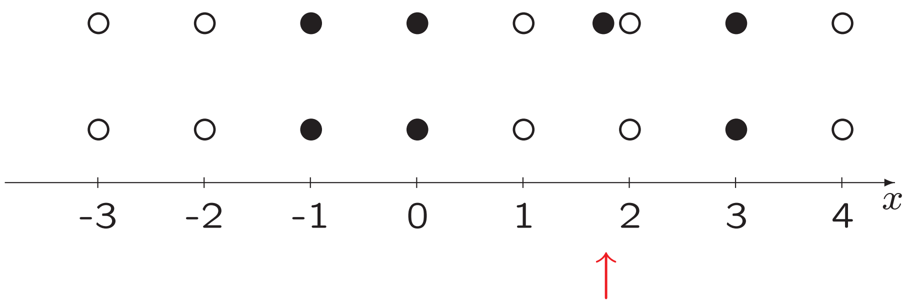

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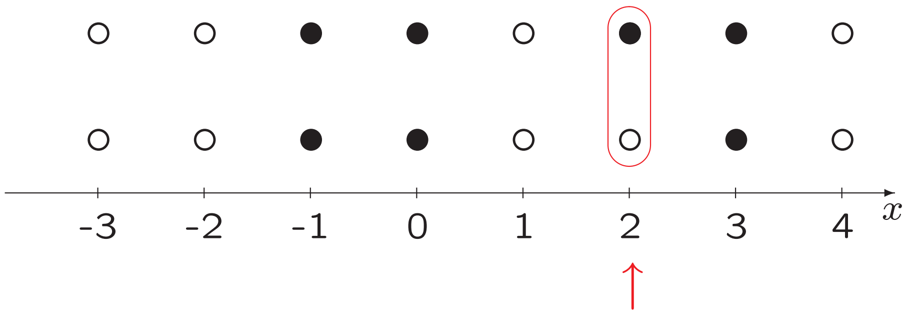

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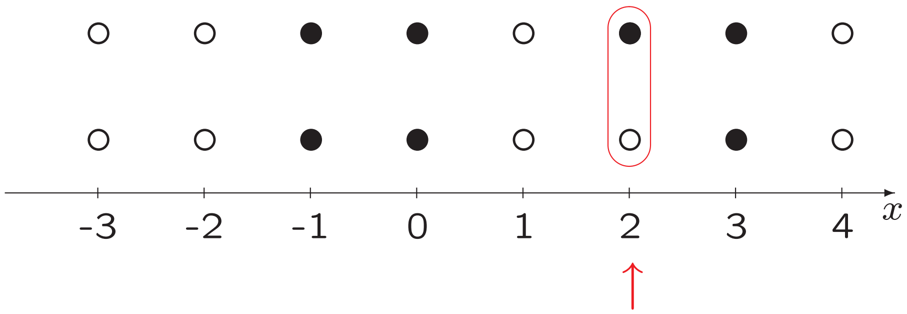

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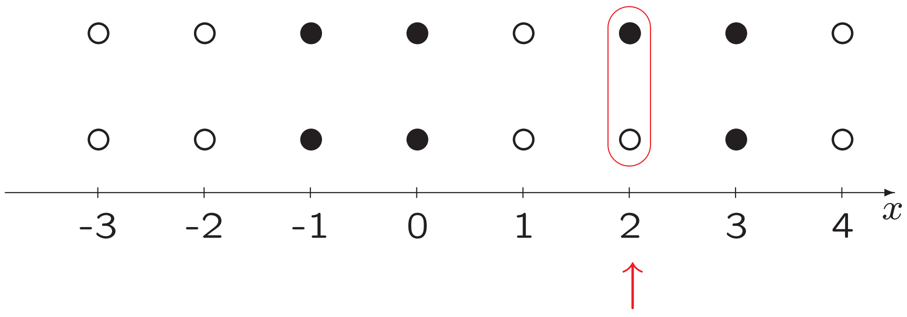

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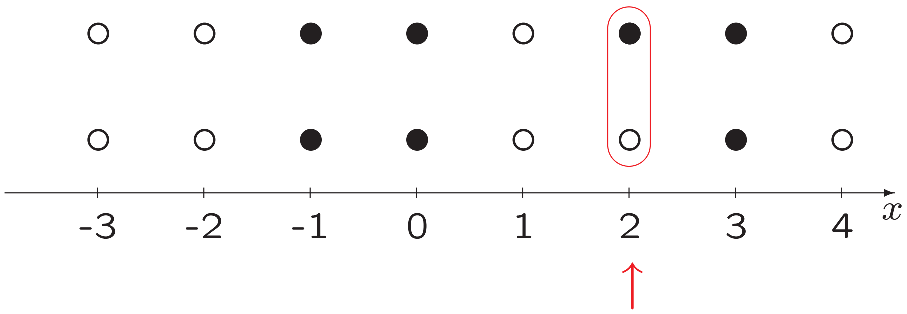

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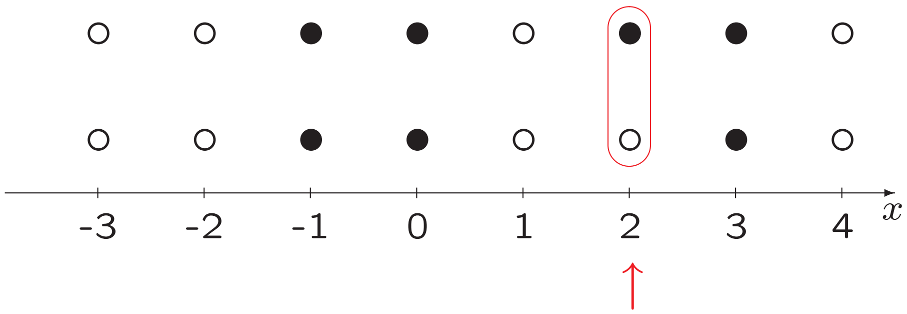

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The proof is based on ideas of Bálint, he said these ideas were standard.

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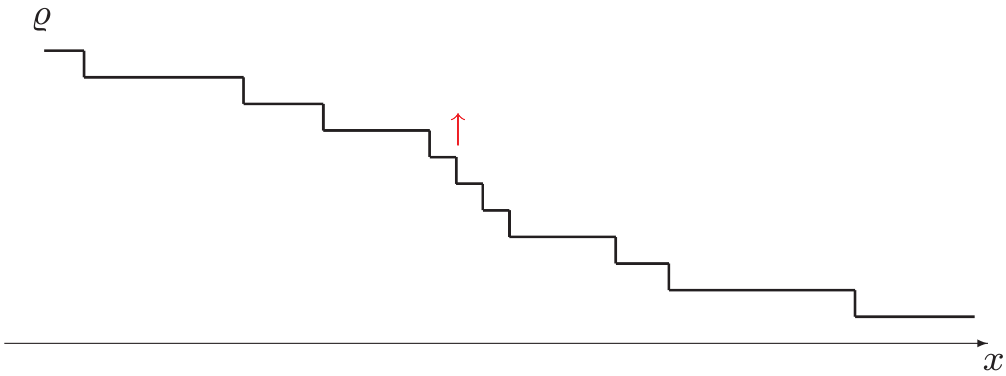

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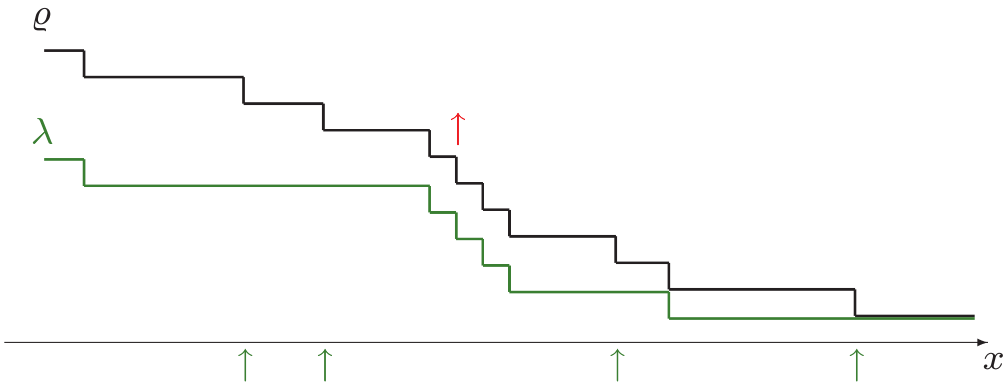

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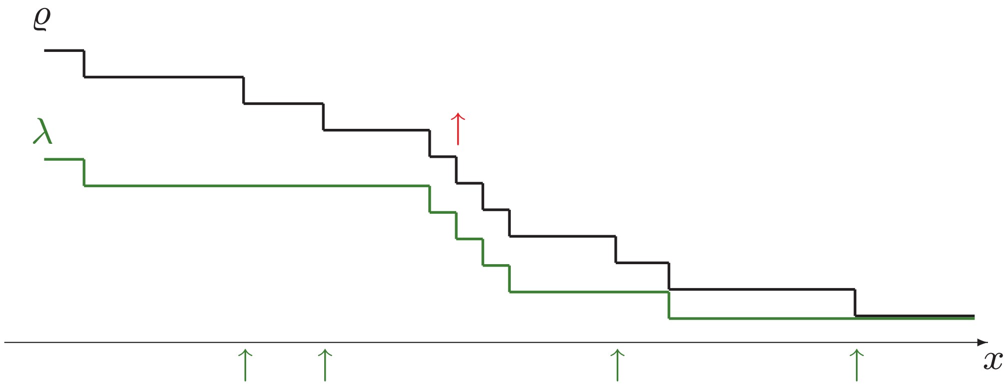

$\mathbf{P}\{Q(t)$ is too large$\}$

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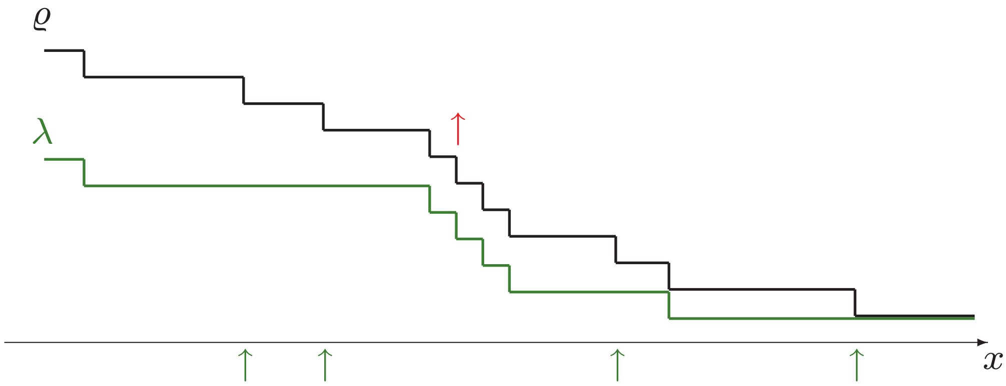

$$\mathbf{P}\{Q(t) \text{ is too large}\}$$
$$\leq \mathbf{P}\{\text{too many } \uparrow\text{'s have crossed } C(\varrho)t\}$$

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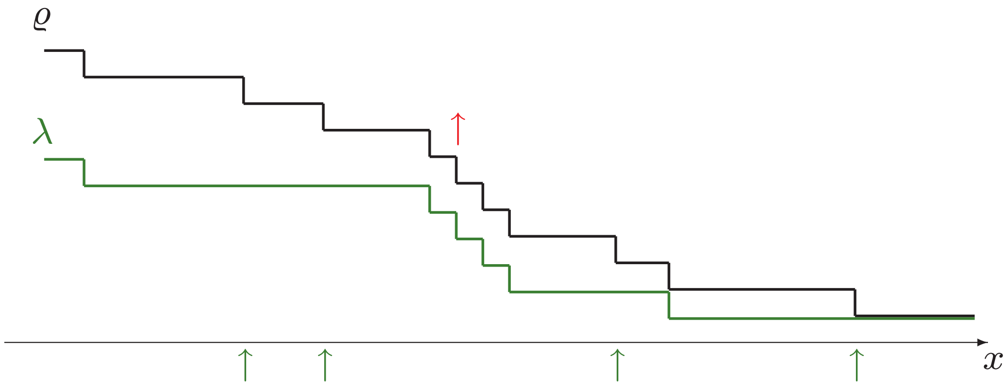

$\mathbf{P}\{Q(t)$ is too large$\}$

$\leq \mathbf{P}\{$too many $\uparrow$'s have crossed $C(\varrho)t\}$

$\leq \mathbf{P}\{h_{C(\varrho)t}(t) - h_{C(\varrho)t}(t)$ is too large$\}$.

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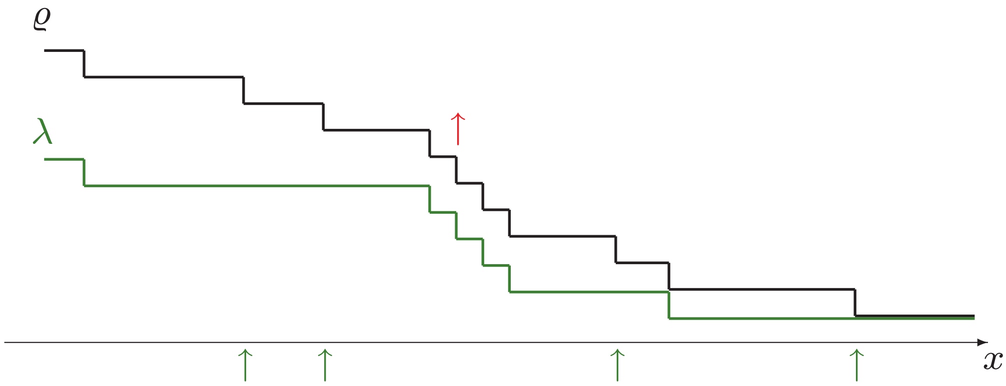

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Optimize "too large" in λ,

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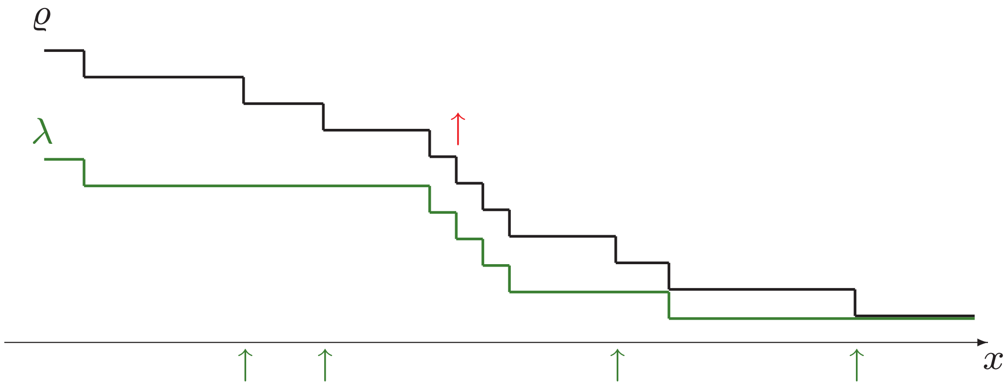

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Optimize "too large" in λ, use a Chebyshev and relate $\mathbf{Var}(h_{C(\varrho)t}(t))$ to $\mathbf{Var}(h_{C(\varrho)t}(t))$.

$$\mathbf{P}\{Q(t) \text{ is too large}\} \le [\dots] \cdot \mathbf{Var}(h_{C(\varrho)t}(t))$$

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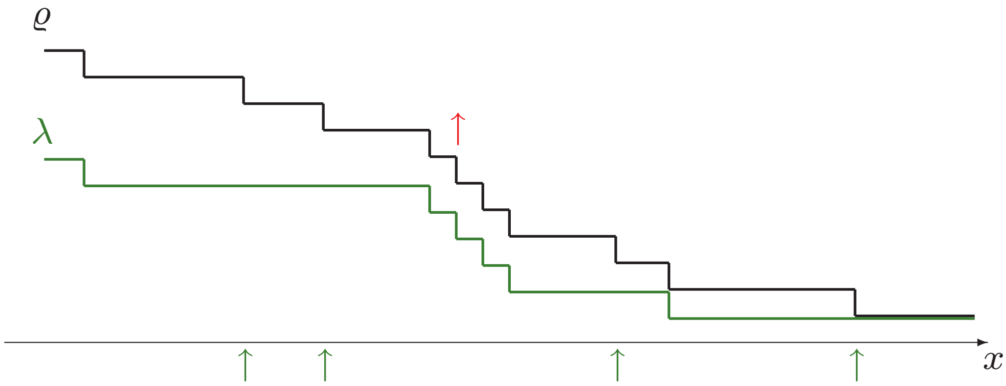

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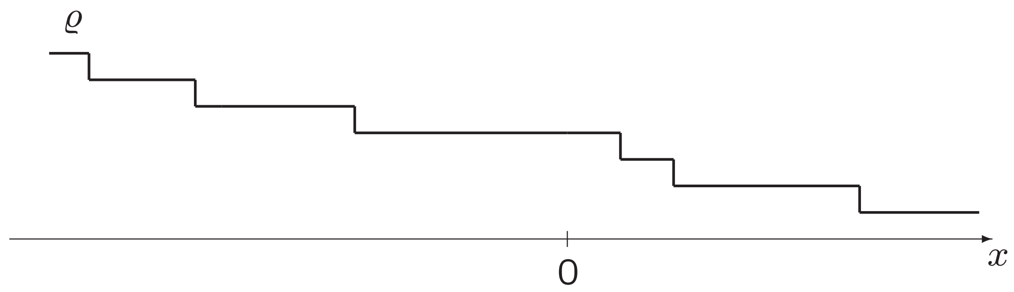
$$\mathbf{P}\{Q(t) \text{ is too large}\} \leq [\dots] \cdot \mathbf{Var}(h_{C(\varrho)t}(t))$$
$$= [\dots] \cdot \mathbf{E}|C(\varrho)t - Q(t)|.$$

Conclude the result for $\mathbf{E}|C(\varrho)t - Q(t)|$.

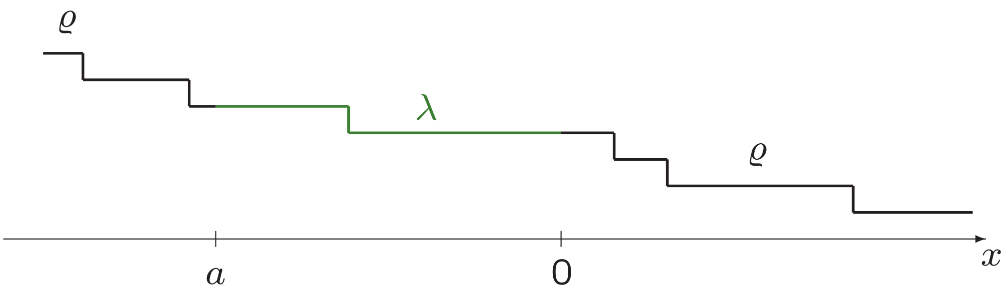
6. The lower bound

There is not much of a difference between this:



6. The lower bound

... and this:



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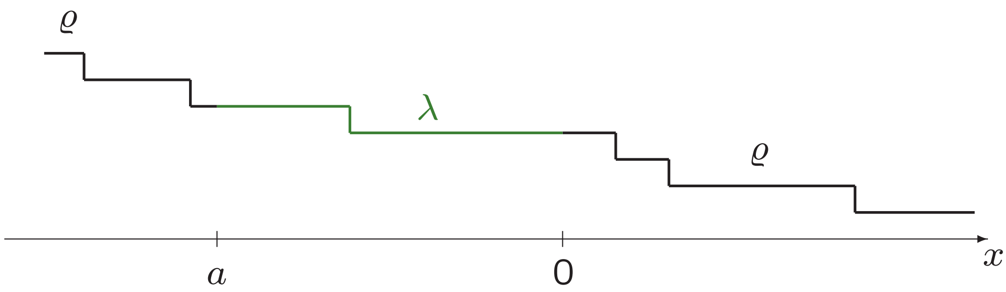

Price to pay: A change of initial measure factor.

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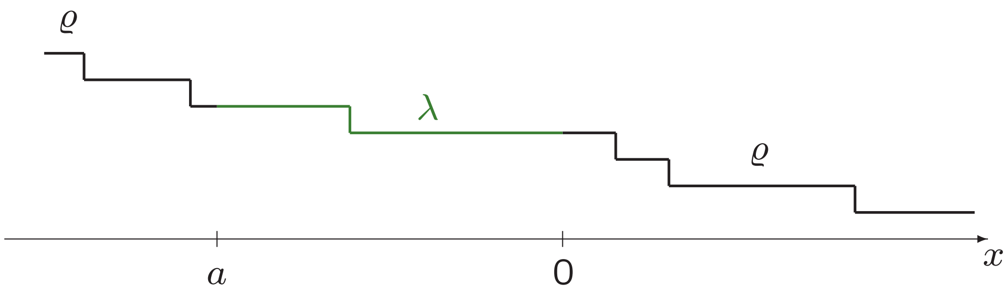

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In return: $h_{C(\varrho)t}(t)$ behaves like $h_{C(\varrho)t}(t)$.

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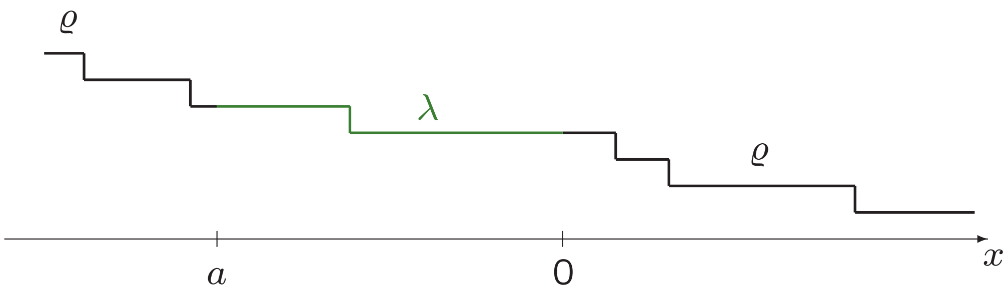

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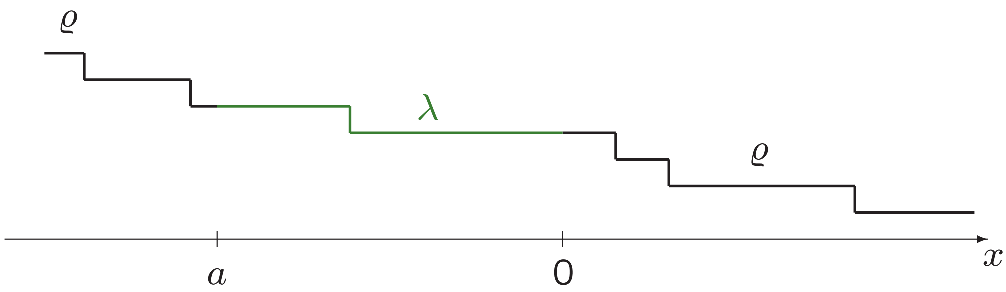

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These have different expectations.
$\rightsquigarrow$ Enough deviation to prove the lower bound if $\varrho - \lambda \simeq t^{-1/3}$, $a \simeq t^{2/3}$.

Thank you.