$t^{1/3}$-order fluctuations in the simple exclusion process

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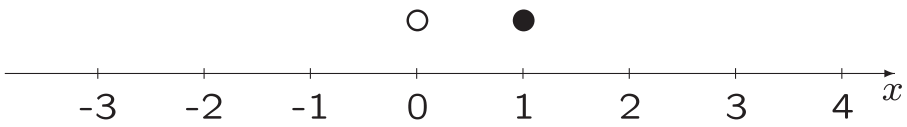
Joint work with
Eric Cator
(Delft University of Technology)
and
Timo Seppäläinen

Madison, March 9

1. The totally asymmetric simple exclusion



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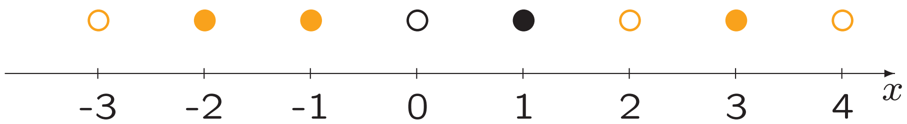

Bernoulli(ϱ) distribution

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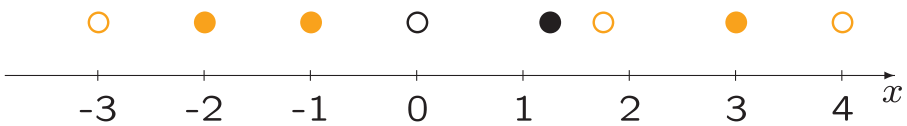

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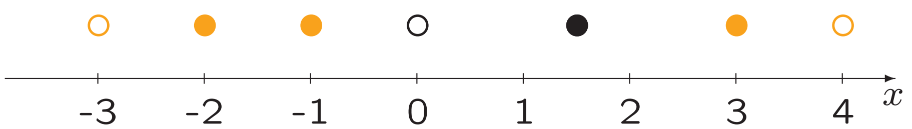

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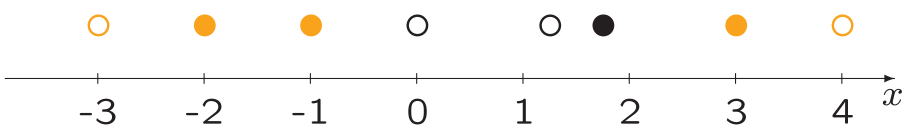

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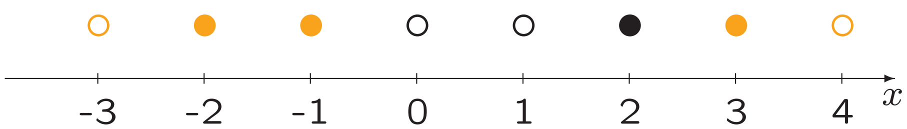

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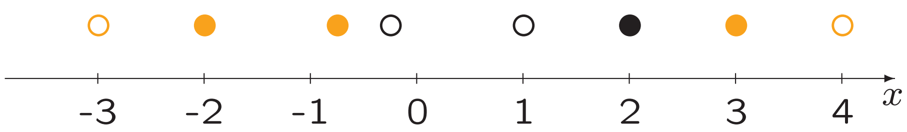

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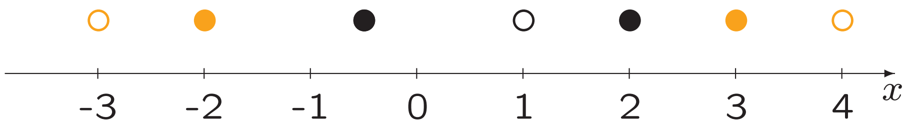

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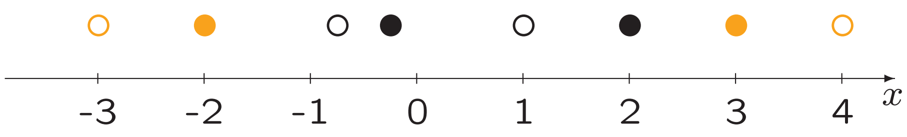

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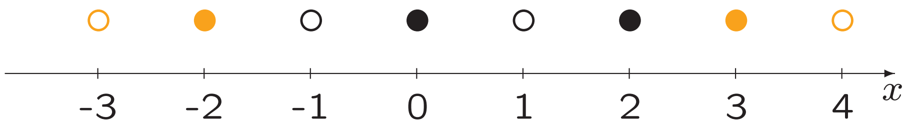

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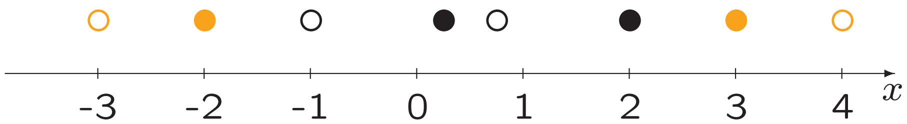

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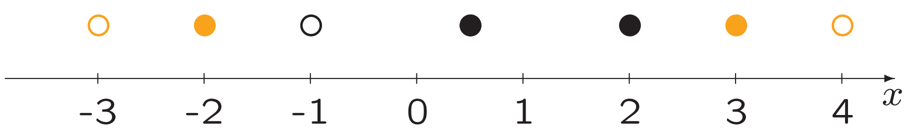

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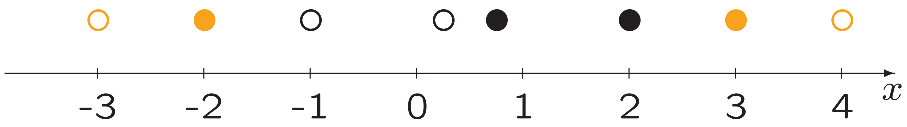

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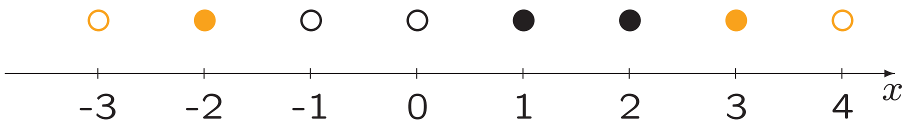

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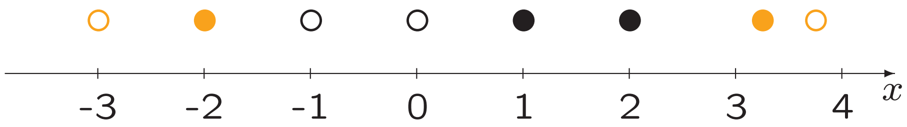

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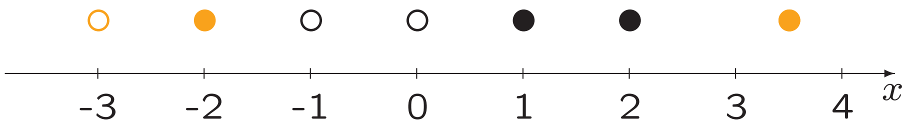

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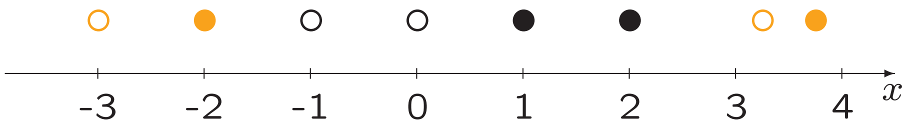

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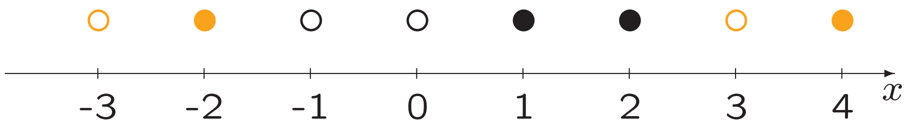

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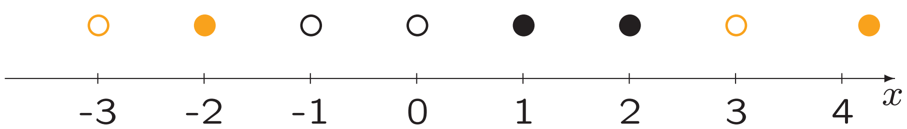

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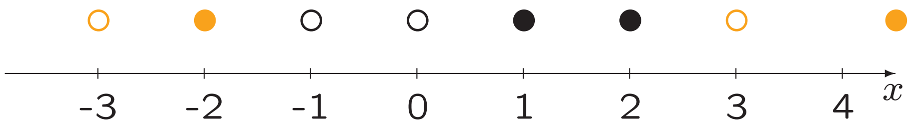

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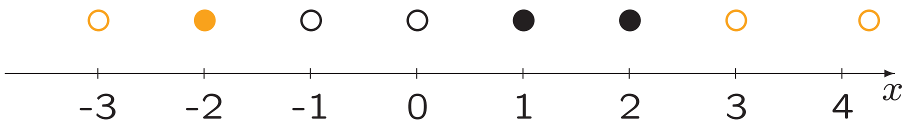

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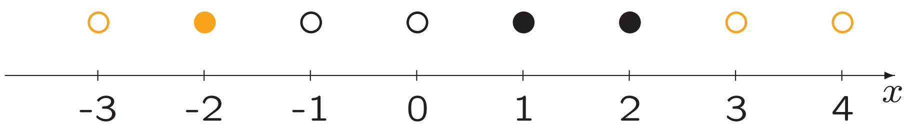

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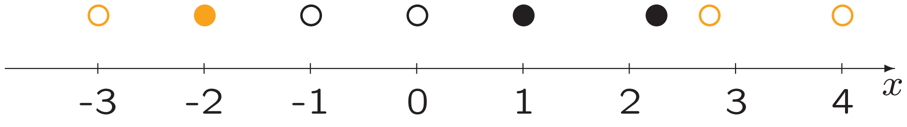

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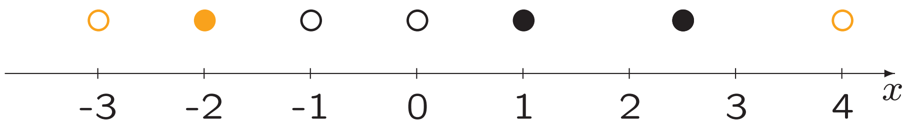

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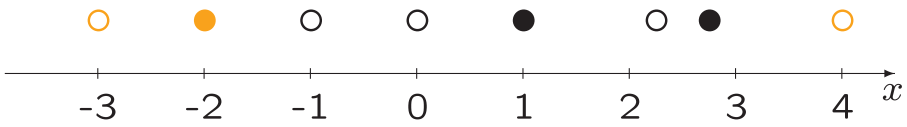

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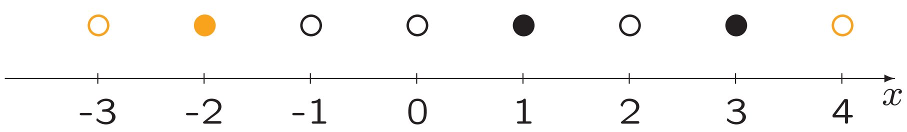

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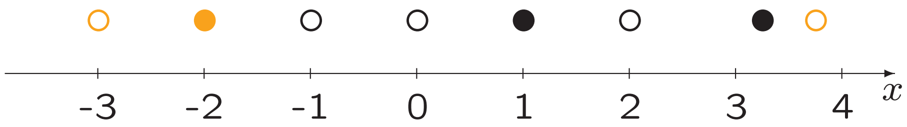

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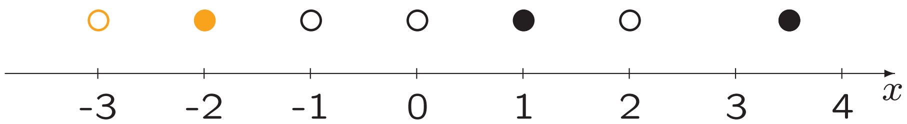

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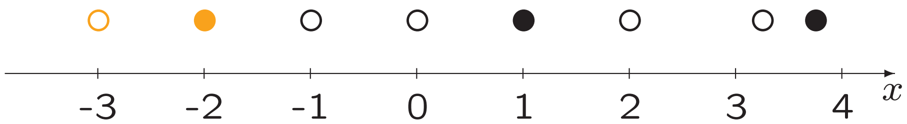

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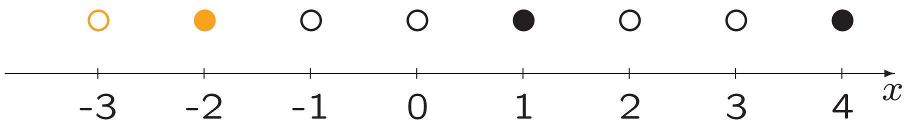

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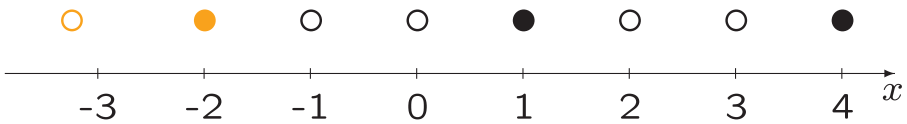

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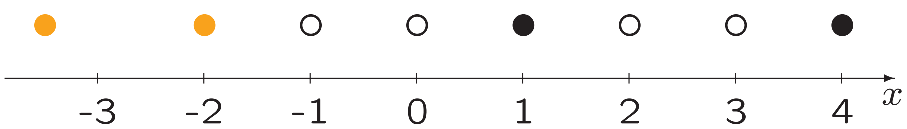

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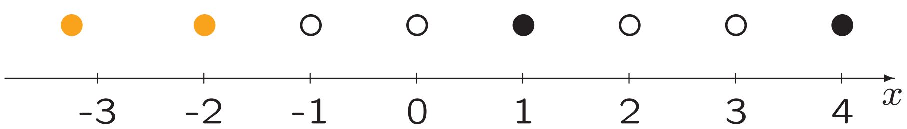

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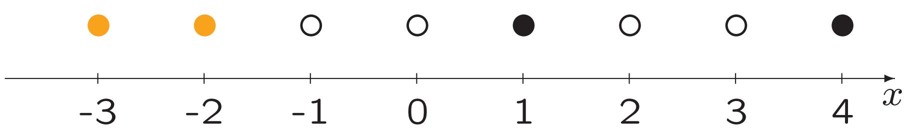

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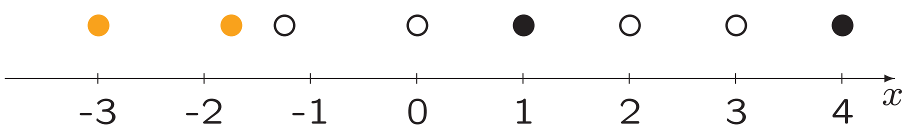

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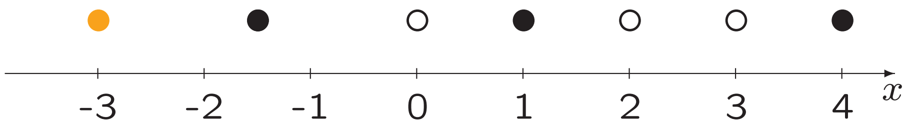

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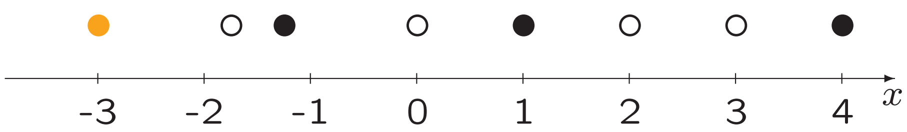

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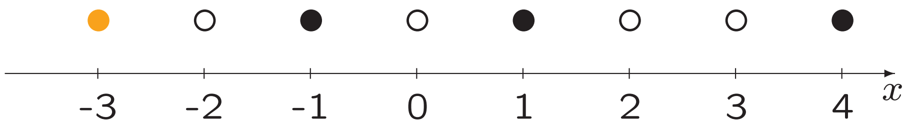

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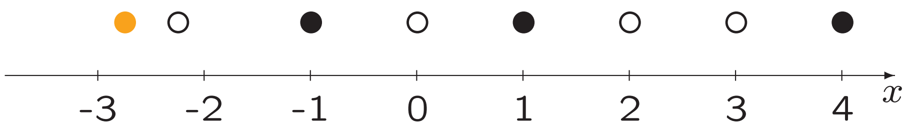

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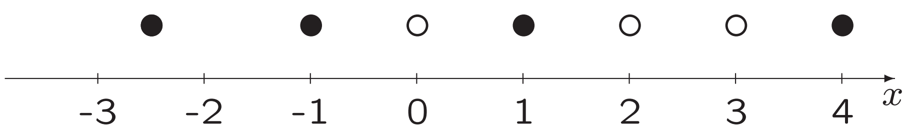

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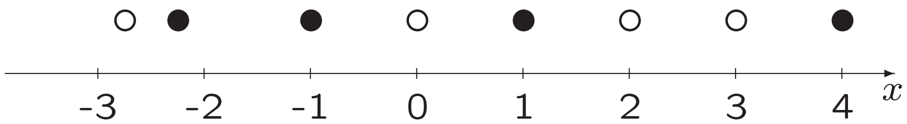

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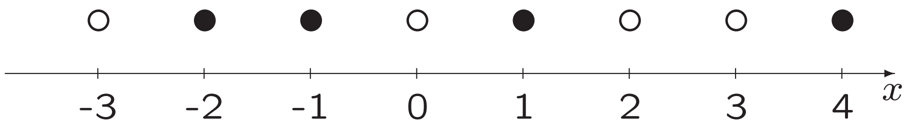

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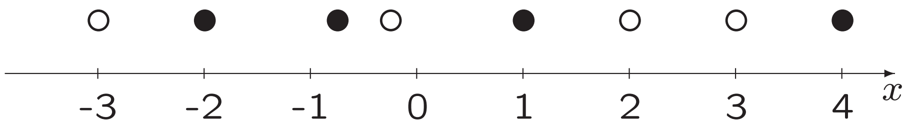

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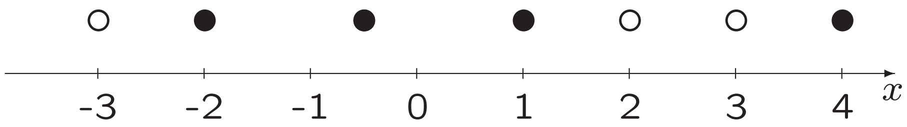

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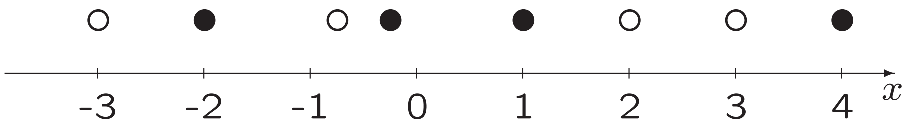

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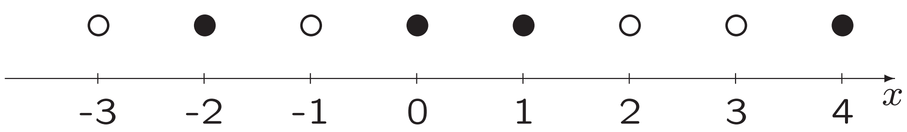

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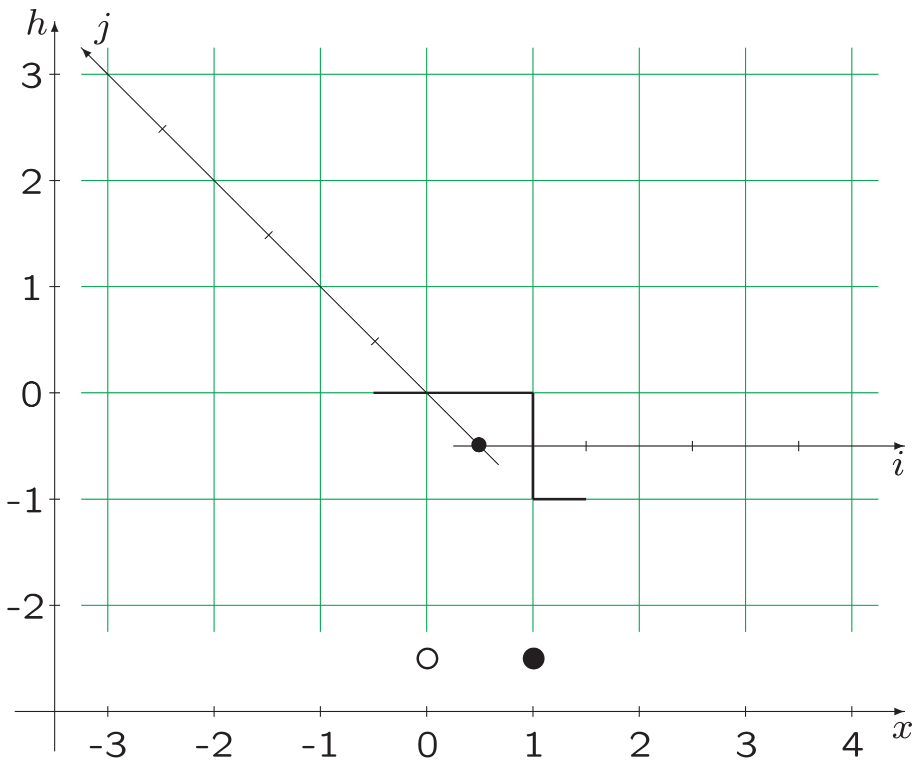

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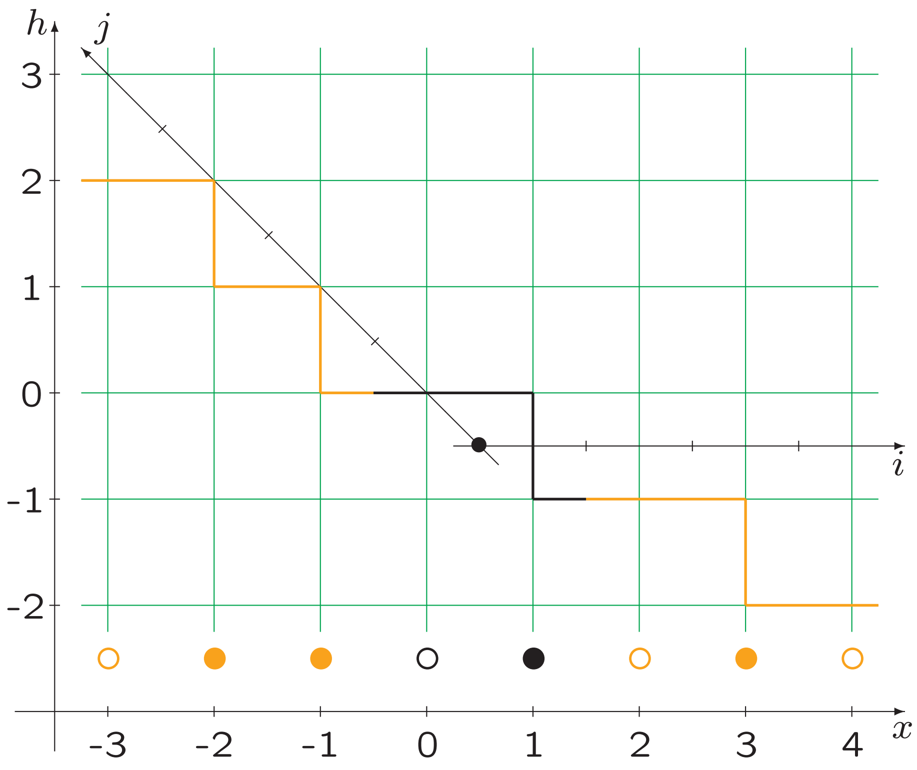

Bernoulli(ϱ) distribution

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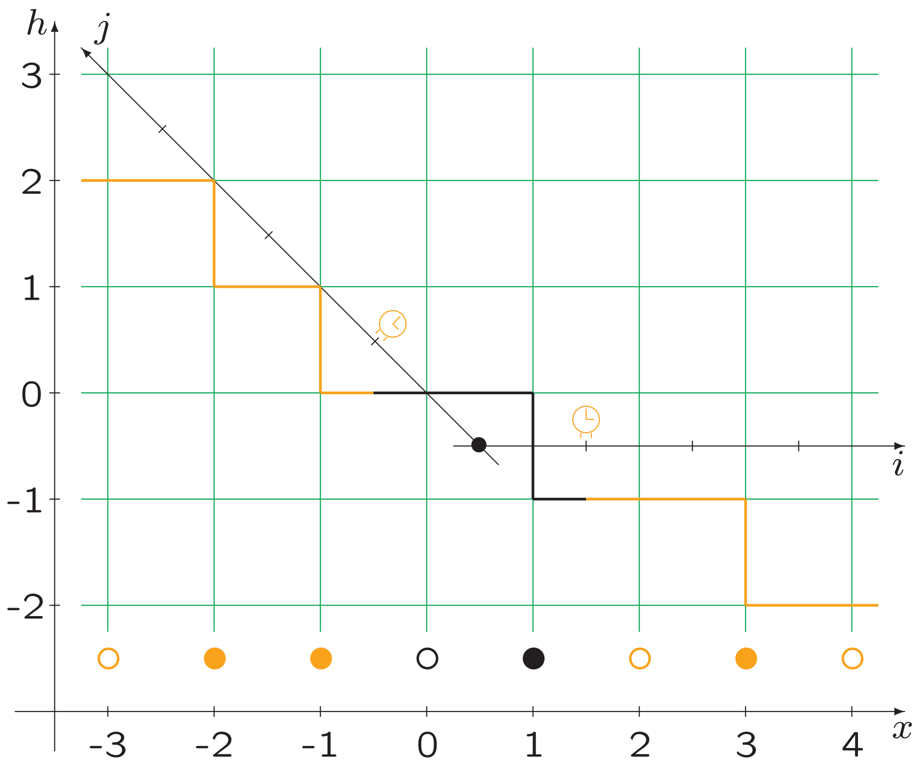

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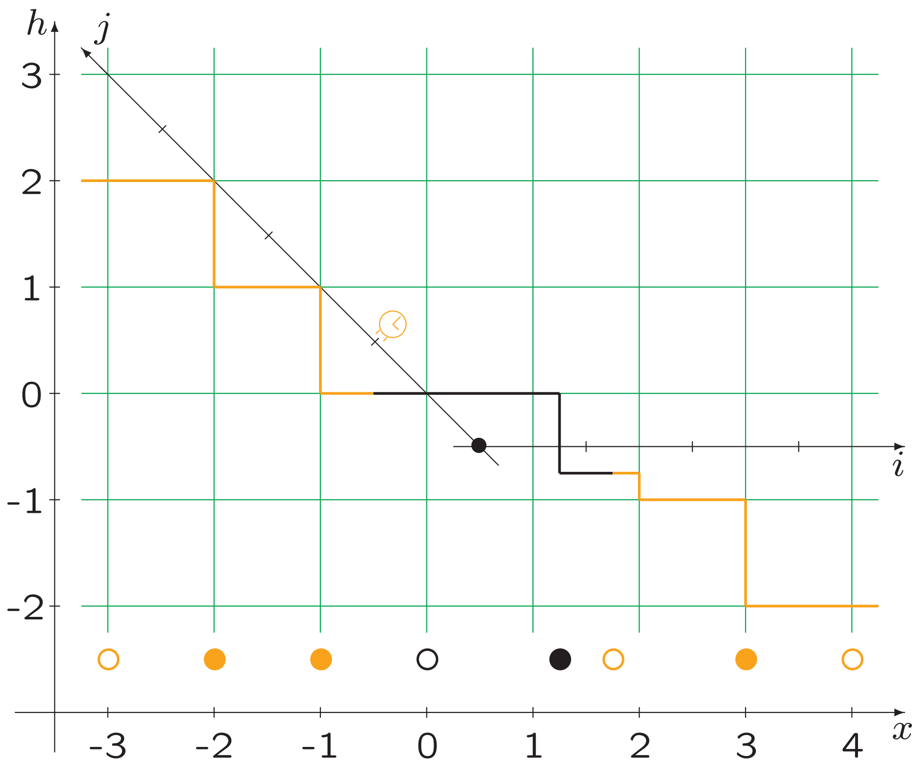

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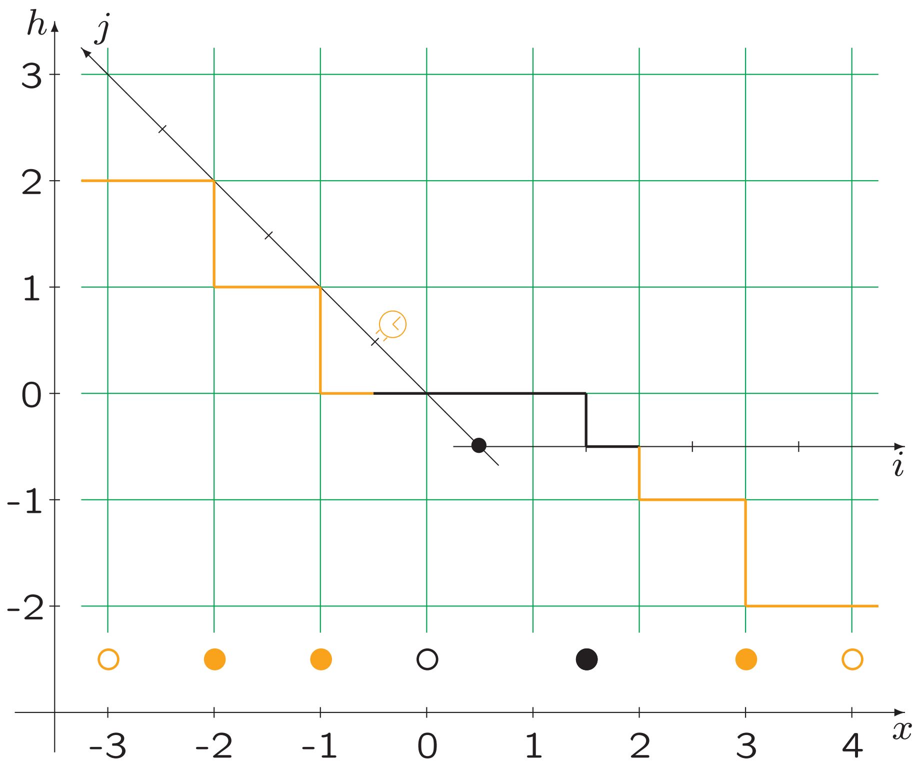

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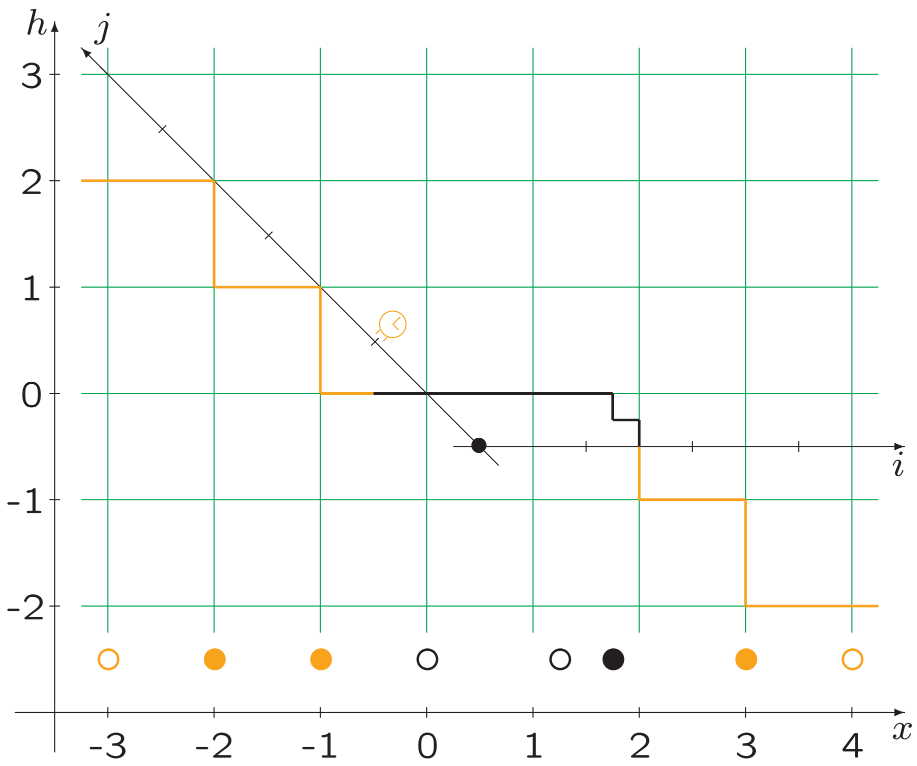

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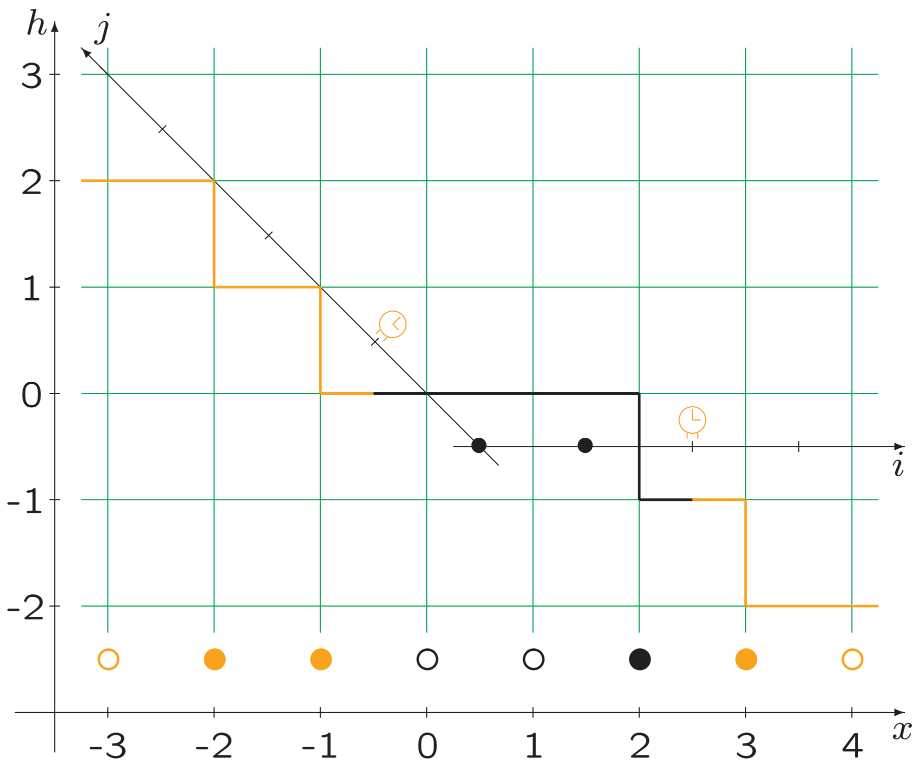

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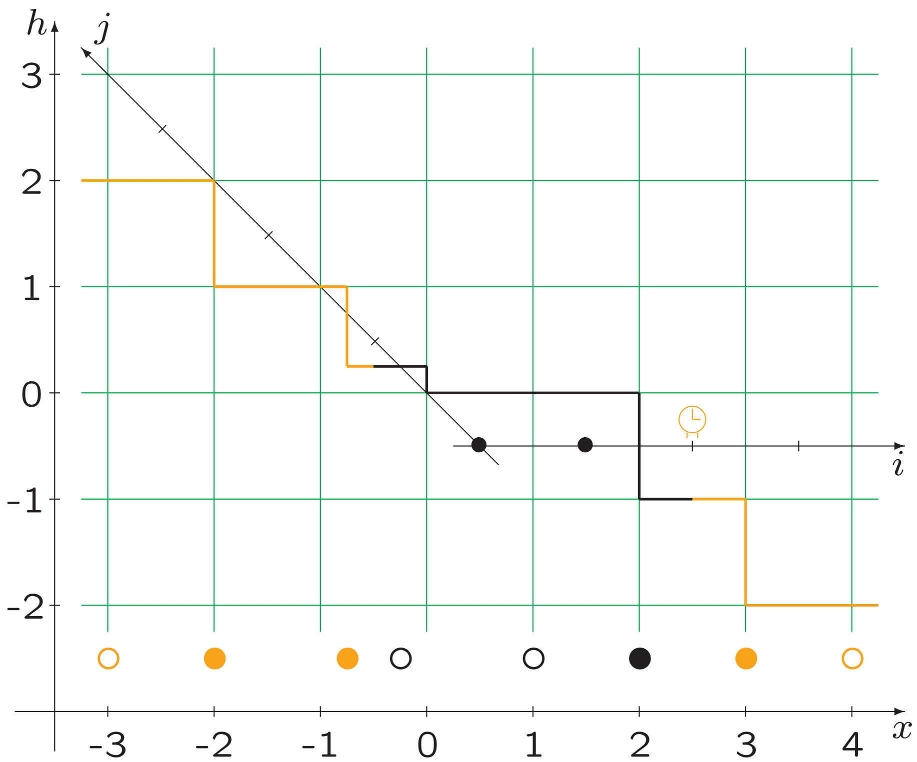

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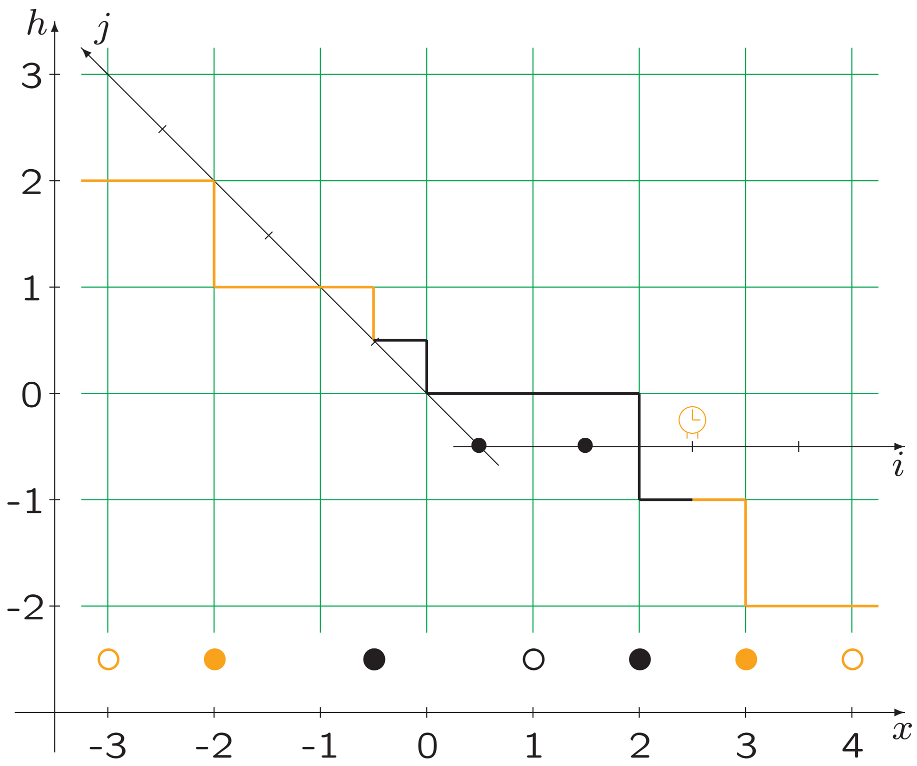

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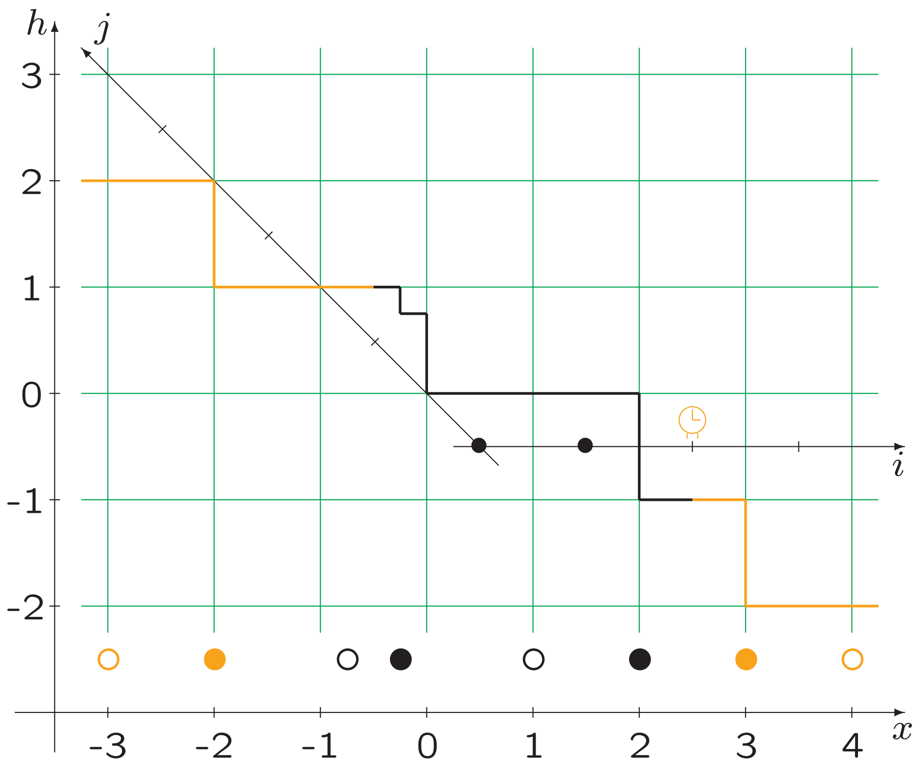

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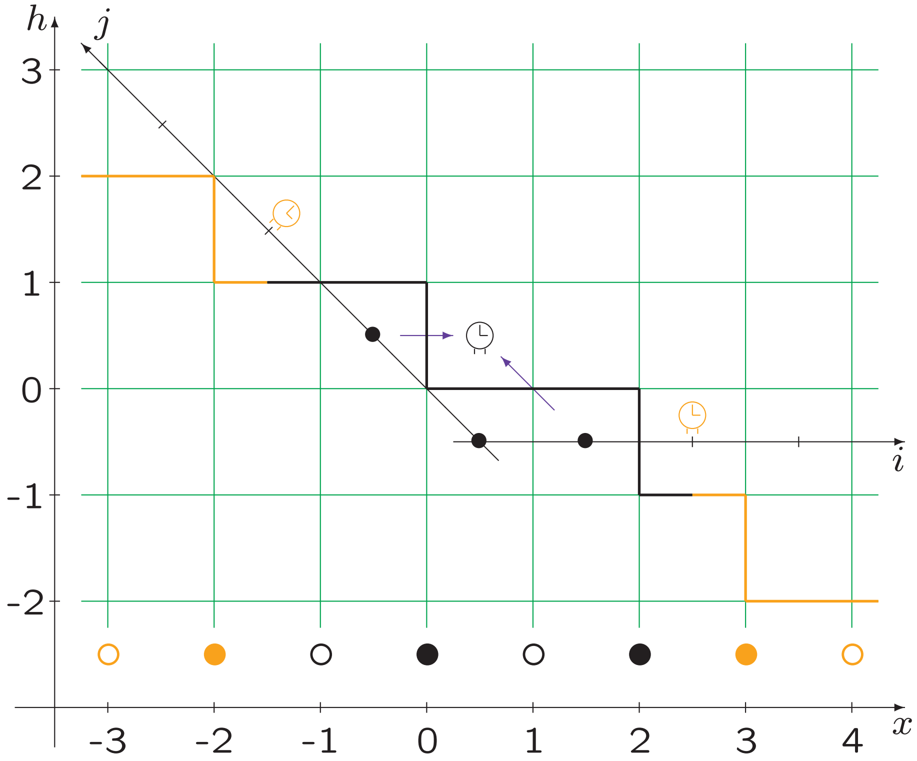

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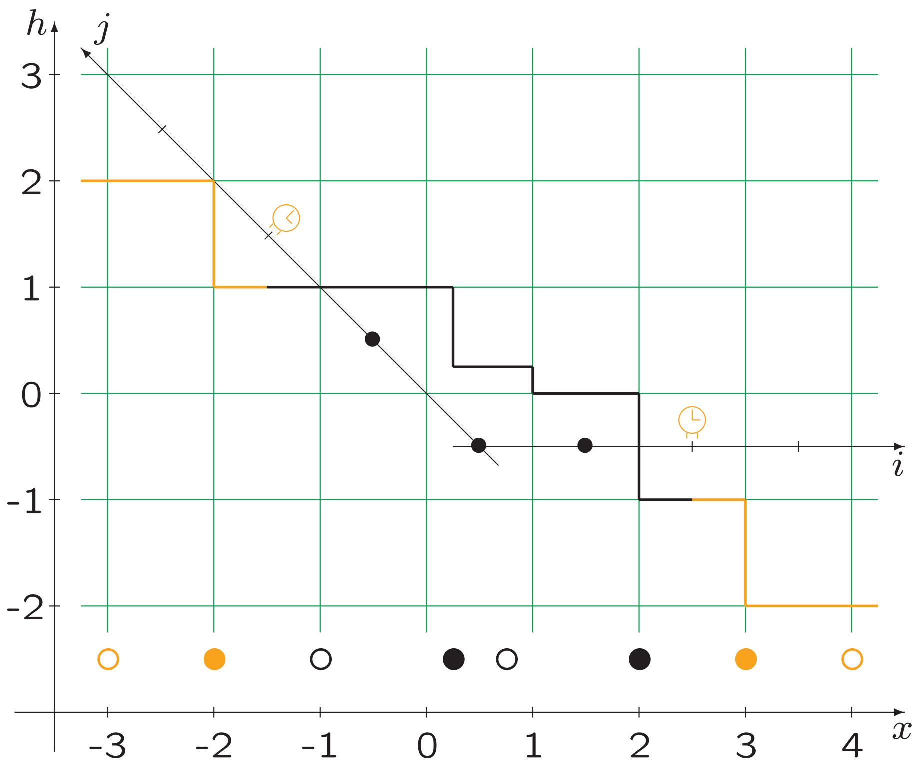

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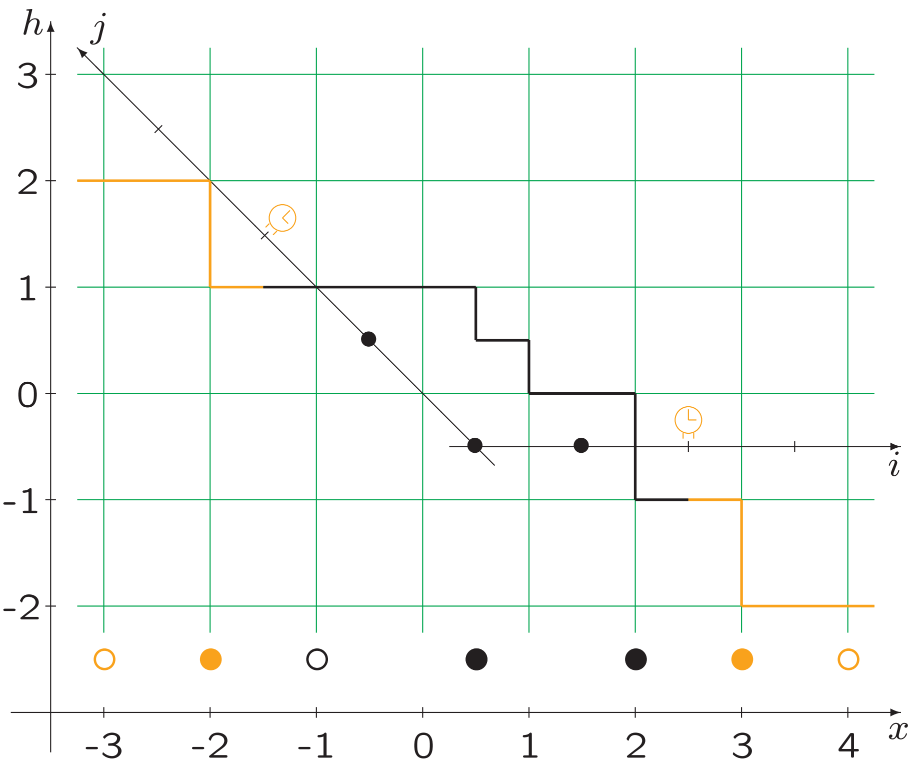

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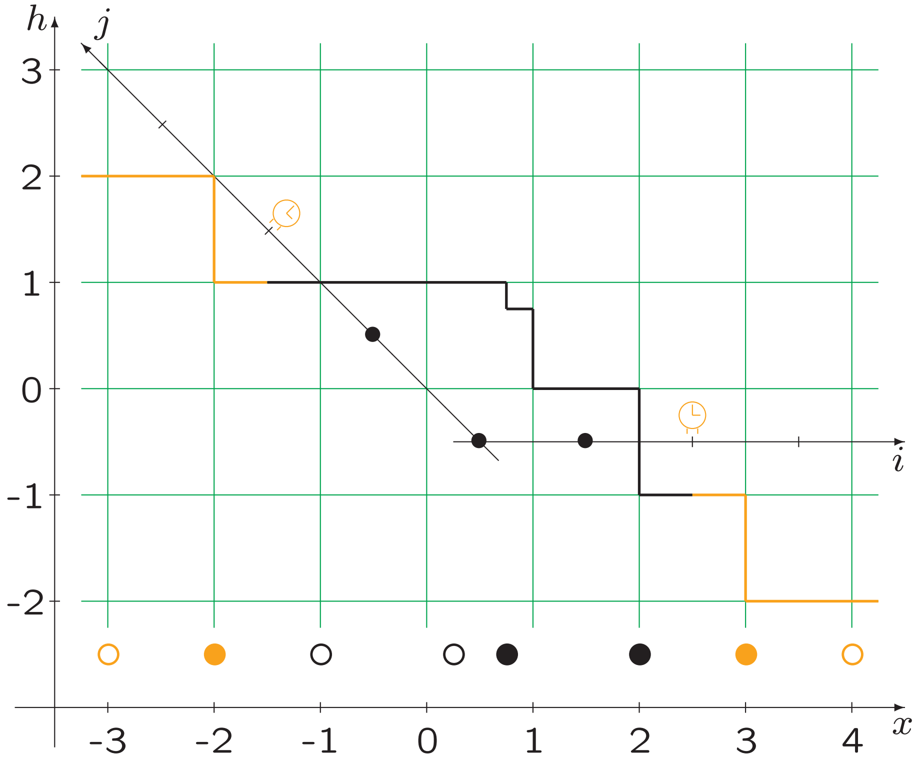

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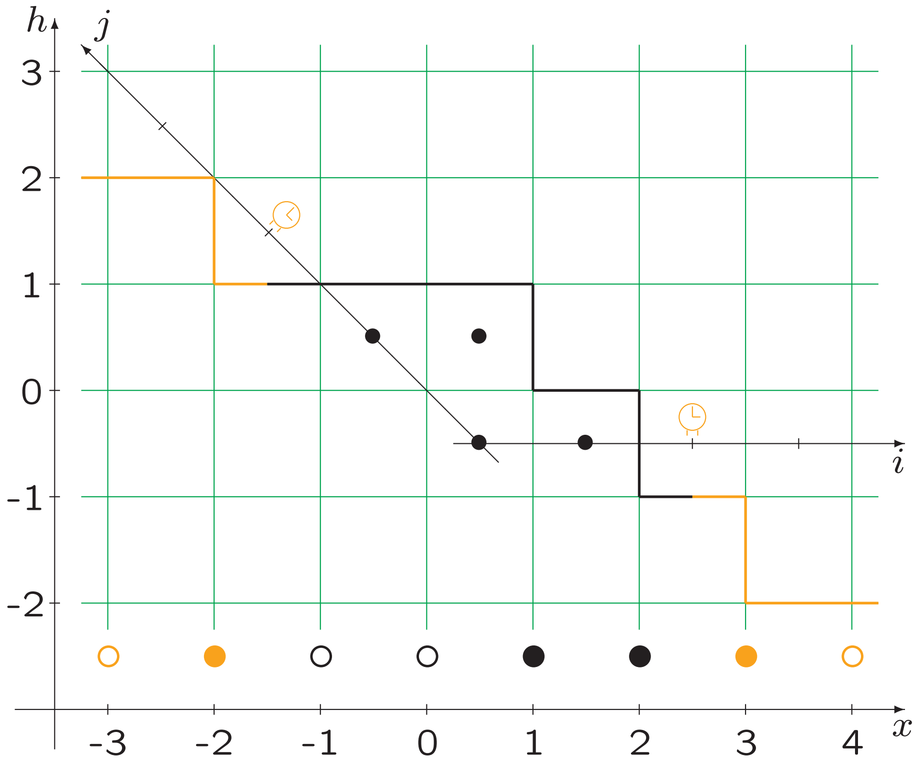

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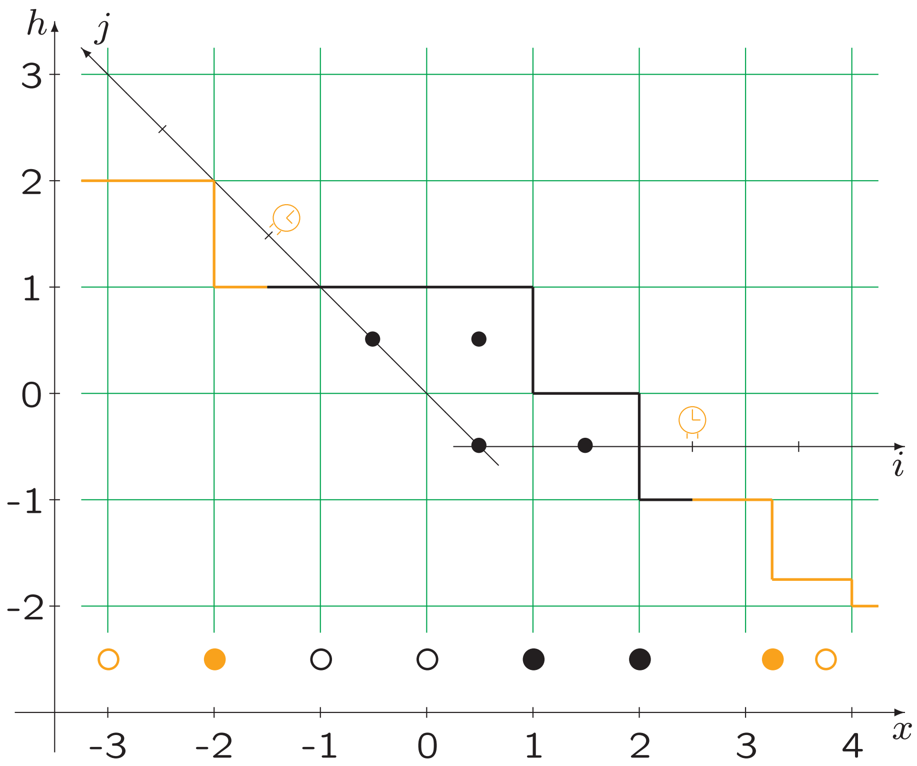

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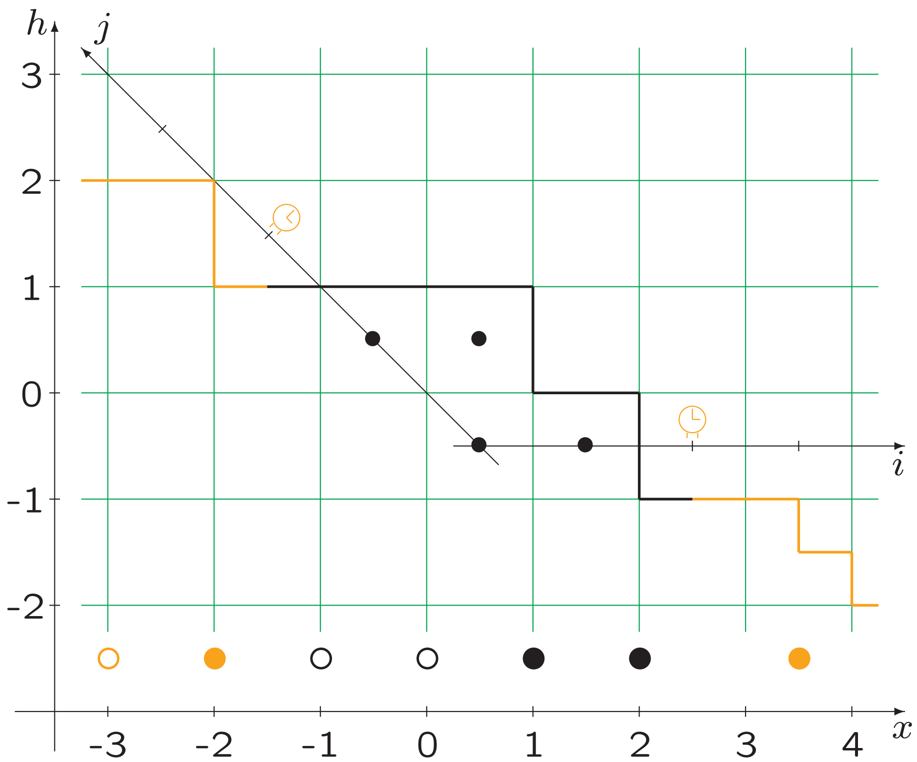

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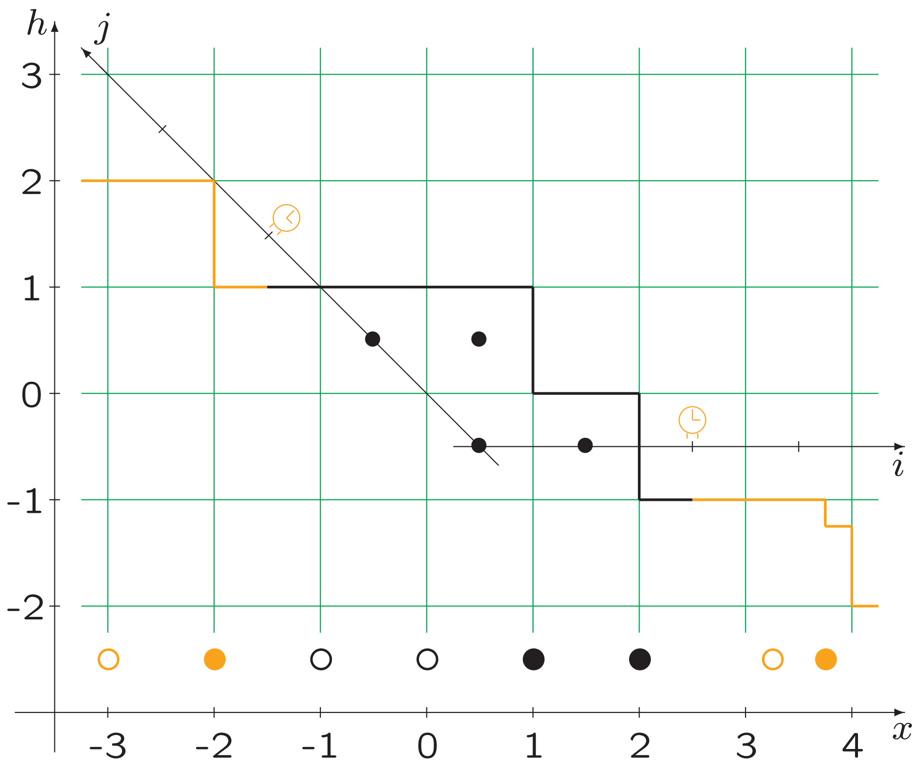

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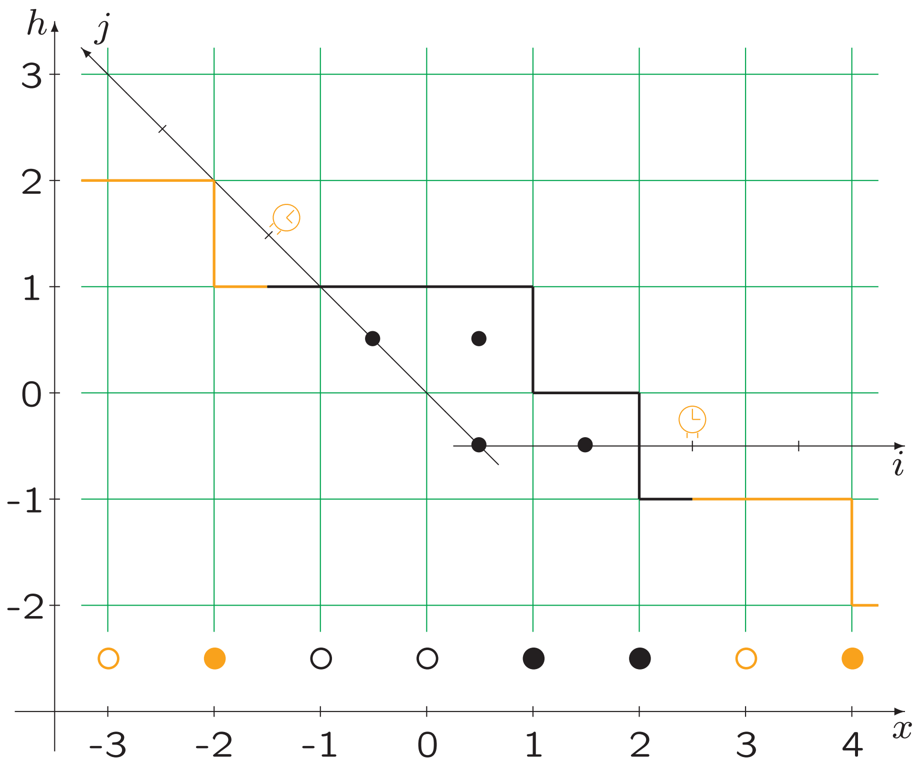

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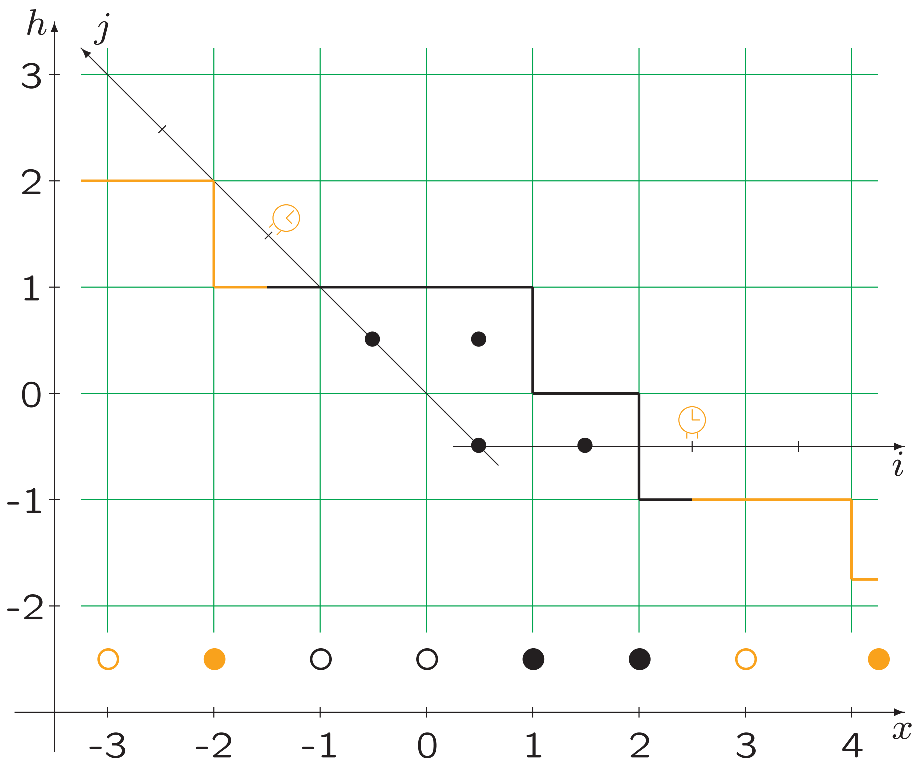

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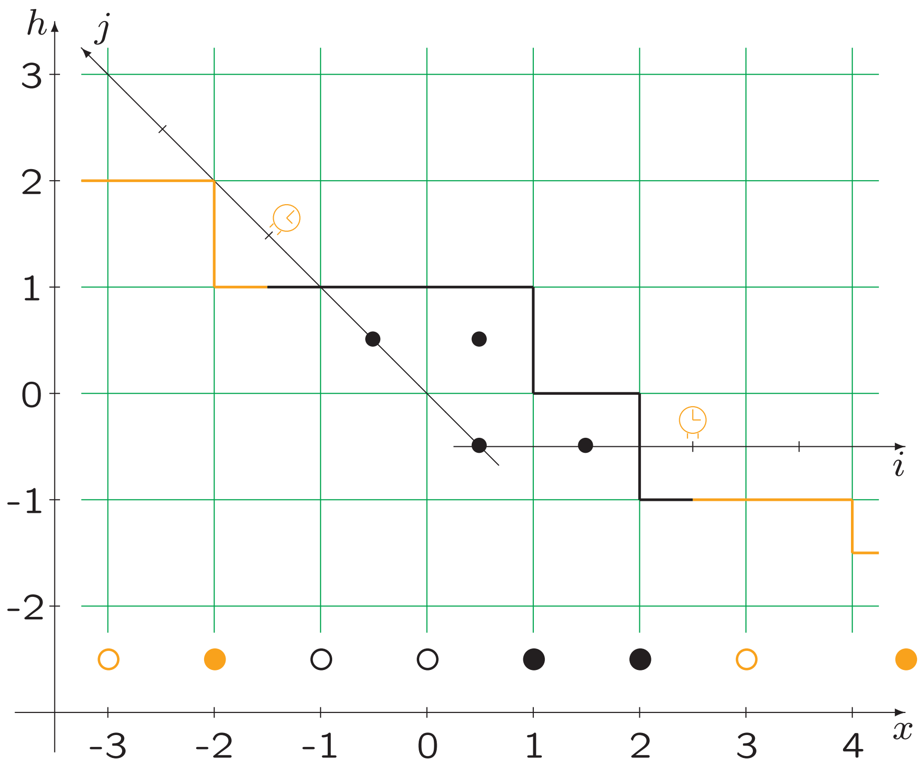

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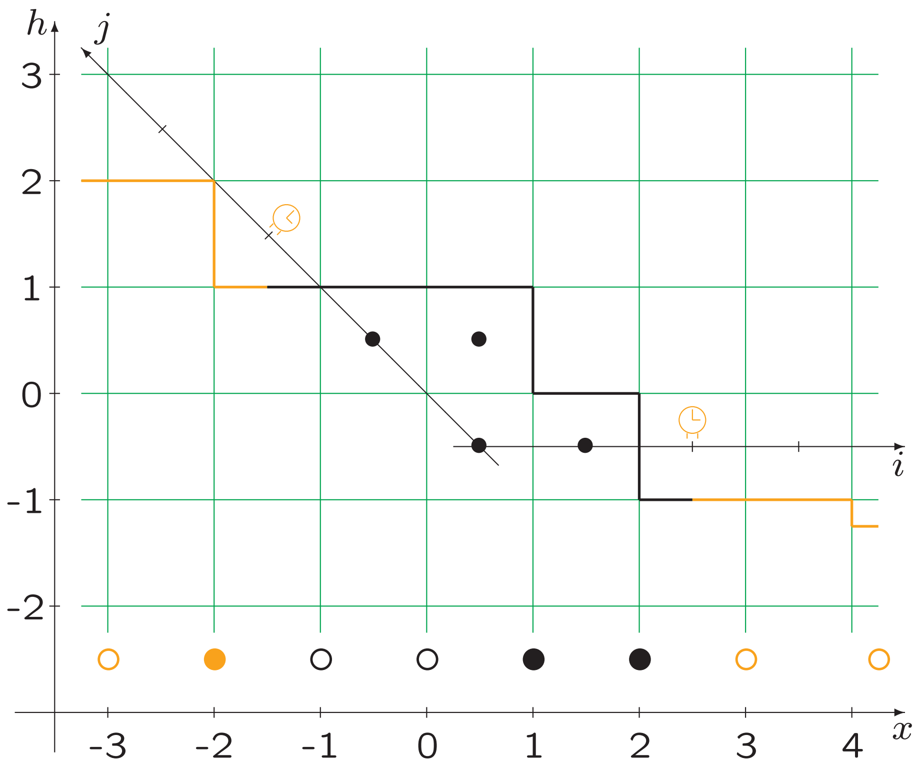

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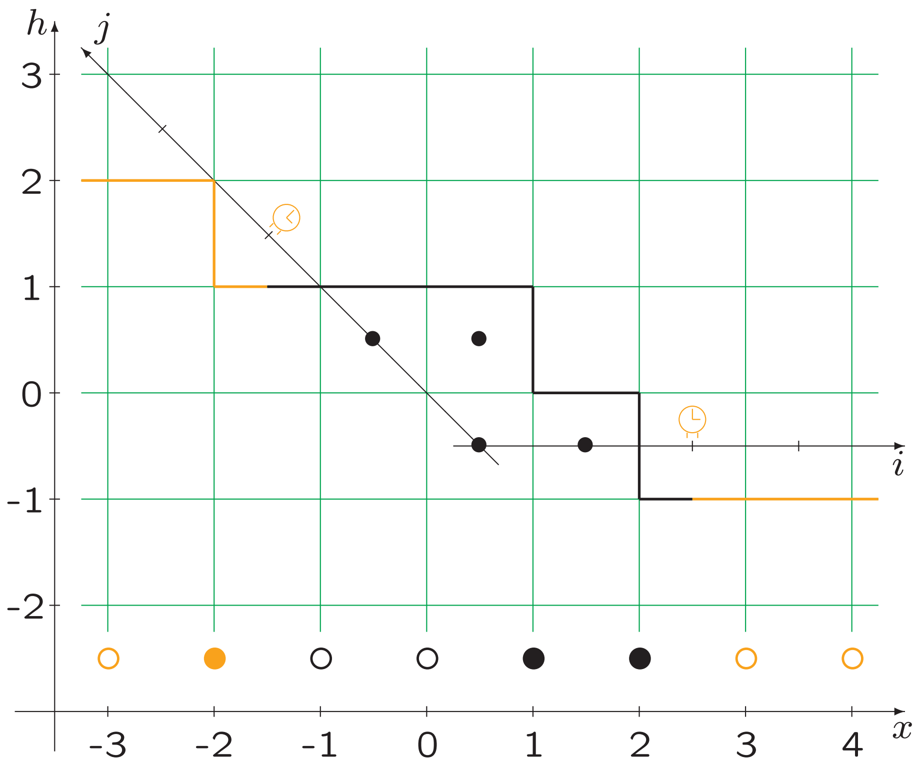

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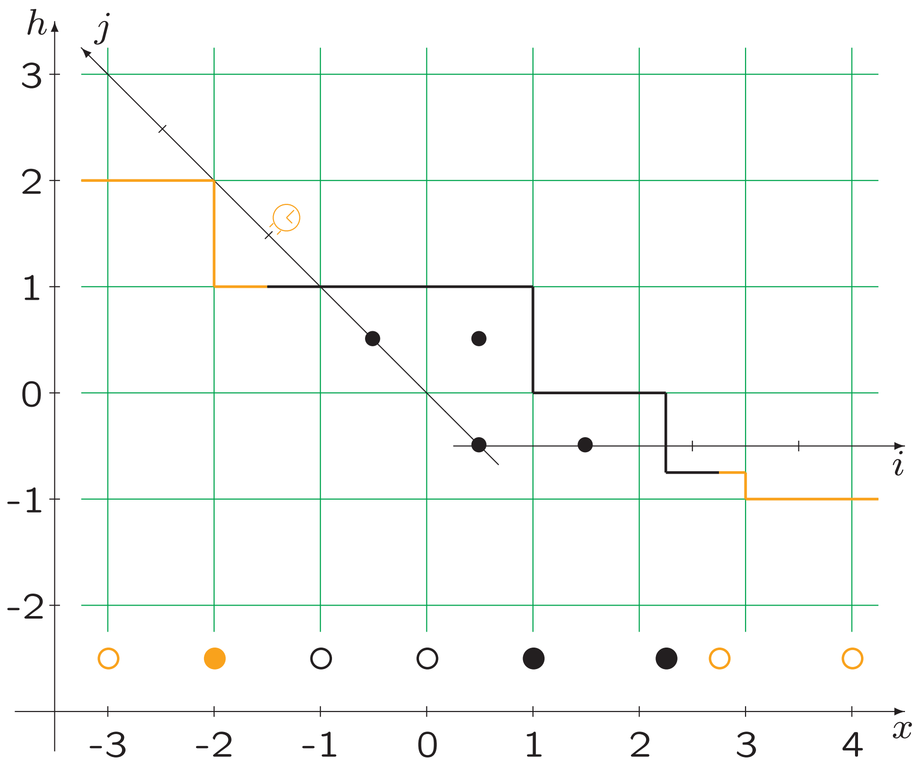

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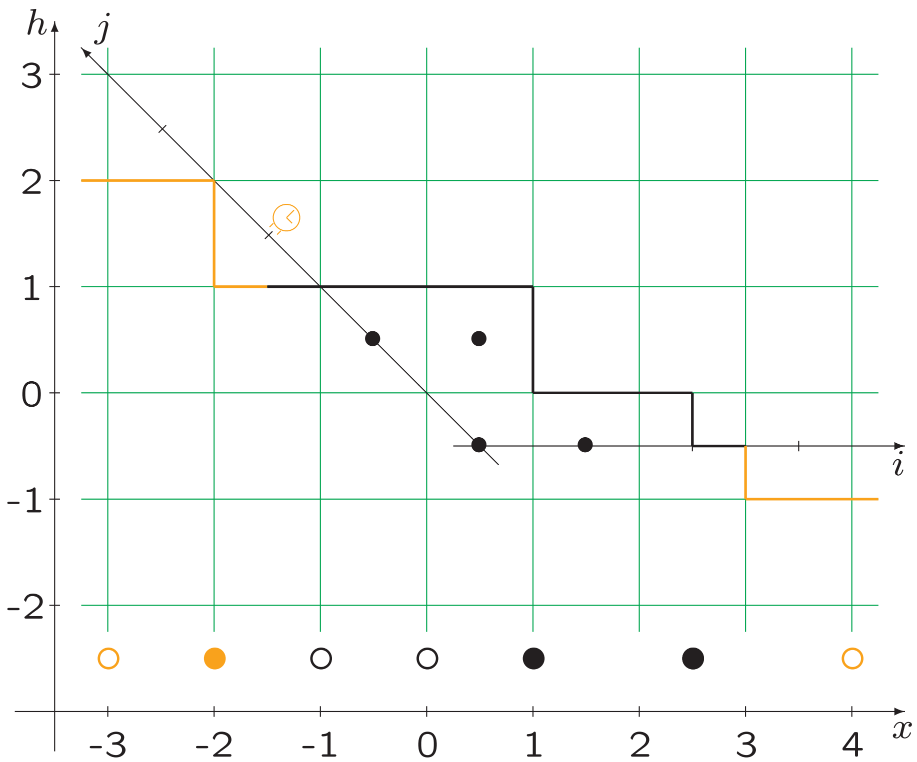

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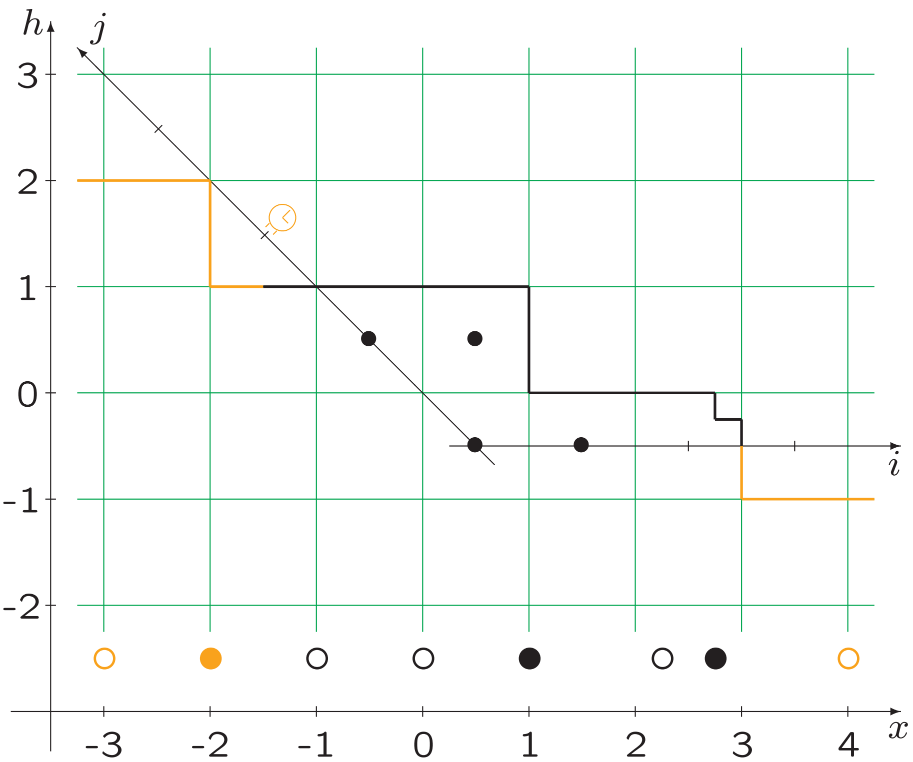

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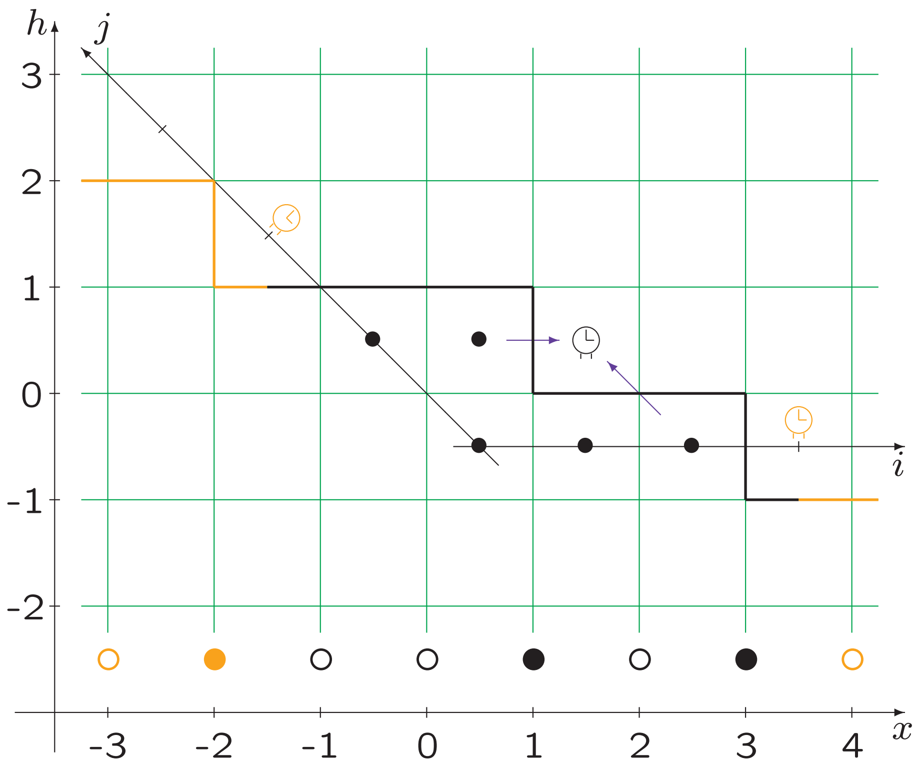

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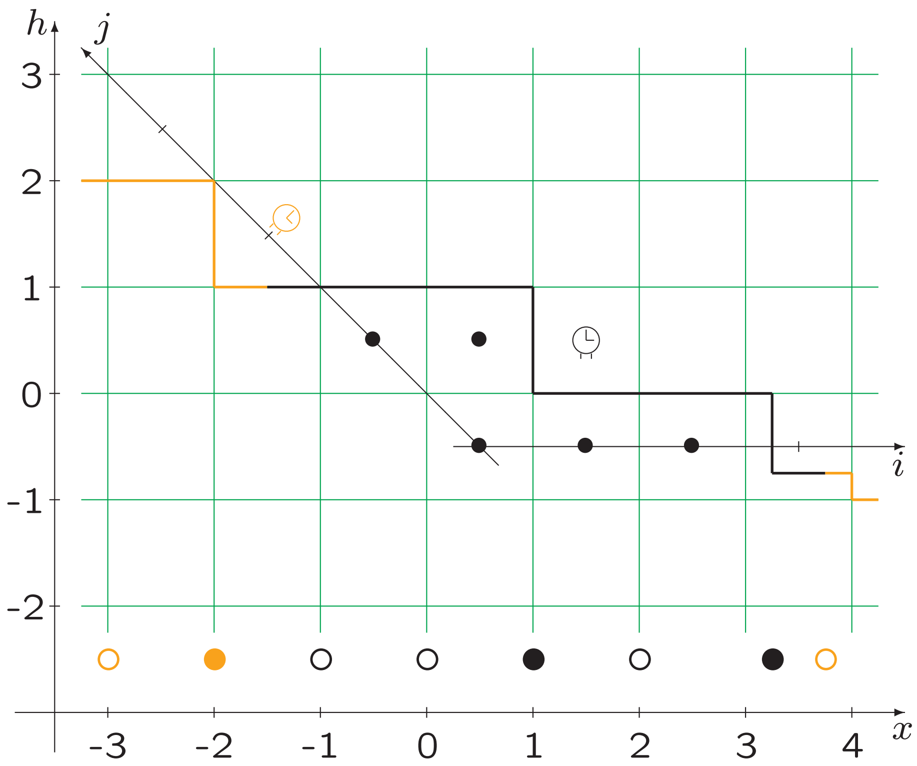

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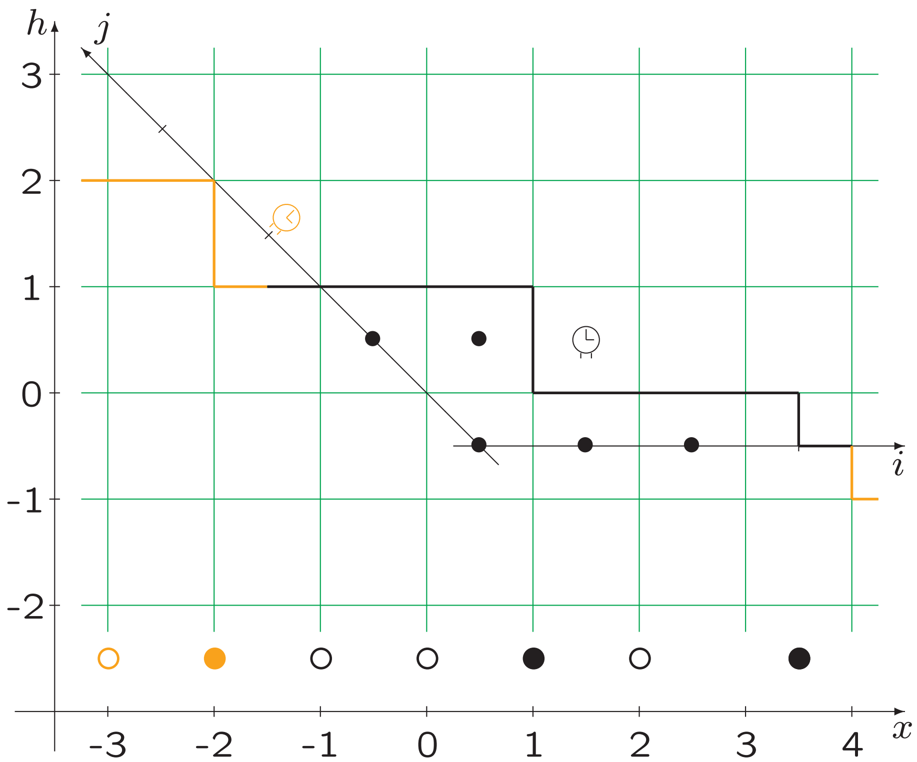

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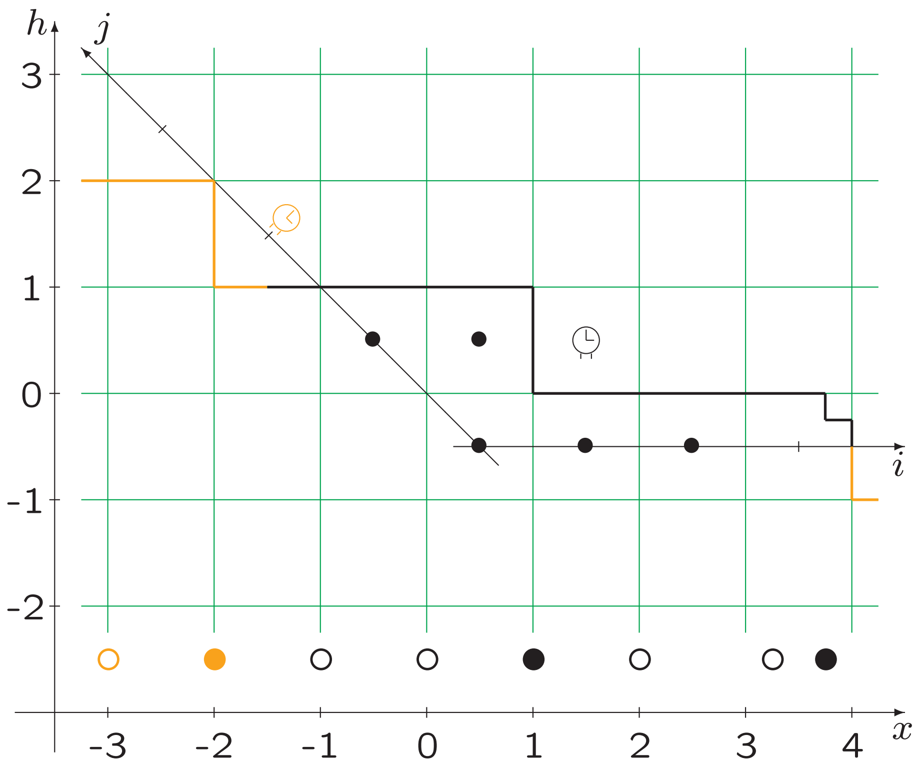

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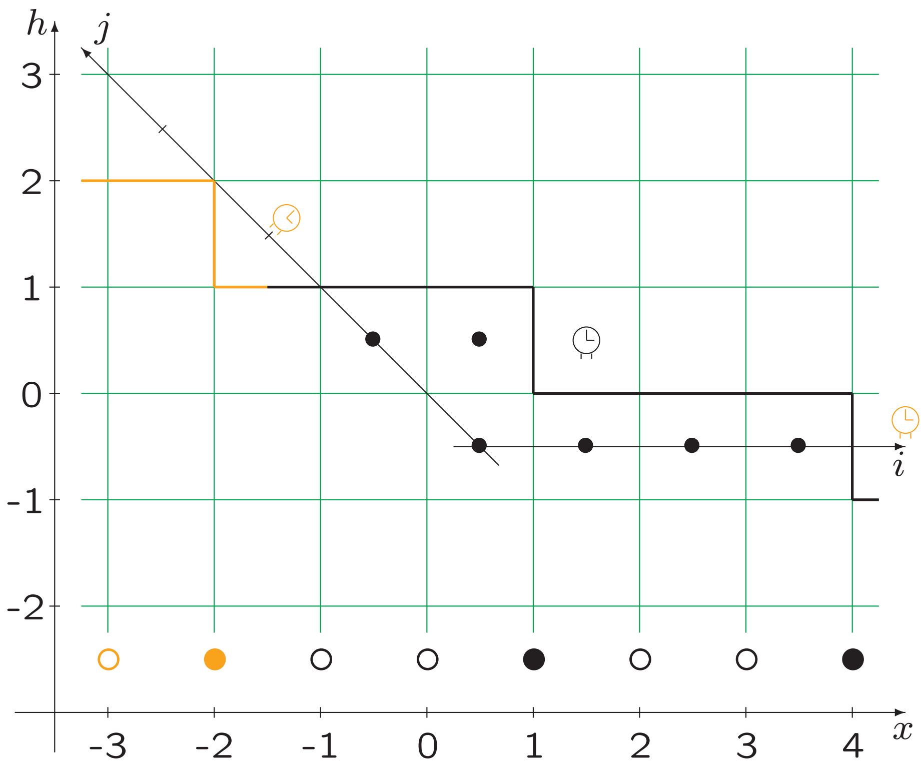

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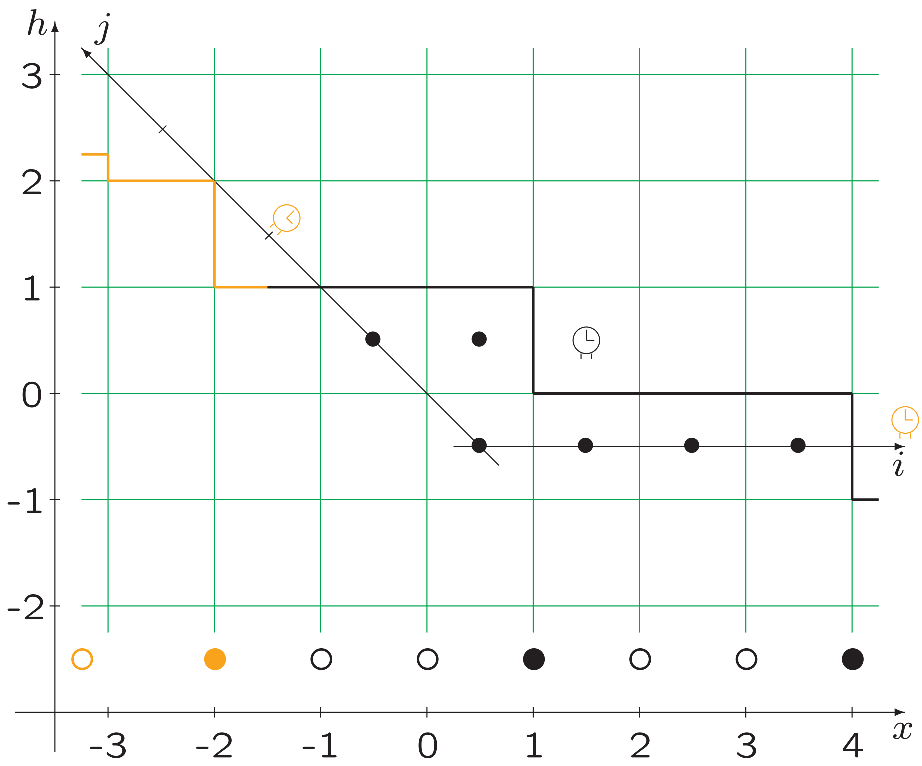

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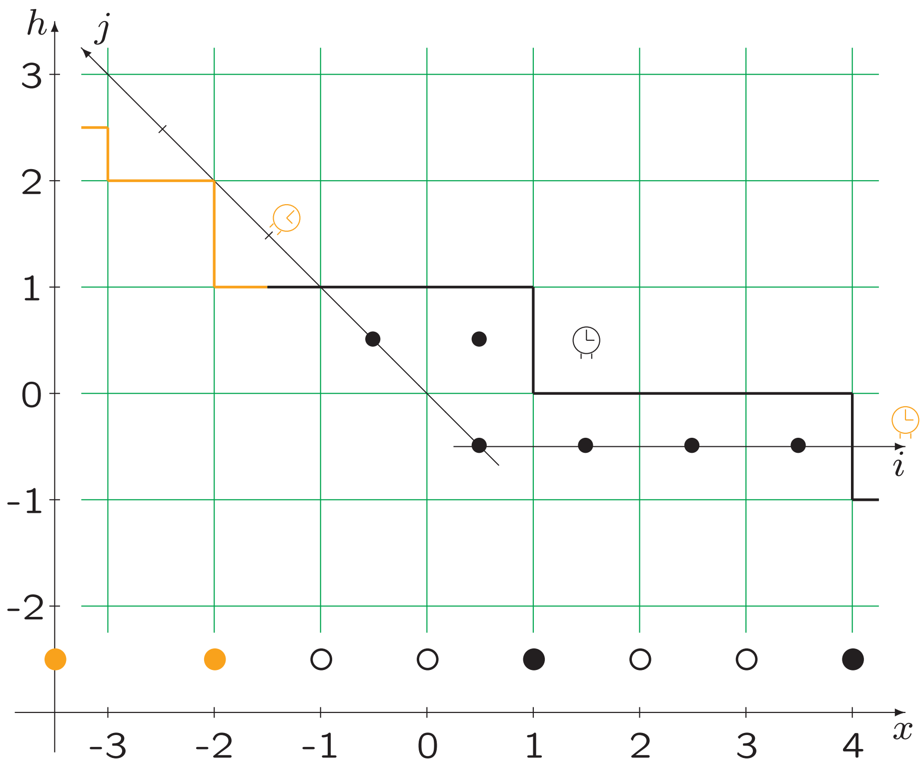

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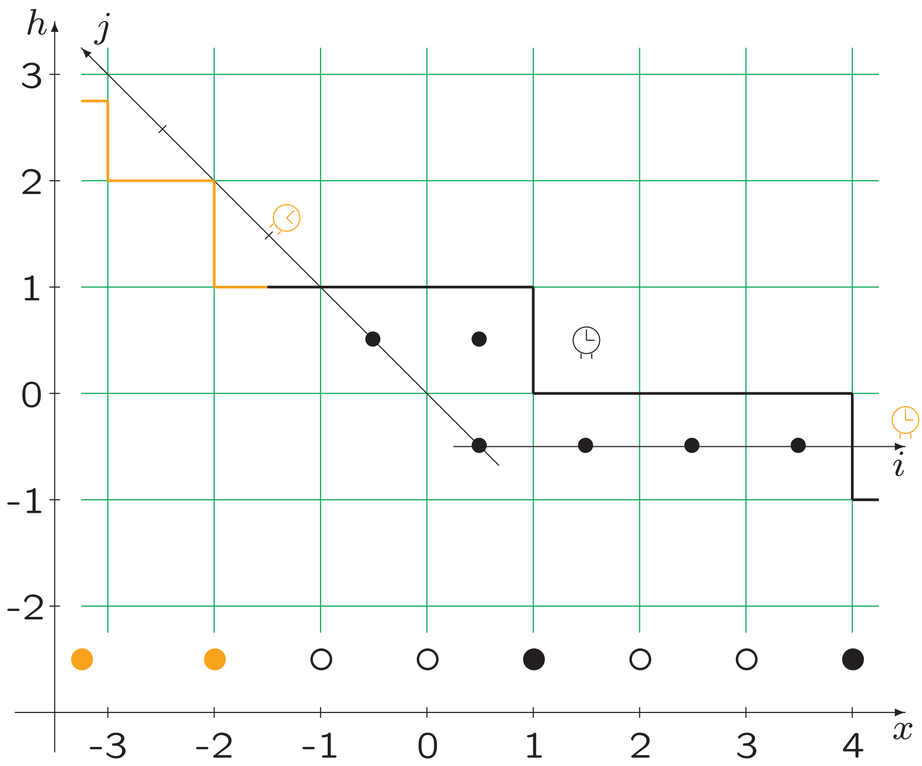

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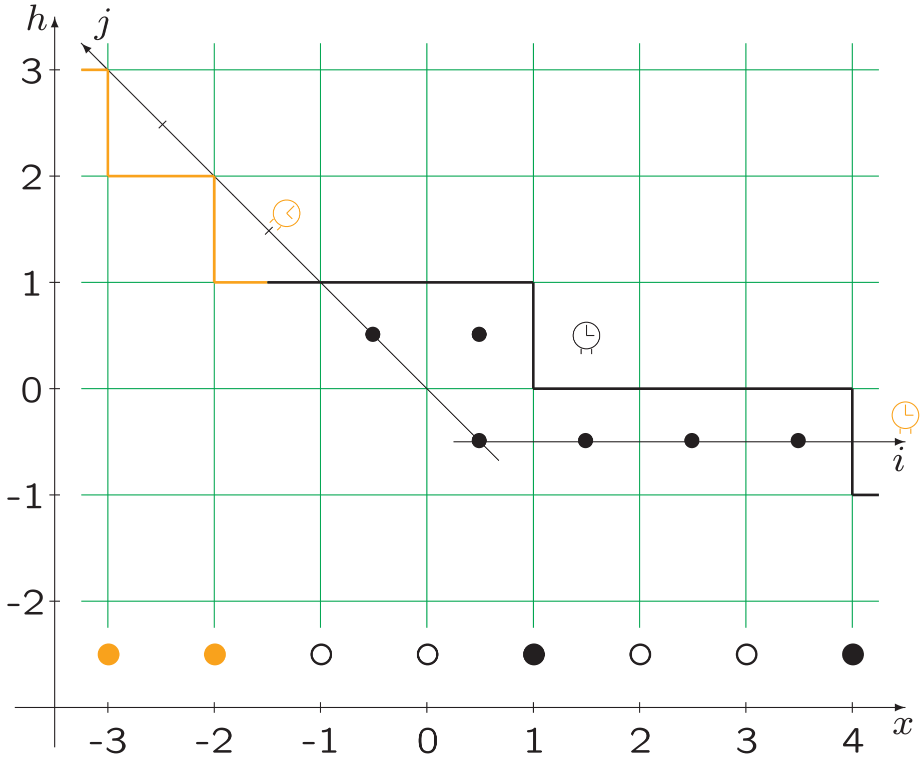

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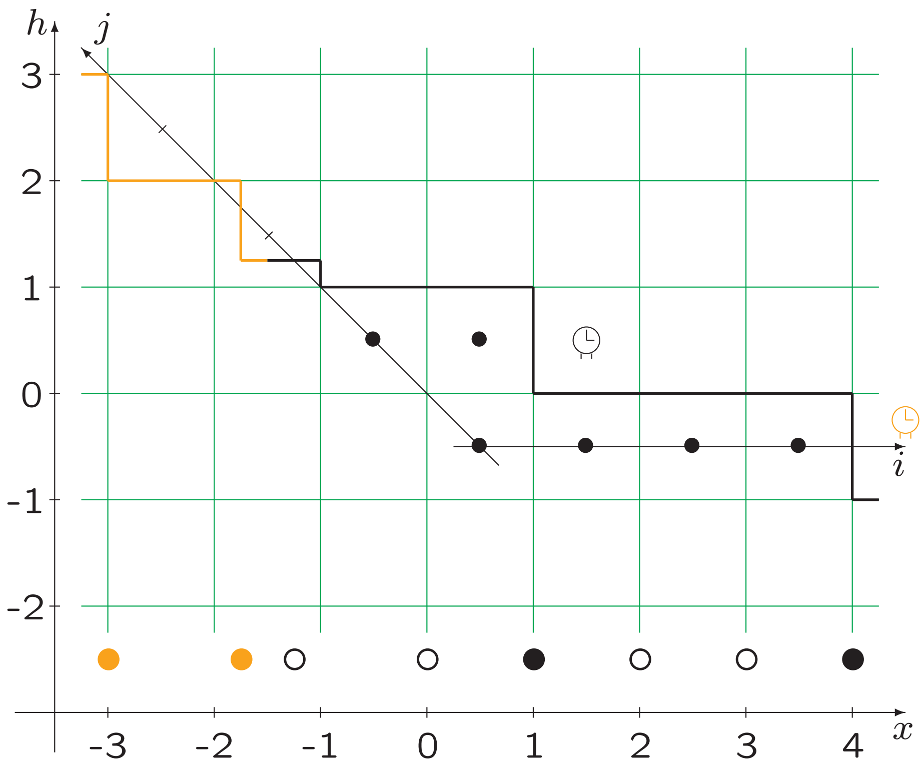

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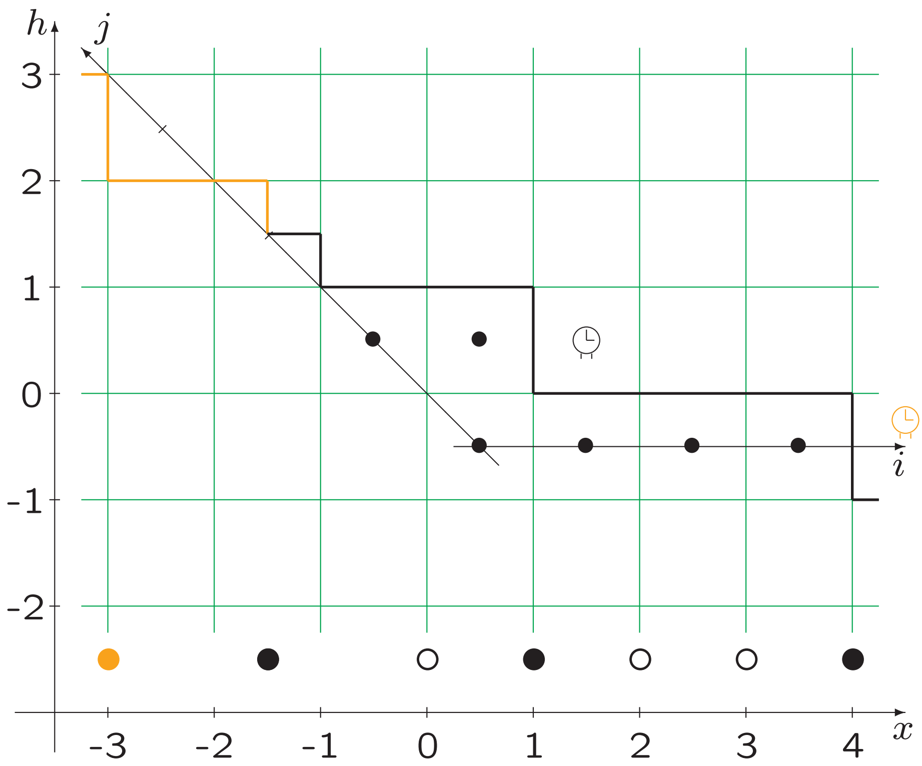

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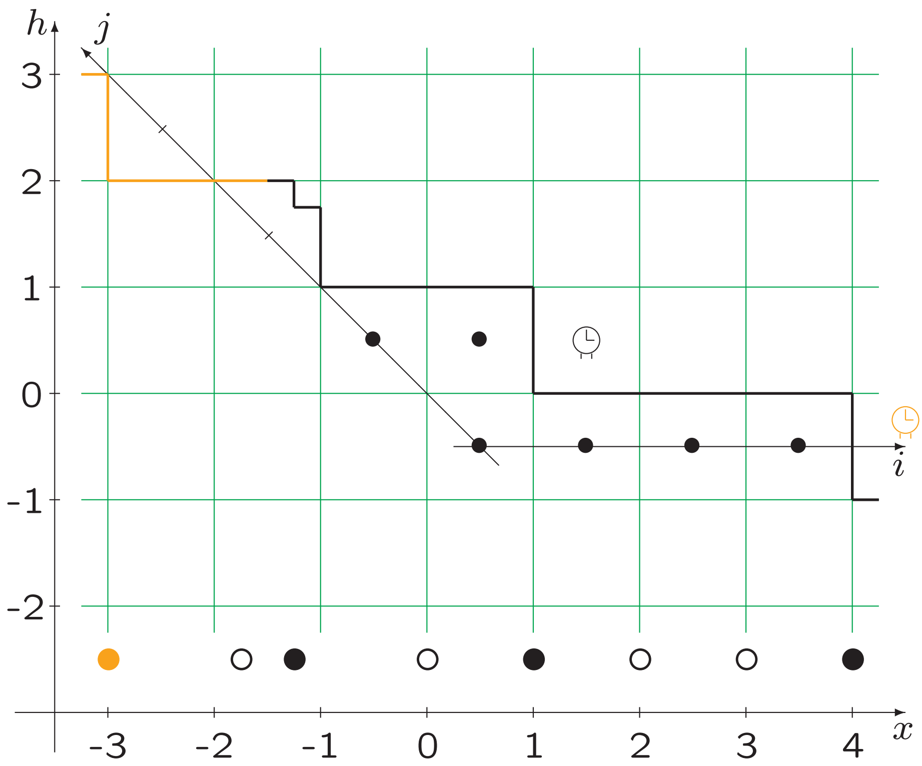

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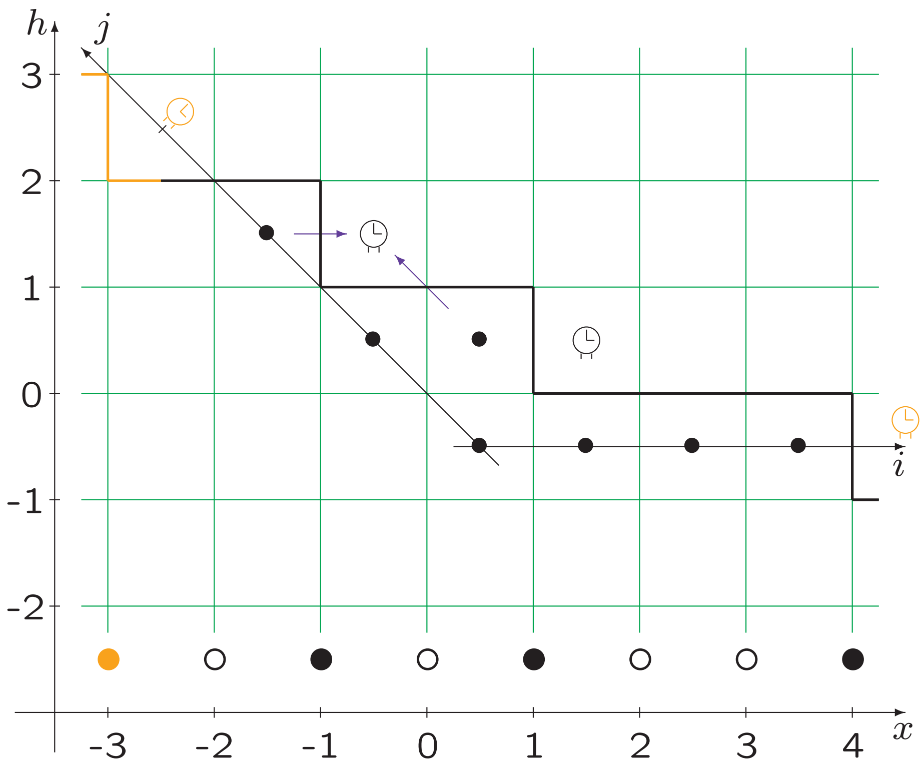

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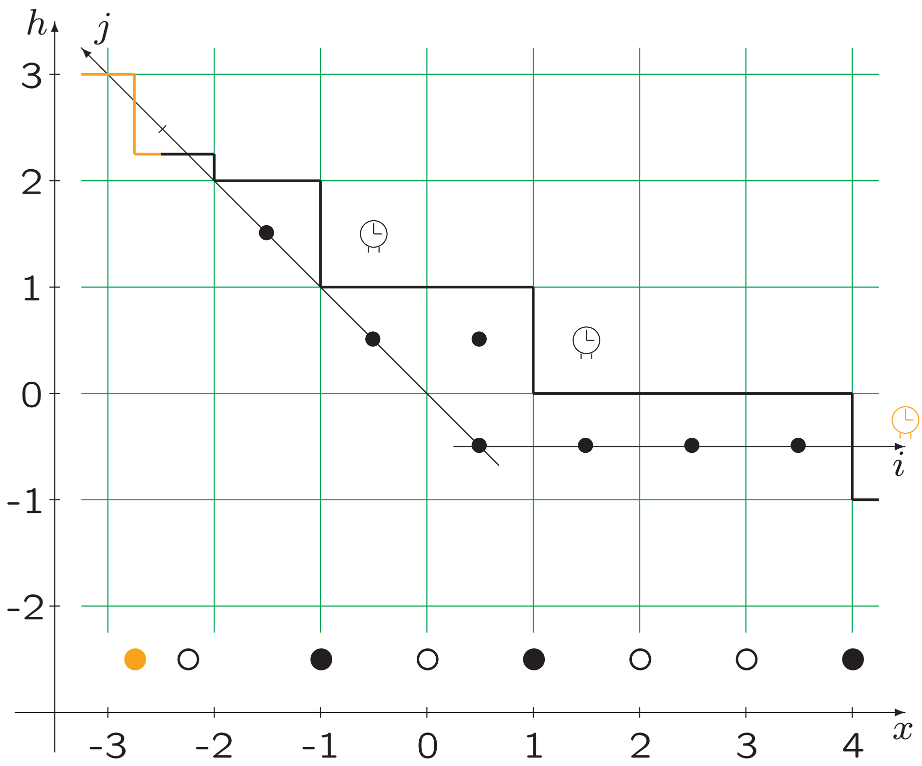

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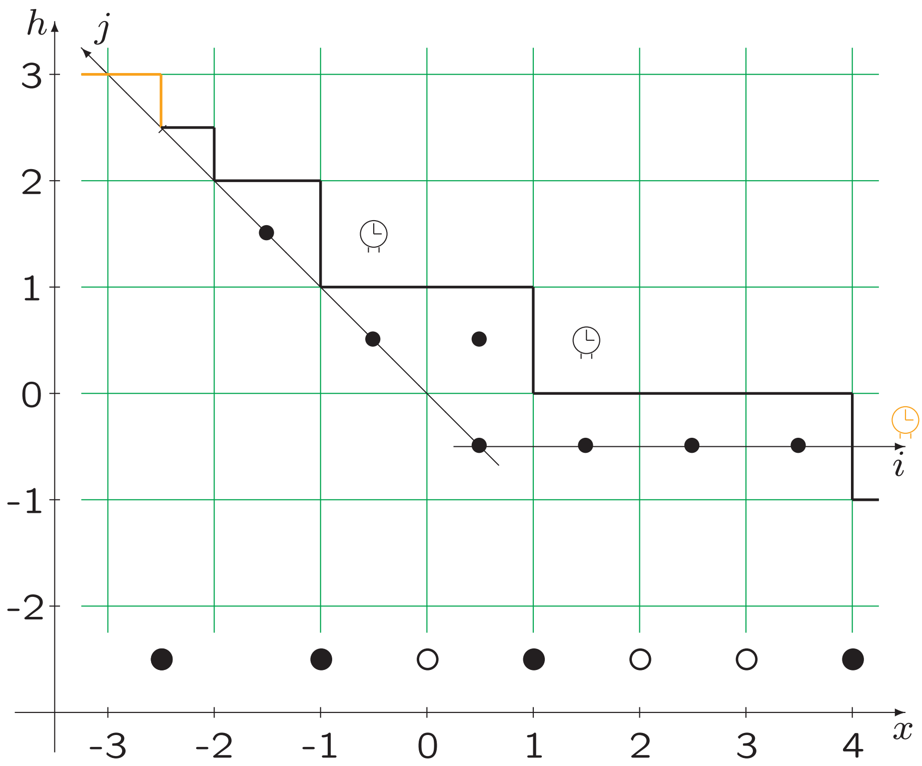

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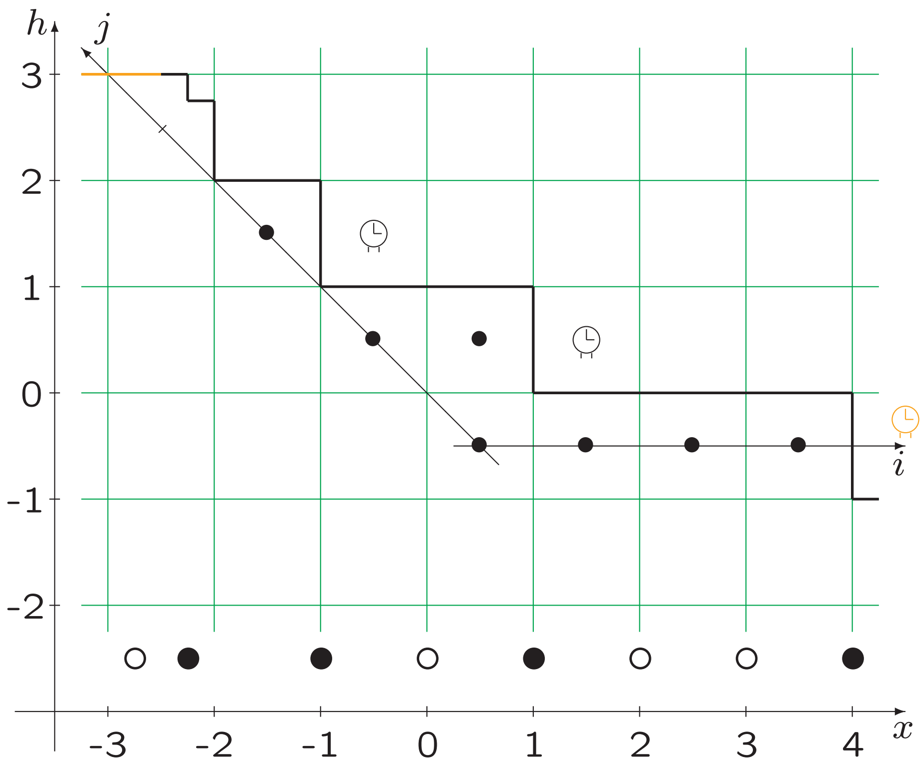

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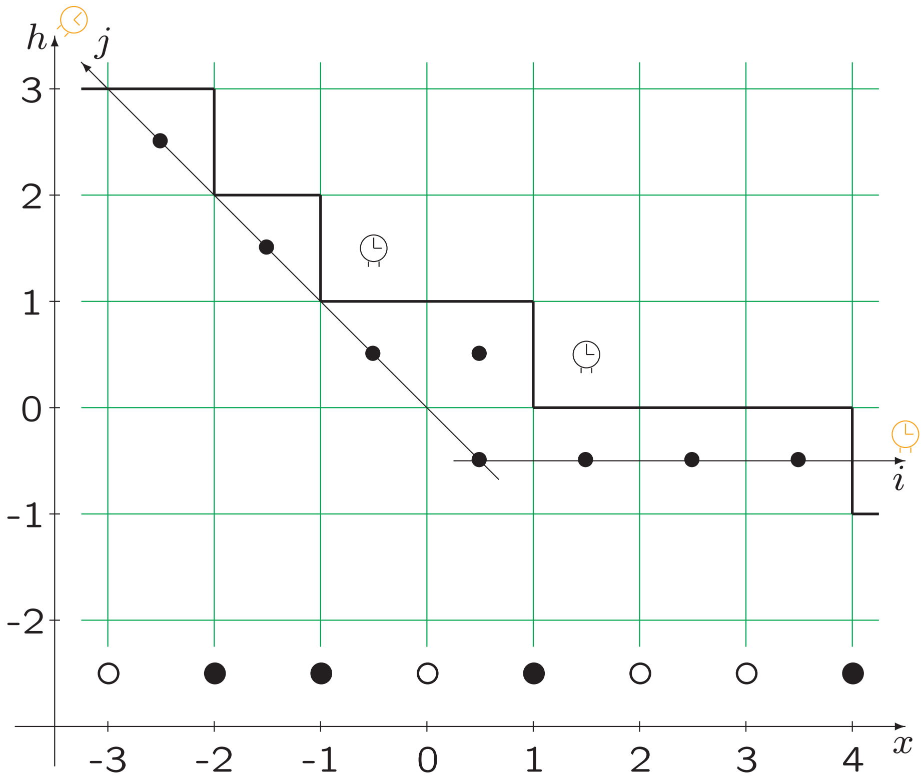

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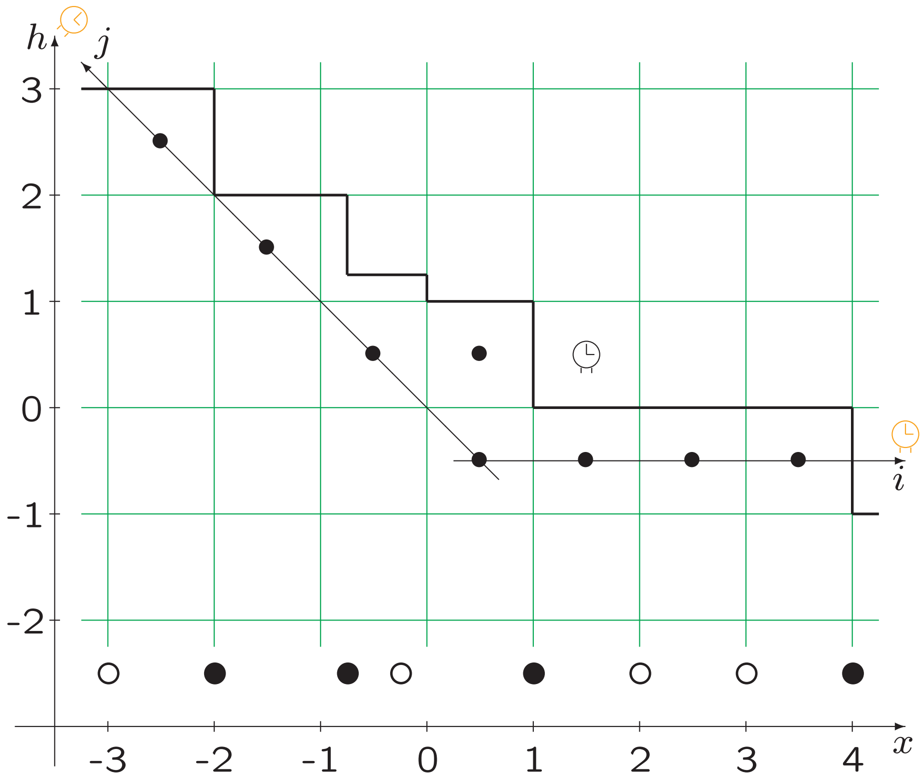

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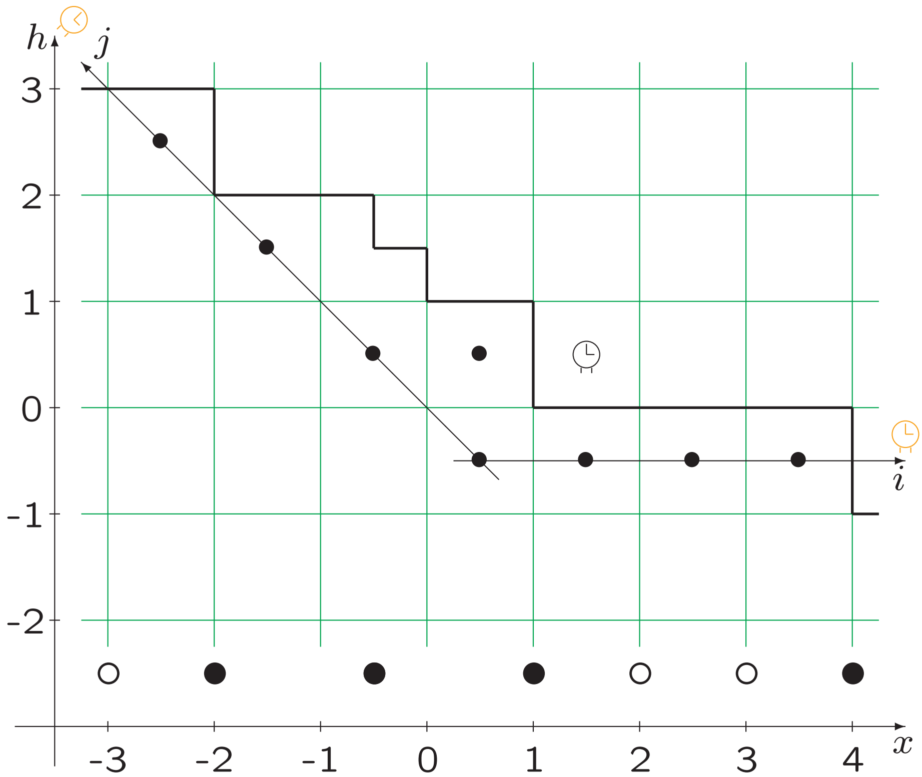

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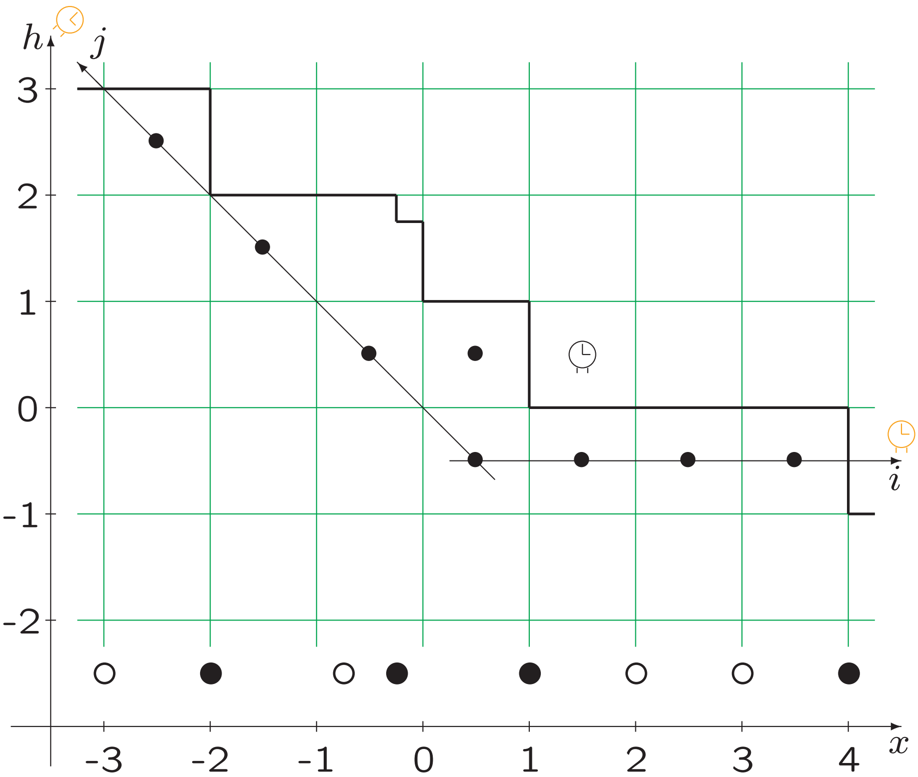

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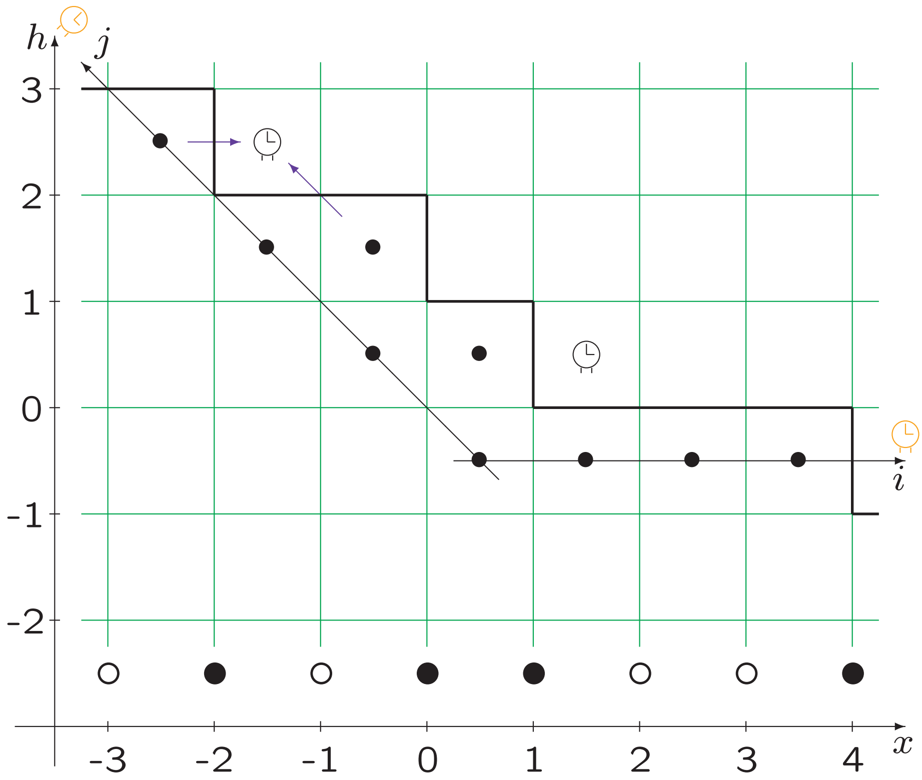
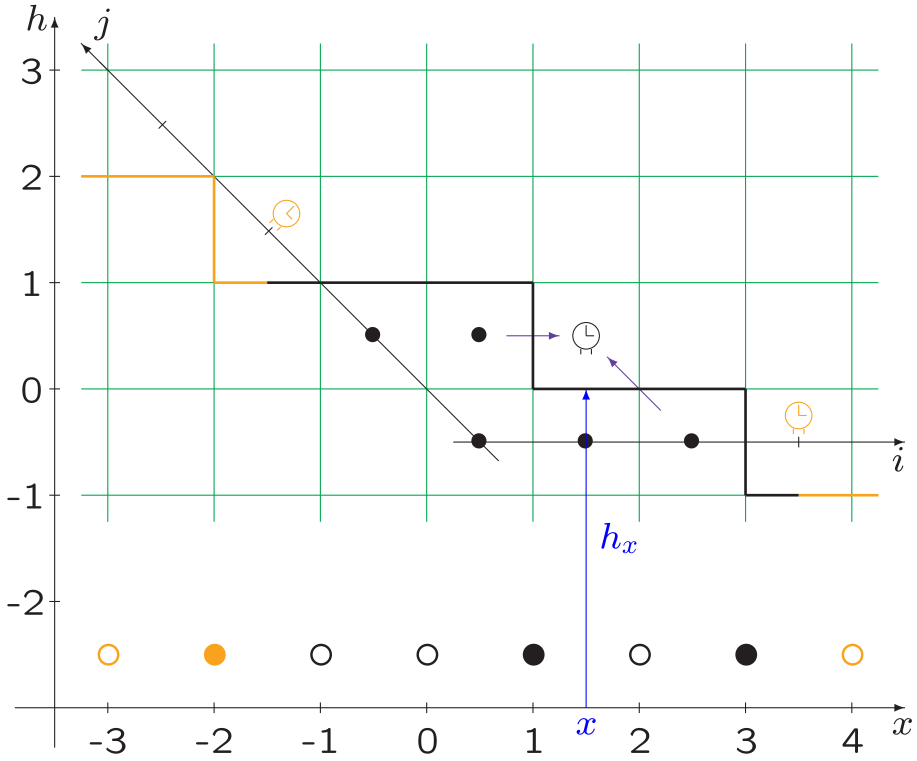
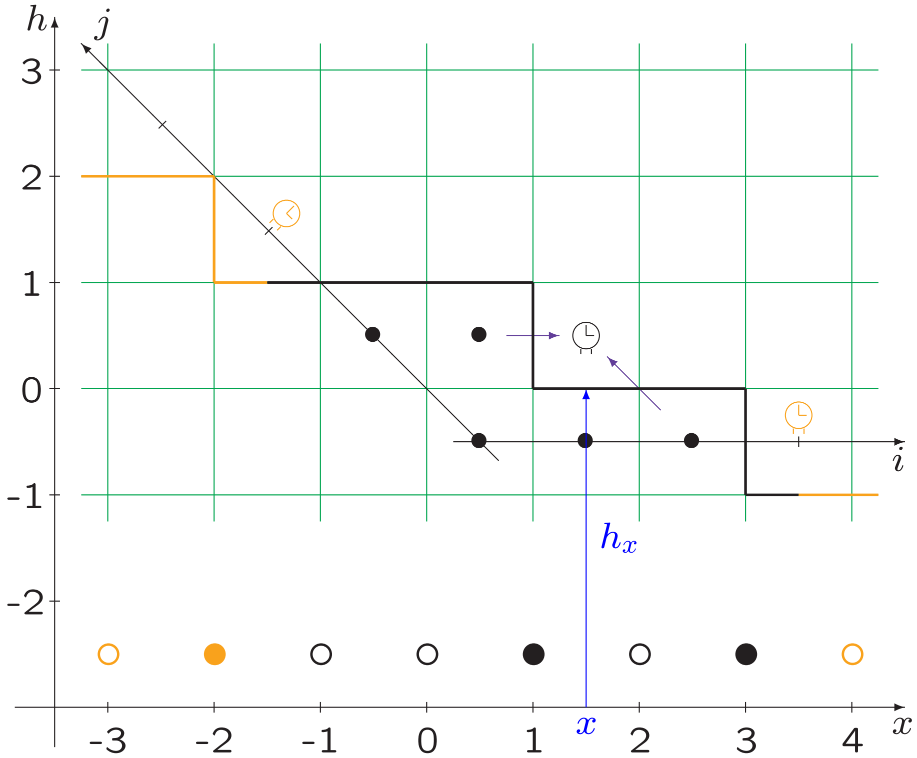


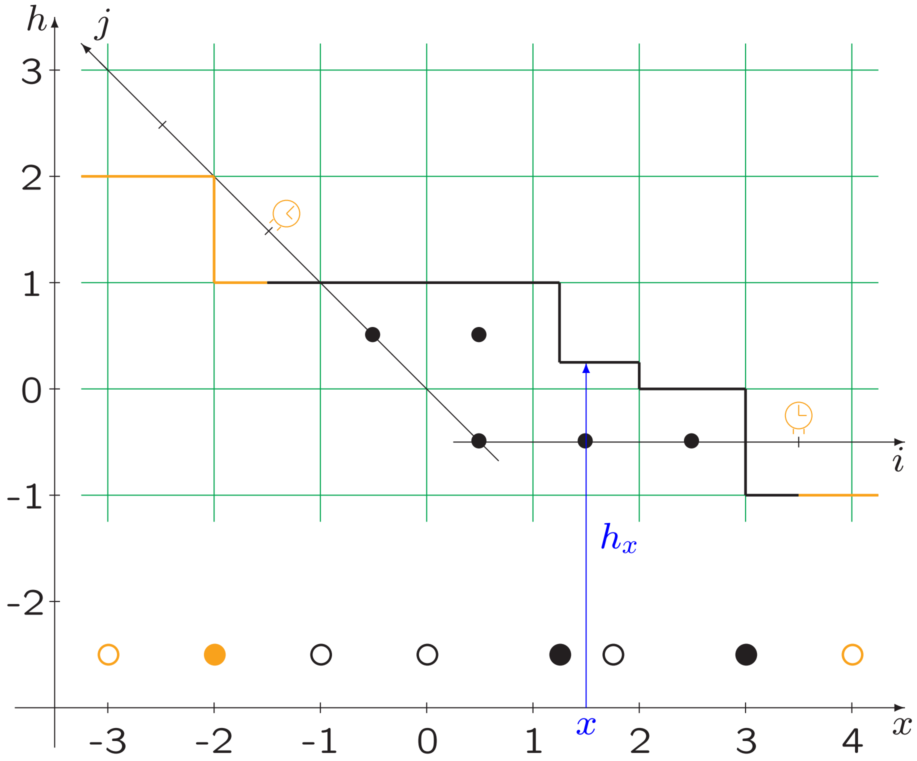


$h_x(t)$ = height of the surface above x.



$h_x(t)$ = height of the surface above x.

$h_x(t) - h_x(0)$ = number of particles passed above x.



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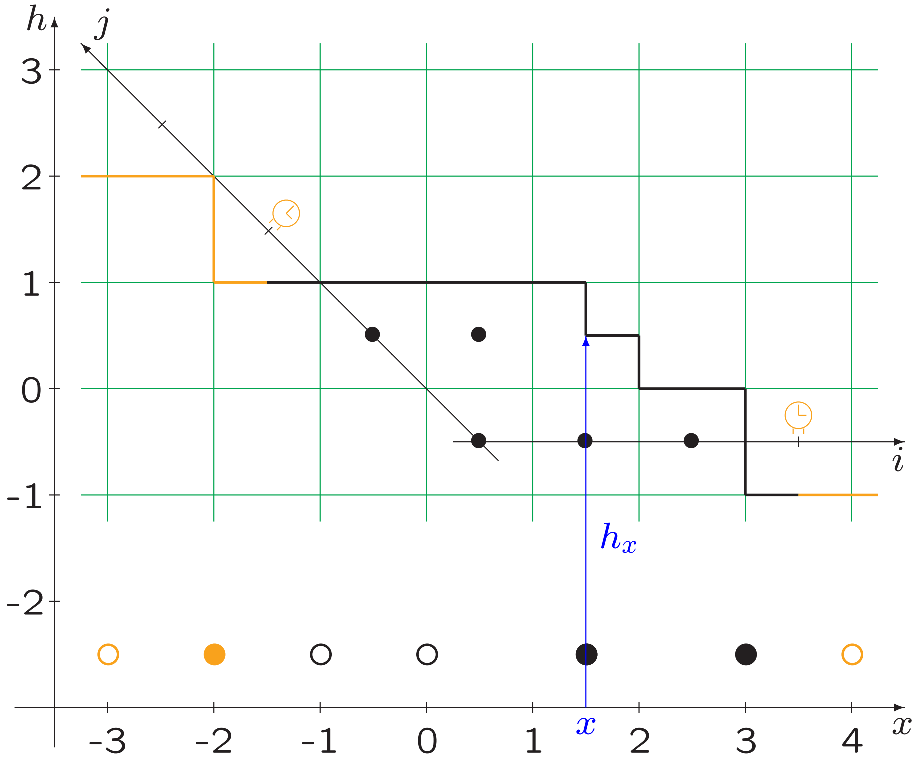

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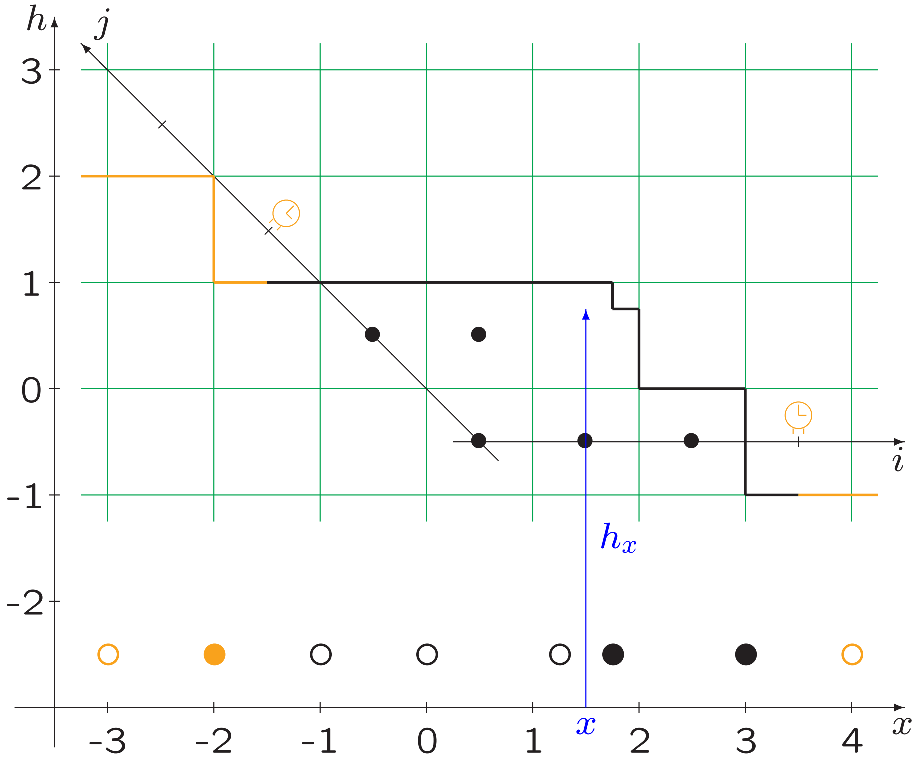

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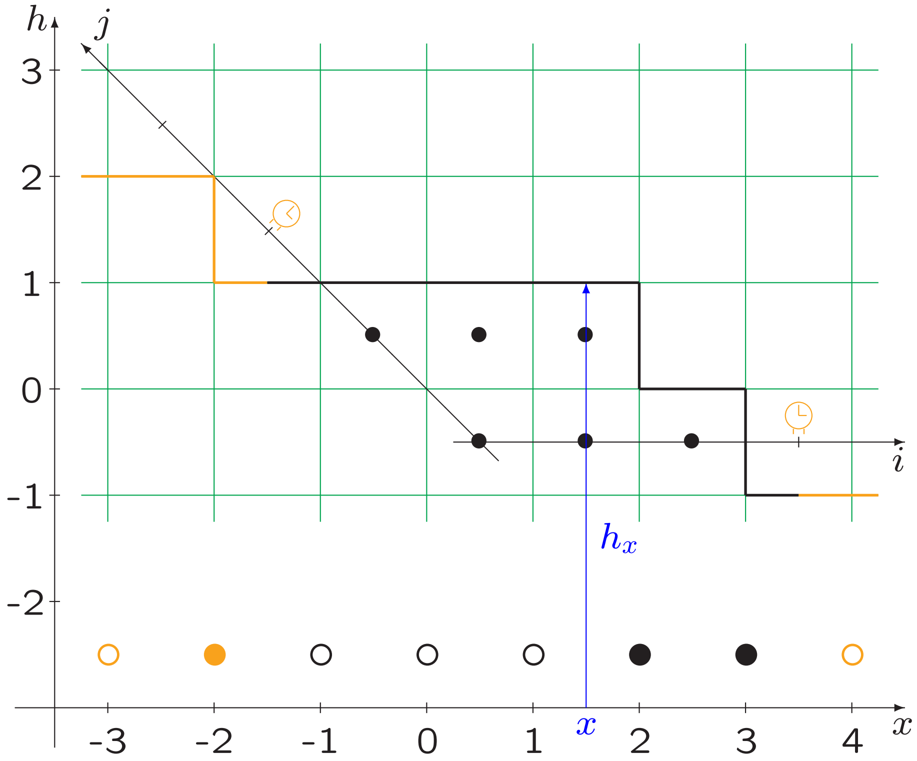

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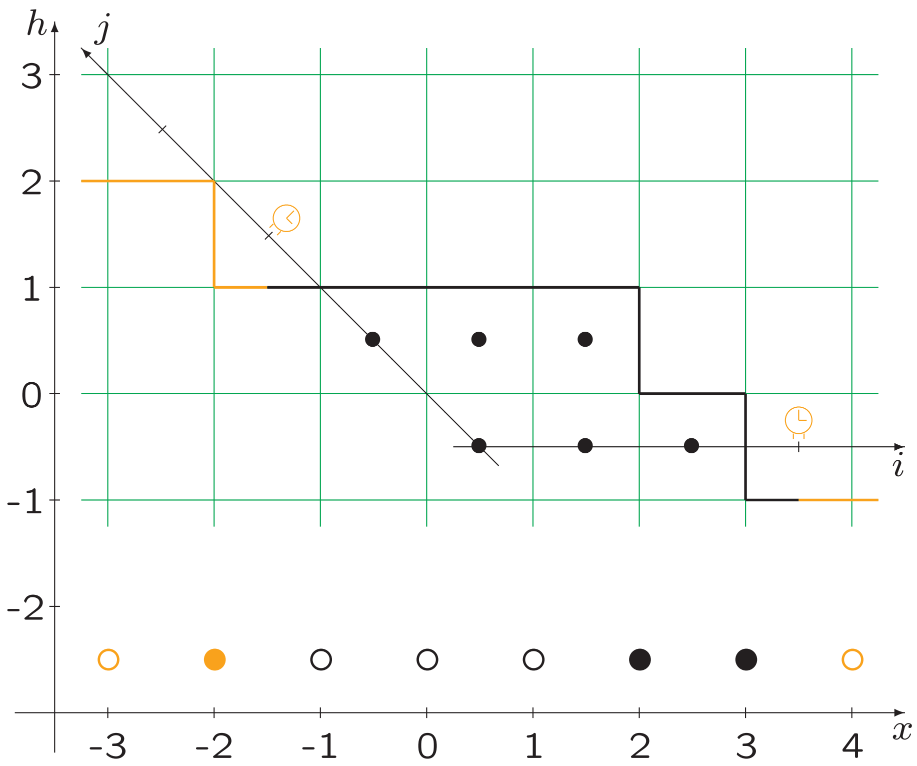
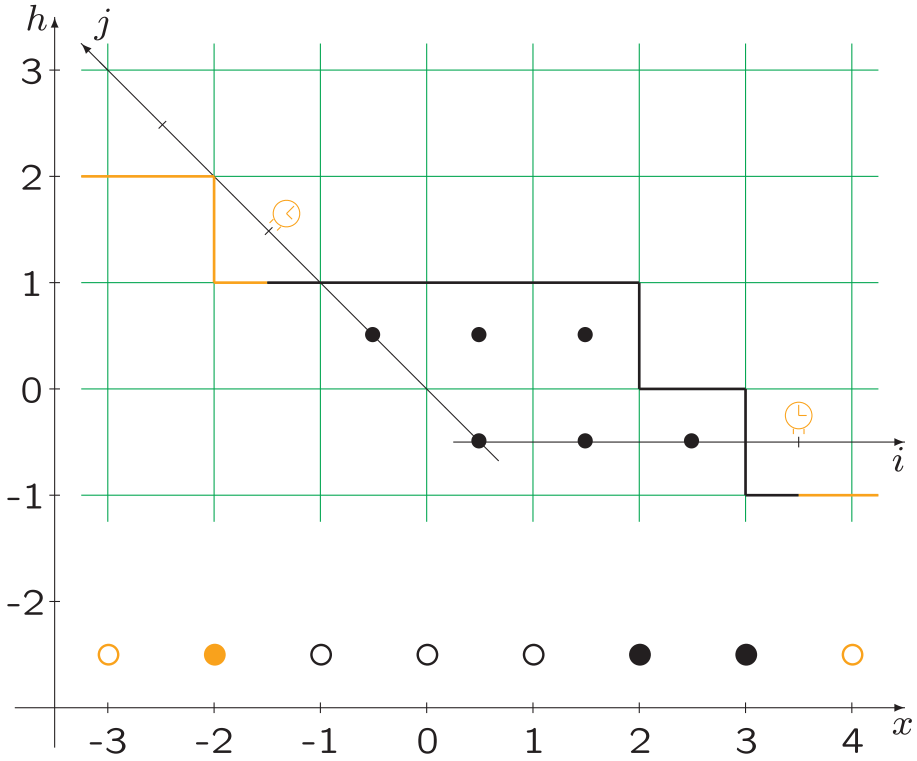


$h_x(t)$ = height of the surface above x.

$h_x(t) - h_x(0)$ = number of particles passed above x.

$h_{Vt}(t)$ = number of particles passed through the moving window at Vt $(V \in \mathbb{R})$.



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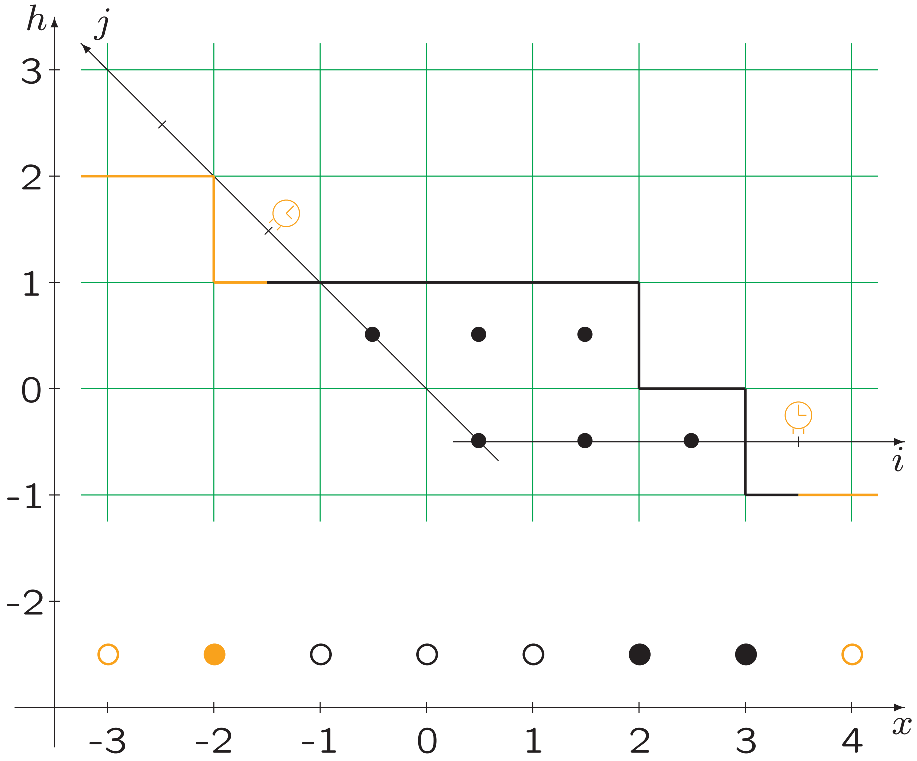
Ferrari - Fontes 1994:

$$\lim_{t \to \infty} \frac{\mathbf{Var}(h_{Vt}(t))}{t} = \text{const} \cdot |V - C(\varrho)|,$$

$C(\varrho)$ coming from the *hydrodynamics* of simple exclusion (characteristic speed).



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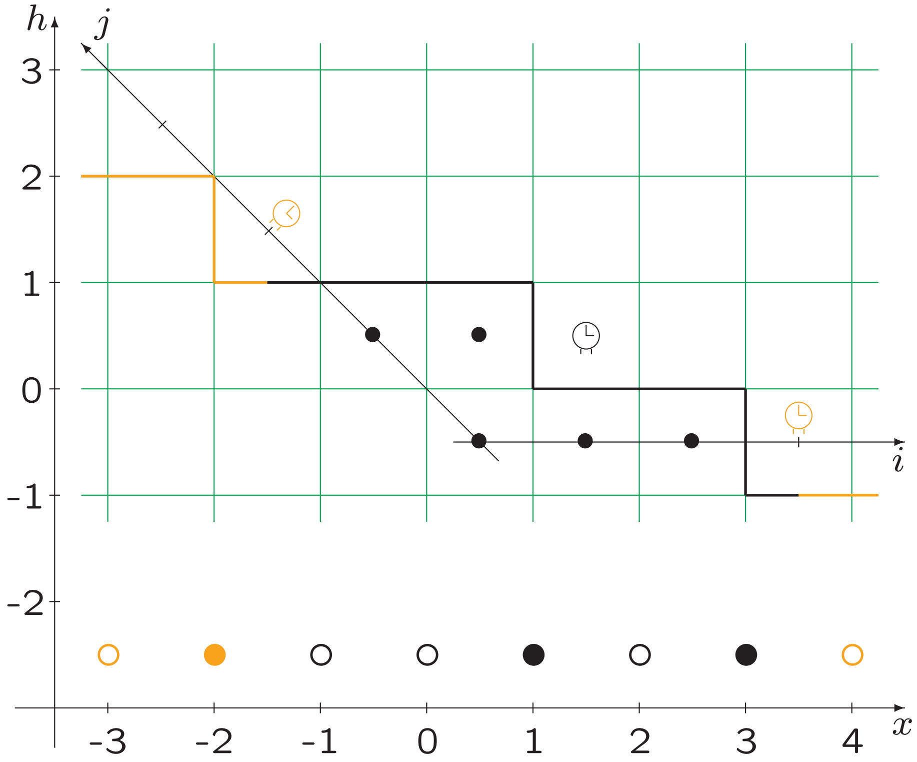
$\rightsquigarrow$ How about $V = C(\varrho)$?



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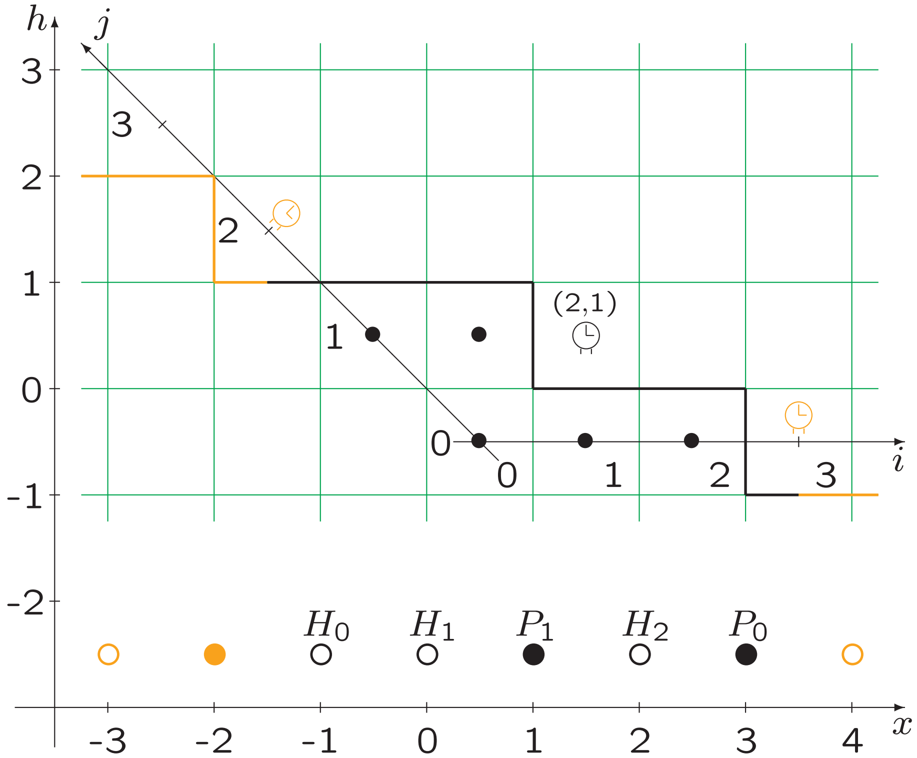

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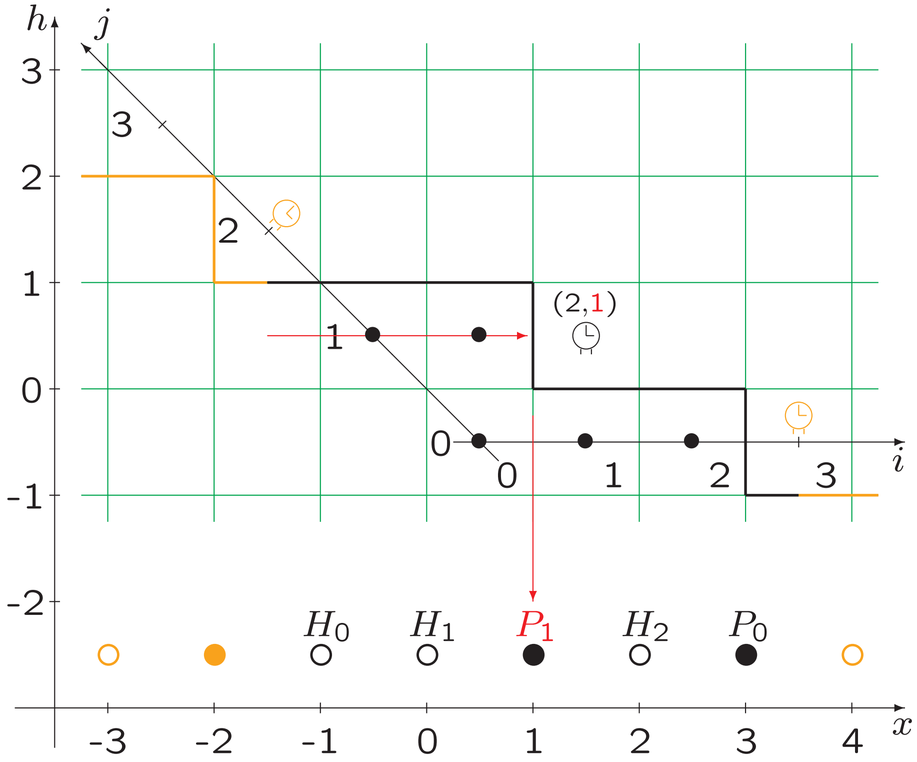

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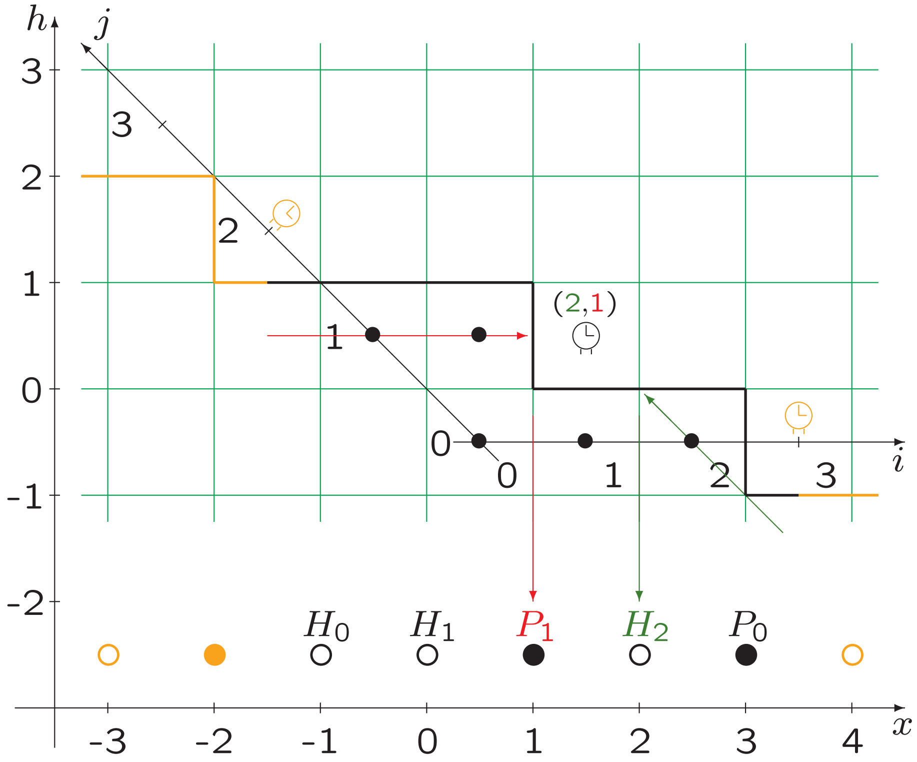
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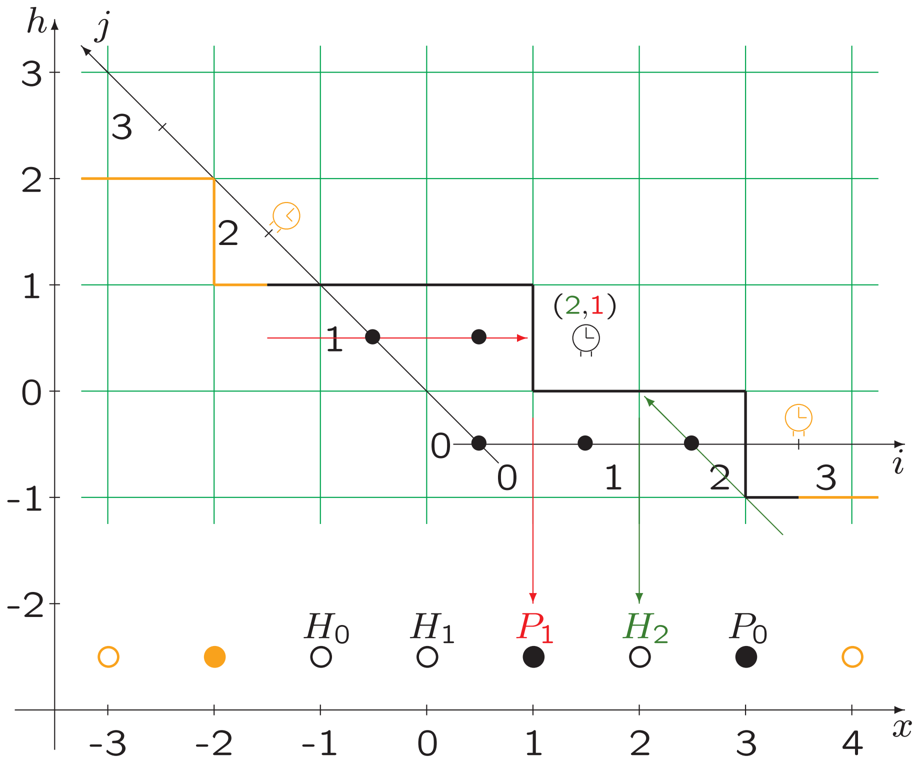
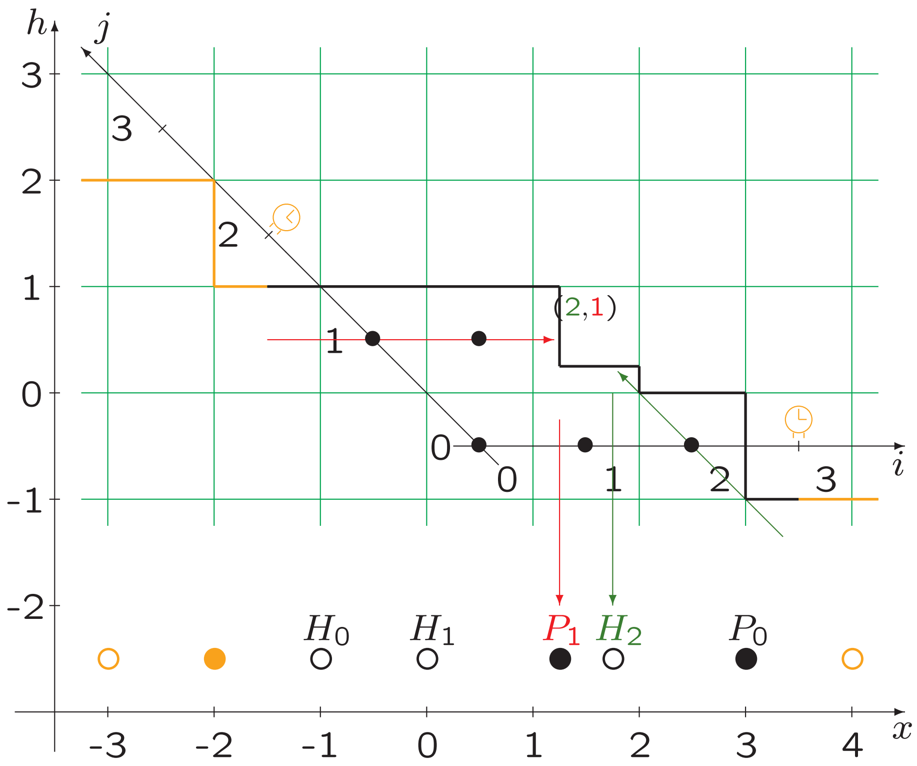


Occupation of (i, j) = jump of P_j over H_i.

Occupation of $(2, 1)$ = jump of P_1 over H_2.



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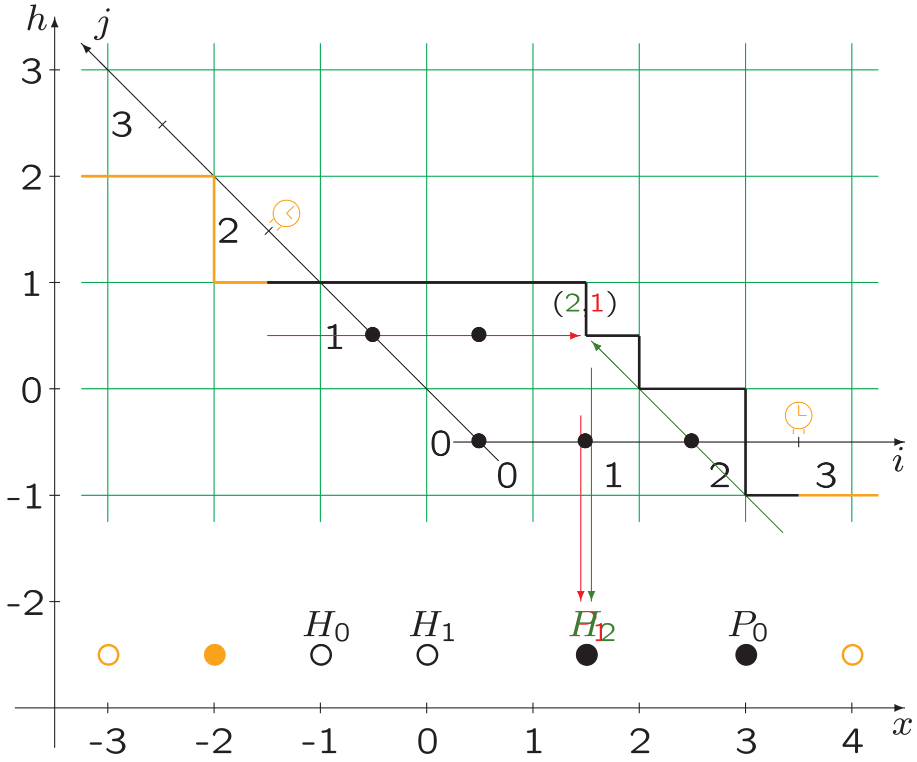

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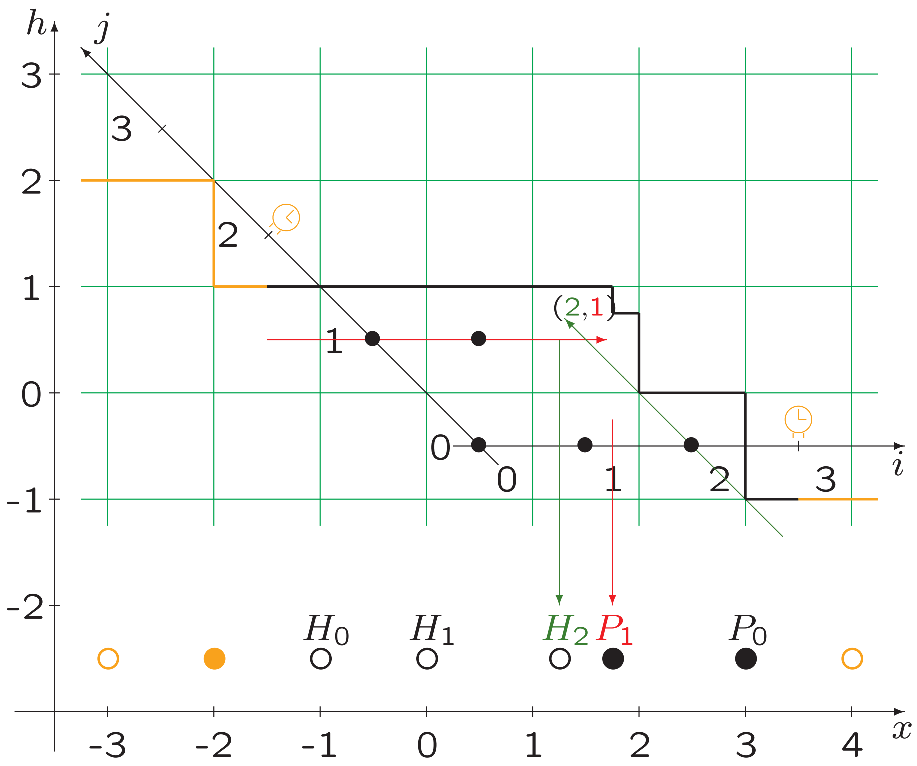

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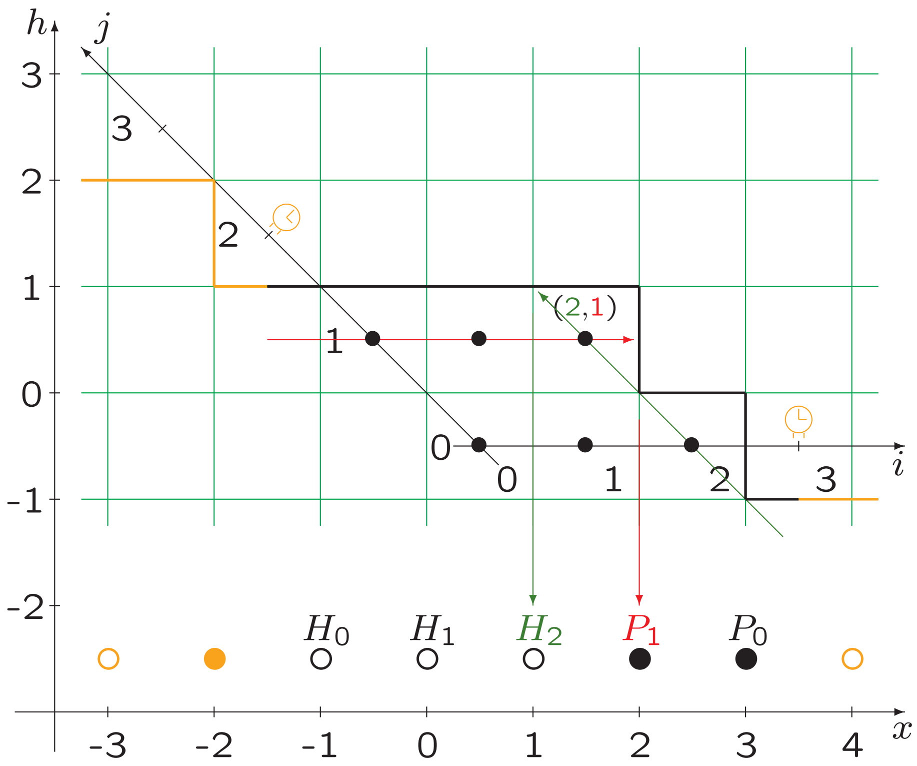

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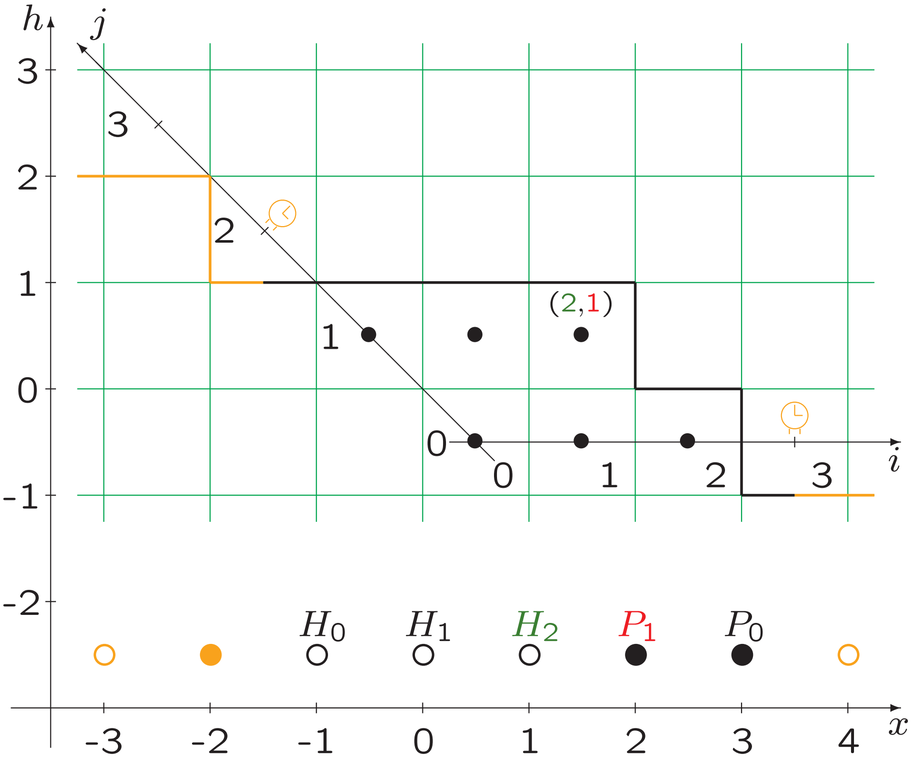

Occupation of (i, j) = jump of P_j over H_i.

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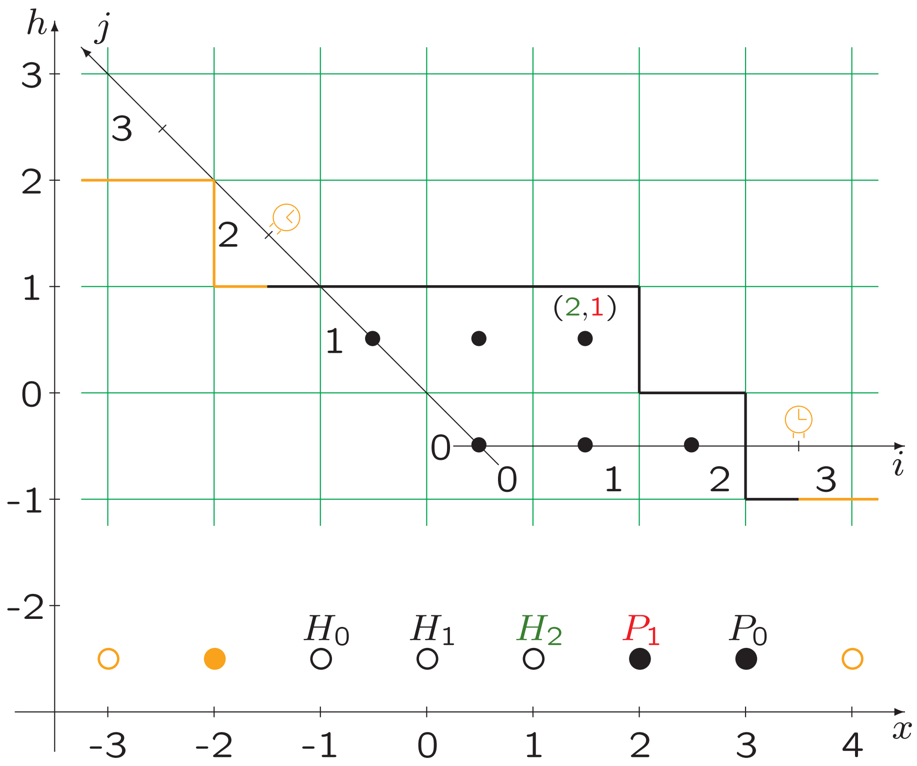
The time when this happens $=: G_{ij}$.



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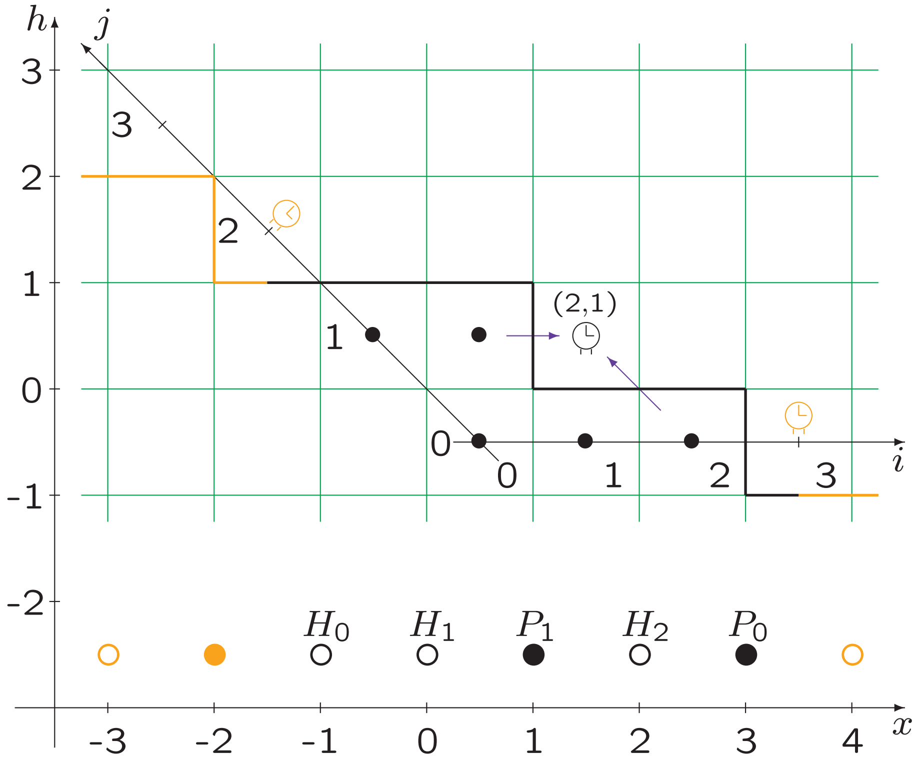
Occupation of $(2, 1)$ = jump of P_1 over H_2.

The time when this happens $=: G_{ij}$.

The characteristic speed $V = C(\varrho)$ translates to

$$m := (1 - \varrho)^2 t \text{ and } n := \varrho^2 t.$$

Will present results on G_{mn}.



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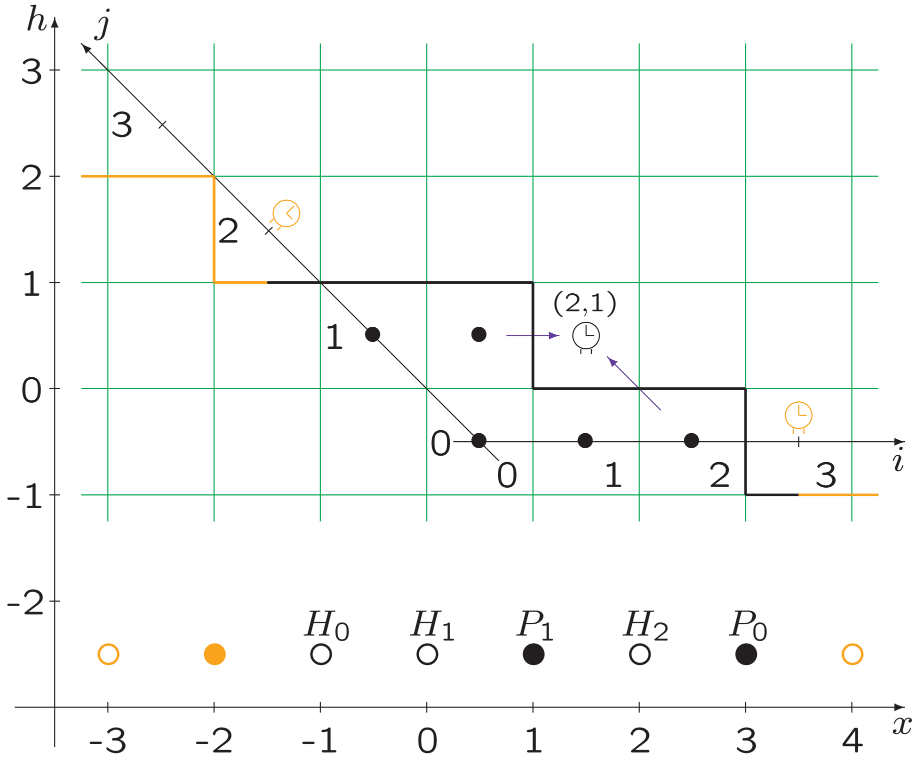
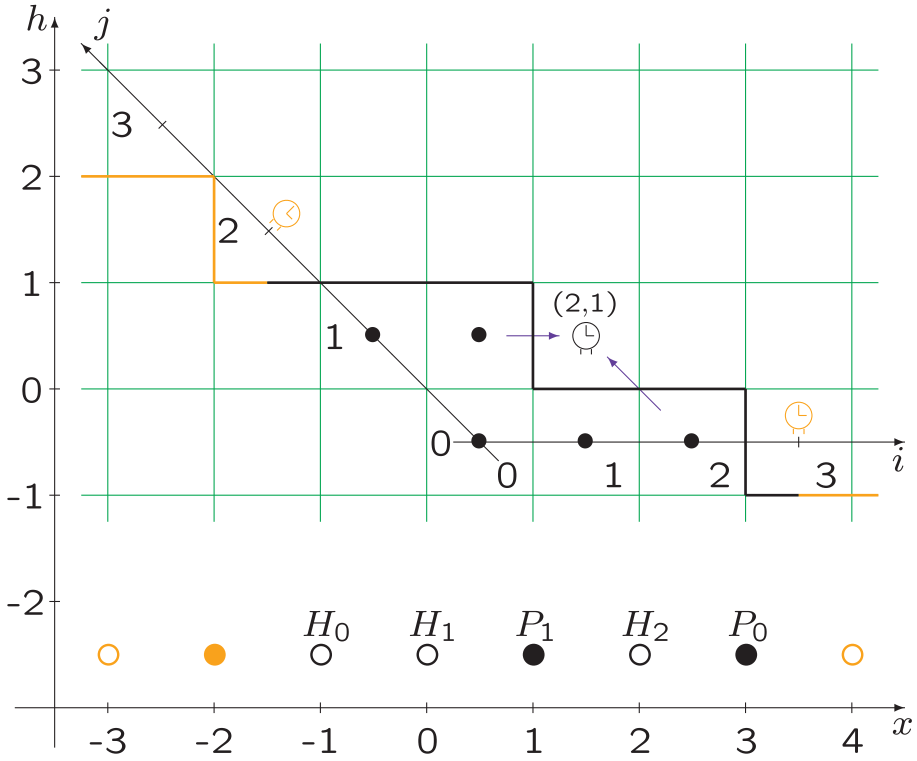


Burke's Theorem:

P_0 jumps according to a Poisson$(1-\varrho)$ process,

governed by the right orange part



Burke's Theorem:

P_0 jumps according to a Poisson$(1-\varrho)$ process,

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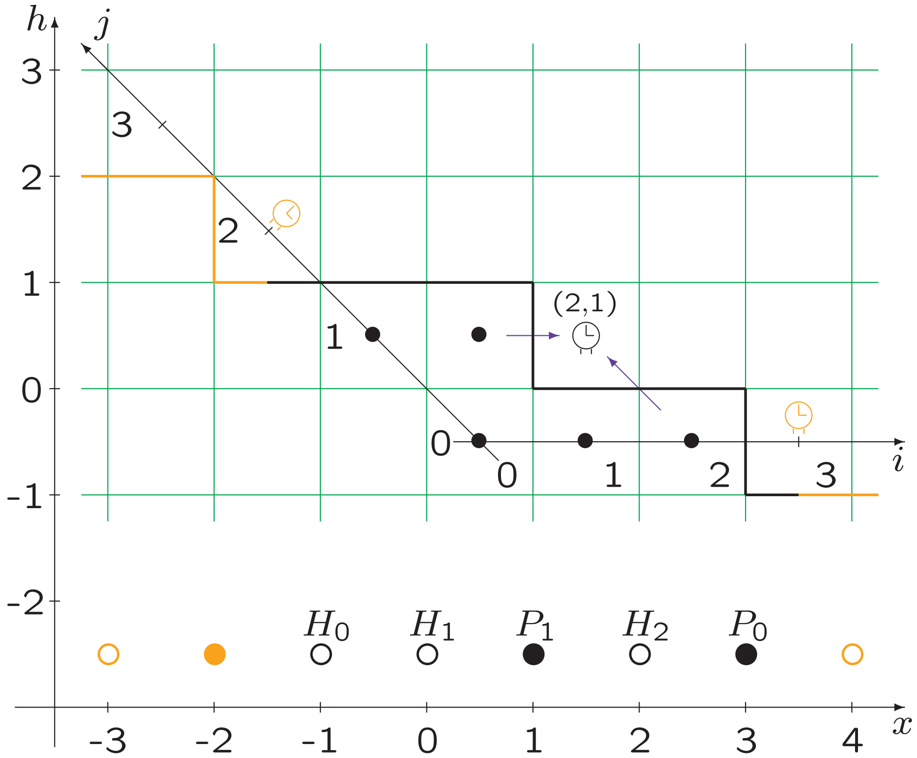
H_0 jumps according to a Poisson(ϱ) process,

governed by the left orange part



Burke's Theorem:

P_0 jumps according to a Poisson($1-\varrho$) process,

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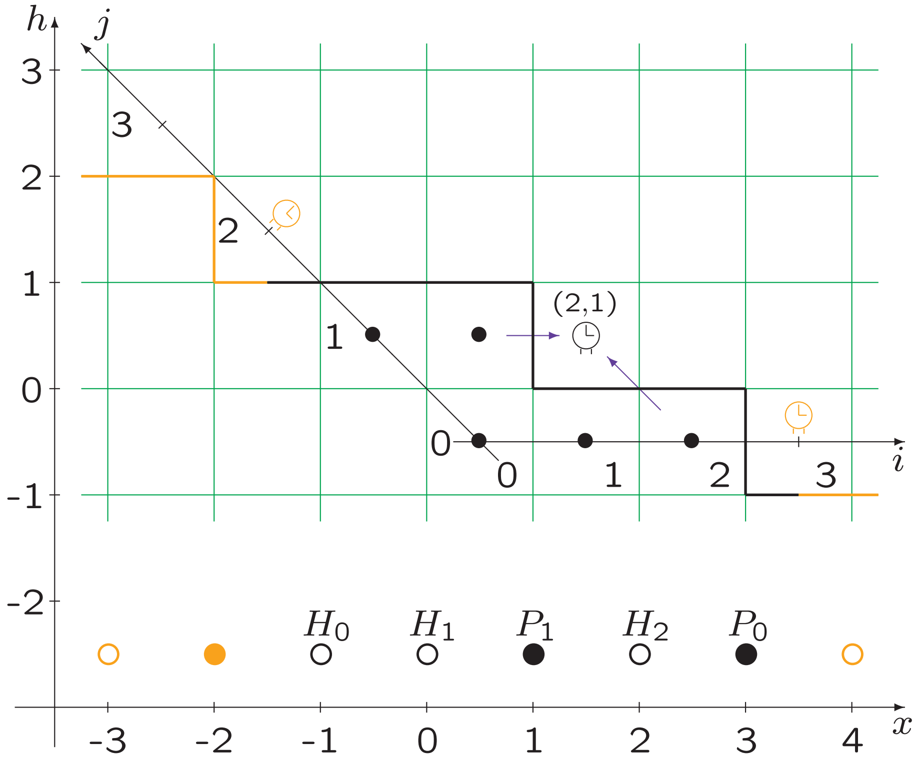
H_0 jumps according to a Poisson(ϱ) process,

governed by the left orange part

independently of the ⏲'s.



<u>Burke's Theorem:</u>

P_0 jumps according to a Poisson$(1-\varrho)$ process,

governed by the right orange part

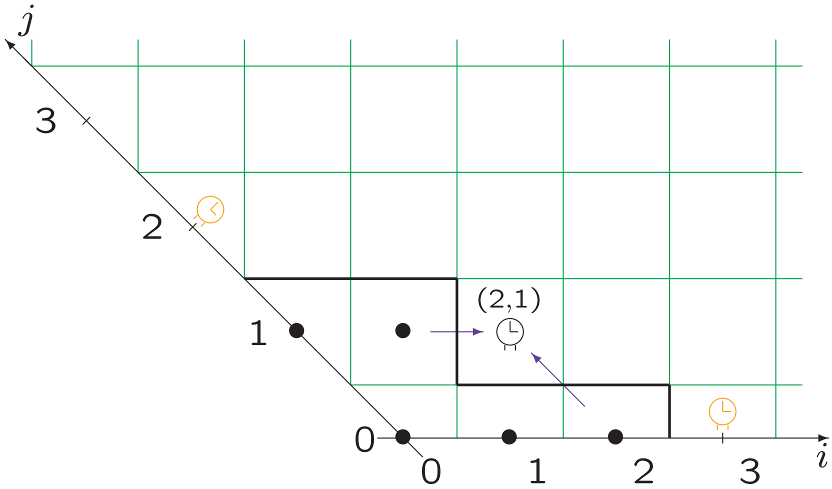
H_0 jumps according to a Poisson(ϱ) process,

governed by the left orange part

independently of the 🕒's.

<u>Therefore:</u>

🕒 $\sim$ Exponential$(1-\varrho)$
🕒 $\sim$ Exponential(ϱ) } independently
🕒 $\sim$ Exponential(1)



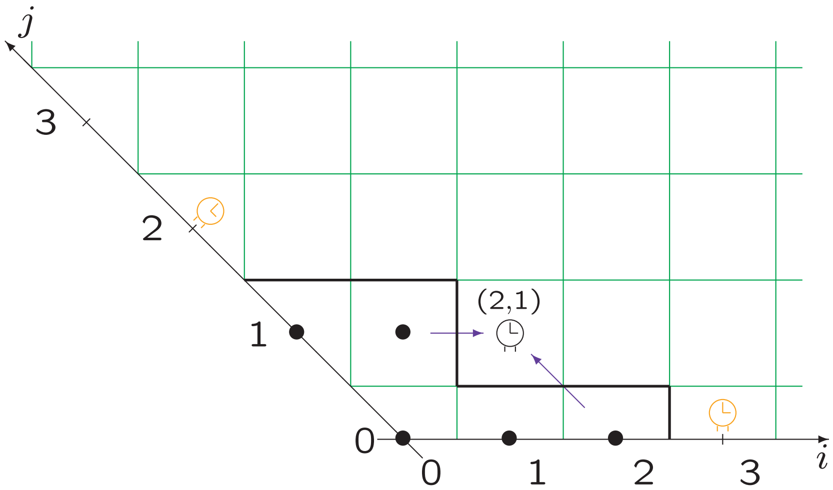
 $\sim$ Exponential$(1 - \varrho)$

 $\sim$ Exponential(ϱ)

 $\sim$ Exponential(1)

(all three) independently

2. The last passage model



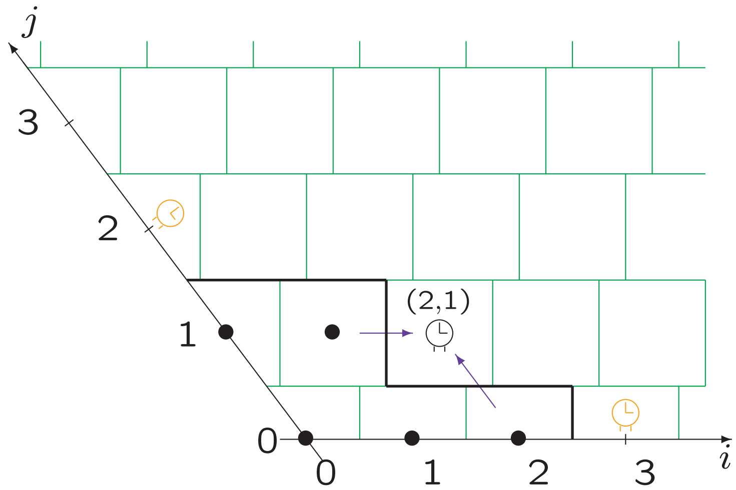
$\sim$ Exponential($1 - \varrho$)

$\sim$ Exponential(ϱ)

$\sim$ Exponential(1)

} independently

2. The last passage model



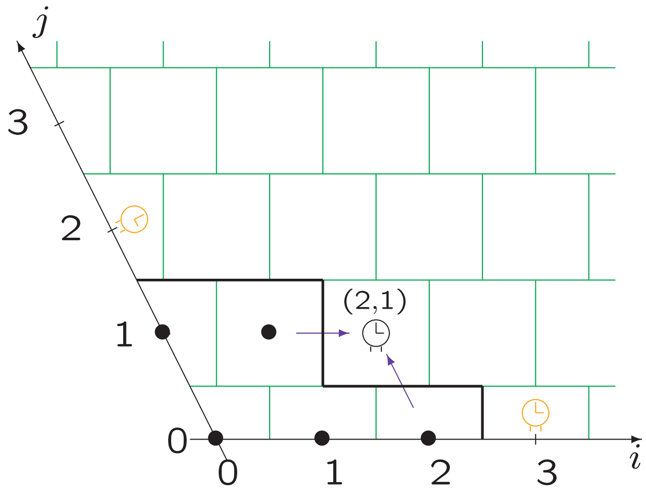
$\sim$ Exponential$(1 - \varrho)$

$\sim$ Exponential(ϱ)

$\sim$ Exponential(1)

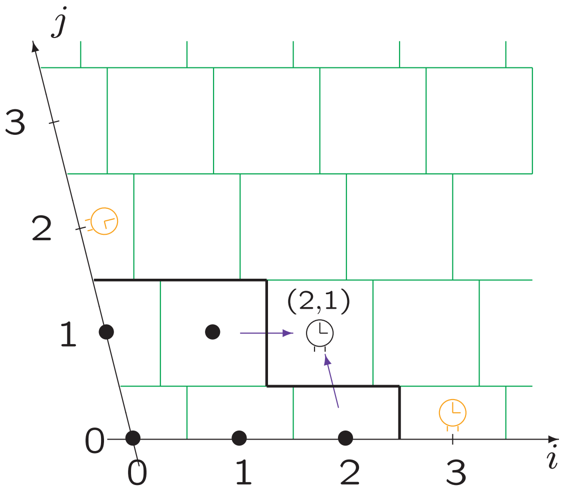
} independently

2. The last passage model



$\sim$ Exponential$(1-\varrho)$
$\sim$ Exponential(ϱ)
$\sim$ Exponential(1)
} independently

2. The last passage model

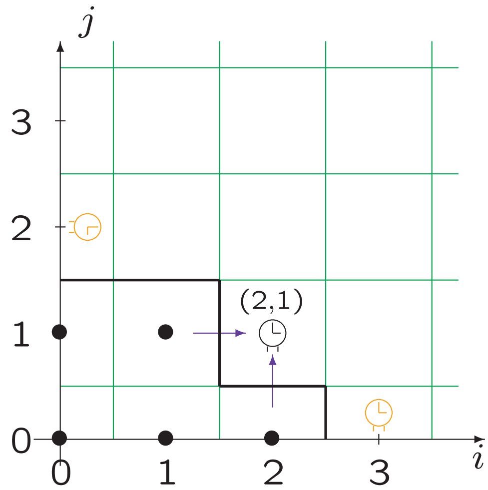


$\sim$ Exponential$(1 - \varrho)$

$\sim$ Exponential(ϱ) } independently

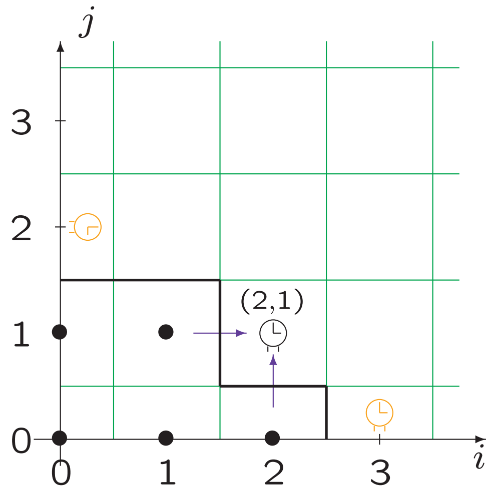
$\sim$ Exponential(1)

2. The last passage model



$\sim$ Exponential$(1-\varrho)$
$\sim$ Exponential(ϱ) independently
$\sim$ Exponential(1)

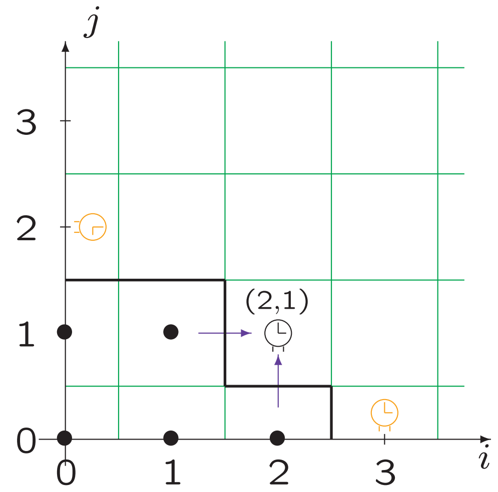
2. The last passage model



$$\left.\begin{array}{l} \sim \text{Exponential}(1-\varrho) \\ \sim \text{Exponential}(\varrho) \\ \sim \text{Exponential}(1) \end{array}\right\} \text{independently}$$

starts ticking when its west neighbor becomes occupied

2. The last passage model

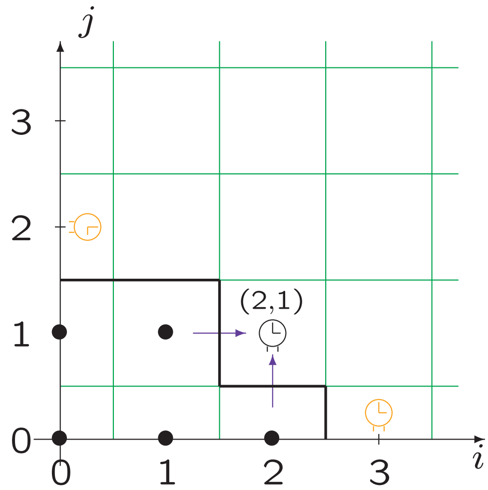


$\sim$ Exponential$(1-\varrho)$
$\sim$ Exponential(ϱ) independently
$\sim$ Exponential(1)

starts ticking when its west neighbor becomes occupied

starts ticking when its south neighbor becomes occupied

2. The last passage model



$$\sim \text{Exponential}(1-\varrho)$$
$$\sim \text{Exponential}(\varrho)$$ independently
$$\sim \text{Exponential}(1)$$

starts ticking when its west neighbor becomes occupied

starts ticking when its south neighbor becomes occupied

starts ticking when both its west and south neighbors become occupied

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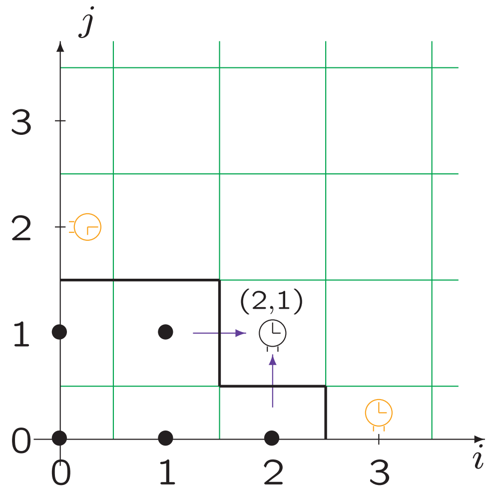

M. Prähofer and H. Spohn 2002

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} independently

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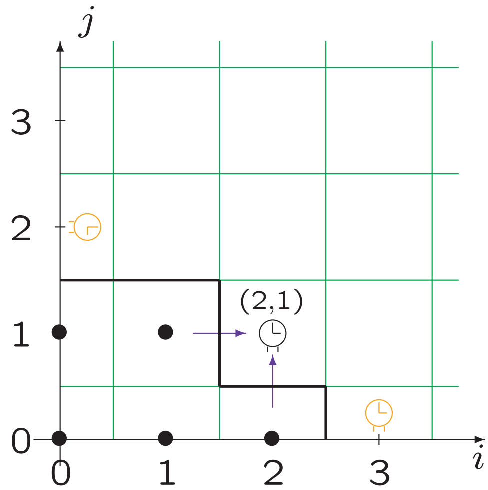

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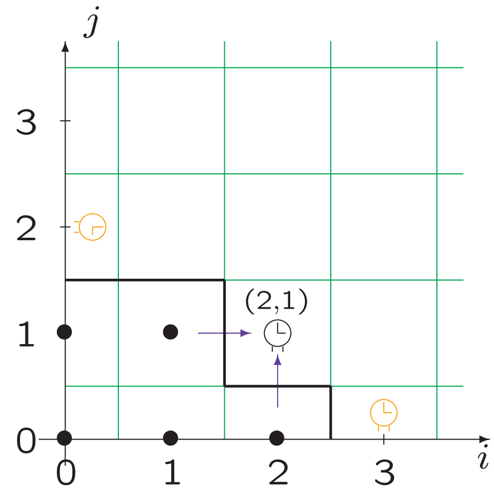

M. Prähofer and H. Spohn 2002

$$\left.\begin{array}{l} \sim \text{Exponential}(1-\varrho) \\ \sim \text{Exponential}(\varrho) \\ \sim \text{Exponential}(1) \end{array}\right\} \text{independently}$$

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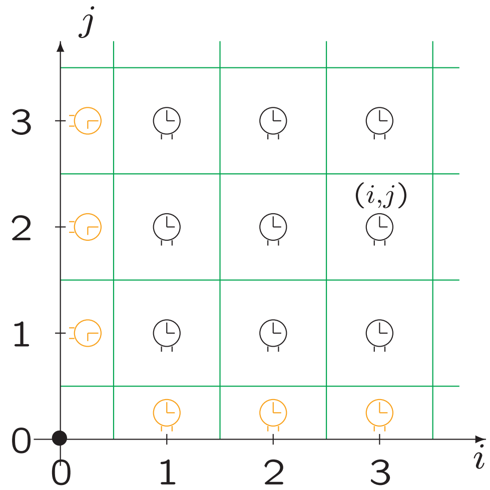

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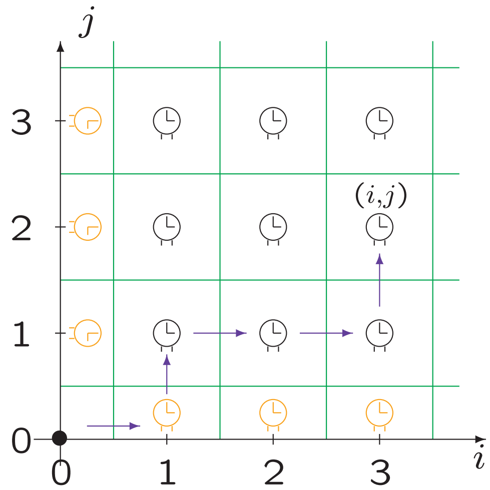

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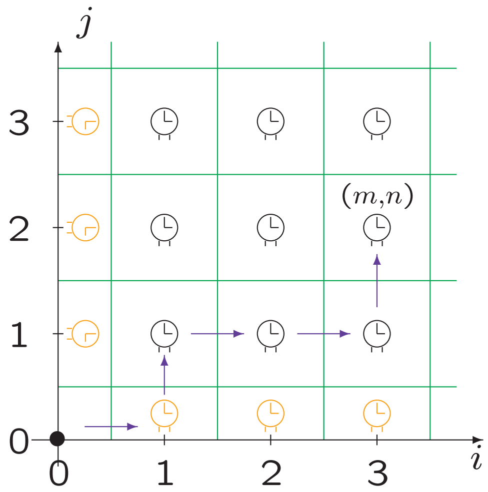
3. Results



On the characteristics

$$m := (1 - \varrho)^2 t \text{ and } n := \varrho^2 t,$$

Theorem:

$$0 < \liminf_{t\to\infty} \frac{\mathbf{Var}(G_{mn})}{t^{2/3}} \le \limsup_{t\to\infty} \frac{\mathbf{Var}(G_{mn})}{t^{2/3}} < \infty.$$

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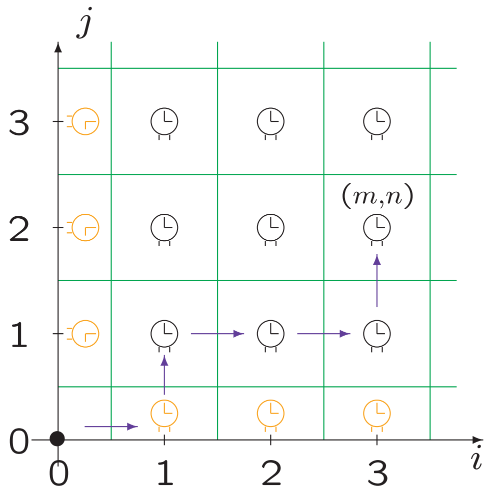

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Johansson (2000) identifies the limiting distribution of $\widetilde{h}_{Vt}(t)/t^{1/3}$ in terms of Tracy-Widom GUE distributions, when  and  $\sim$ Exponential(1) (rarefaction fan).

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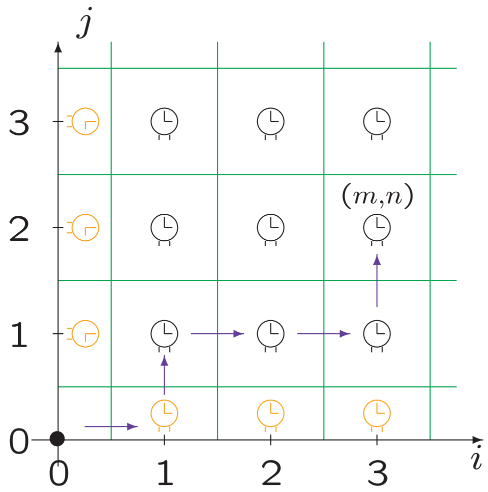

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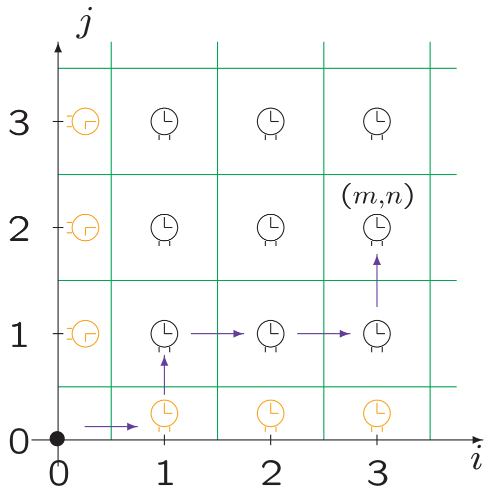

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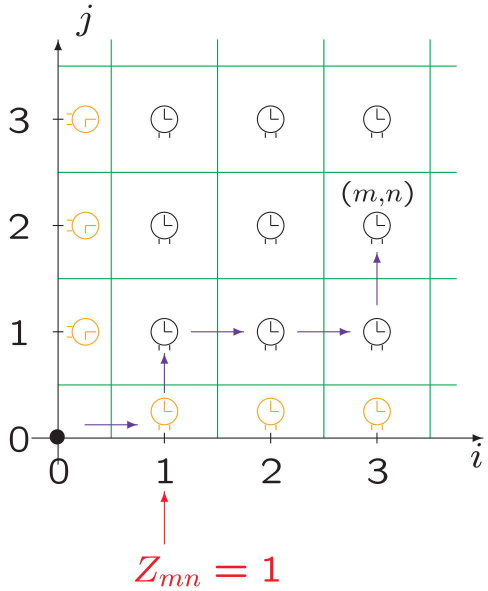
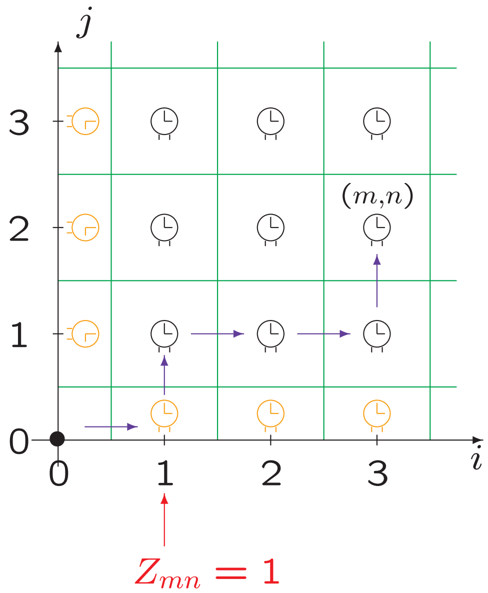
Their method: RSK correspondence, random matrices.



Z_{mn} is the exit point of the longest path to

$$(m,\ n) = ((1-\varrho)^2 t,\ \varrho^2 t).$$



Z_{mn} is the exit point of the longest path to

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Theorem:

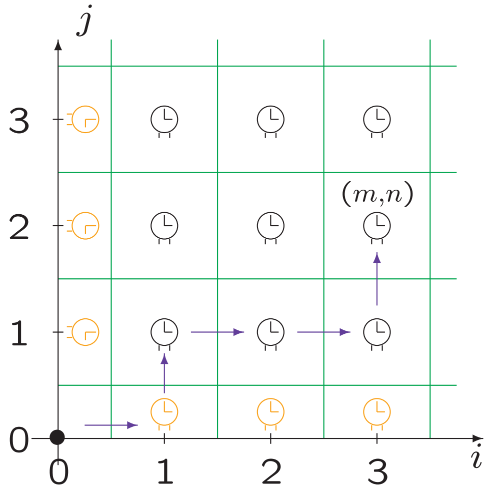
For all large t and all $a > 0$,

$$\mathbf{P}\{Z_{mn} \geq a t^{2/3}\} \leq C a^{-3}.$$

Given $\varepsilon > 0$, there is a $\delta > 0$ such that

$$\mathbf{P}\{1 \leq Z_{mn} \leq \delta t^{2/3}\} \leq \varepsilon$$

for all large t.



Equilibrium:

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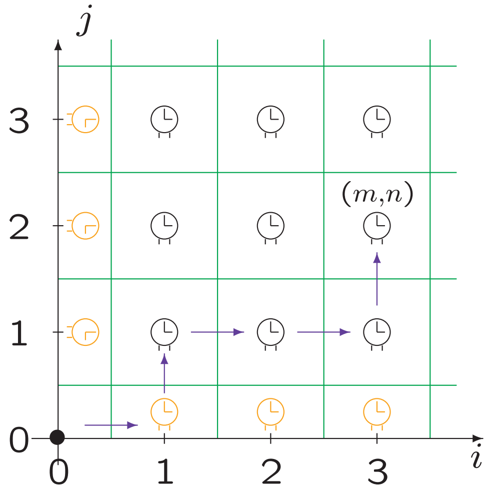

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Rarefaction fan:

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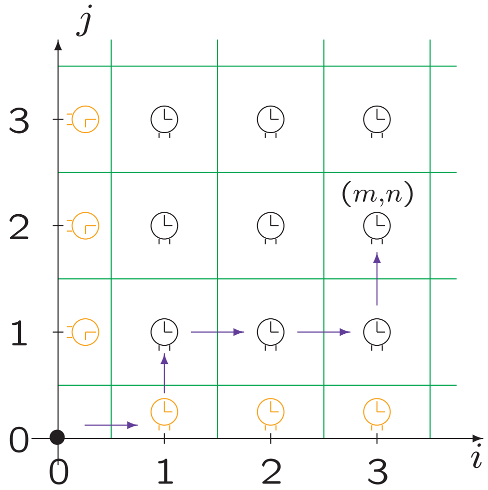

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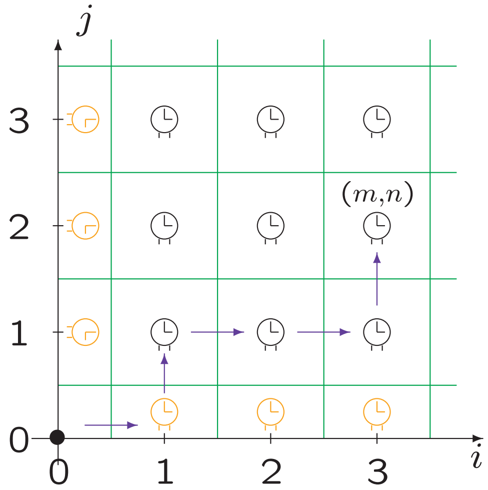
Theorem:

For $0 < \alpha < 1$ and all $t > 1$,

$$\mathbf{P}\{|G_{mn} - t| > at^{1/3}\} \leq Ca^{-3\alpha/2}.$$



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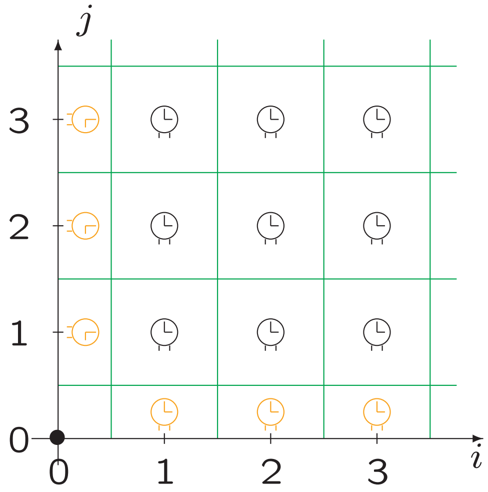
Also transversal $t^{2/3}$-deviations of the longest path.

Method:

Find a similar proof for Hammersley's process, and copy it.

E. Cator and P. Groeneboom 2005.

4. Last passage equilibrium



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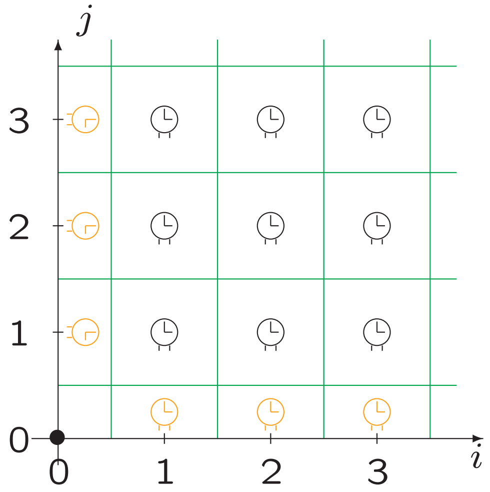

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G-increments:

$$I_{ij} := G_{ij} - G_{\{i-1\}j} \quad \text{for } i \geq 1,\ j \geq 0, \qquad \text{and}$$
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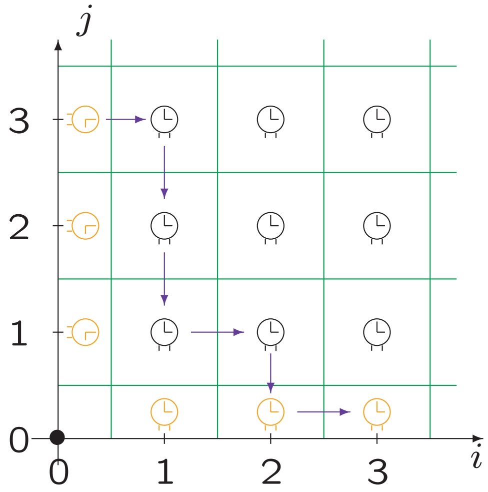

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$\rightsquigarrow$ Any fixed southeast path meets *independent* increments

$$I_{ij} \sim \text{Exponential}(1-\varrho) \quad \text{and}$$
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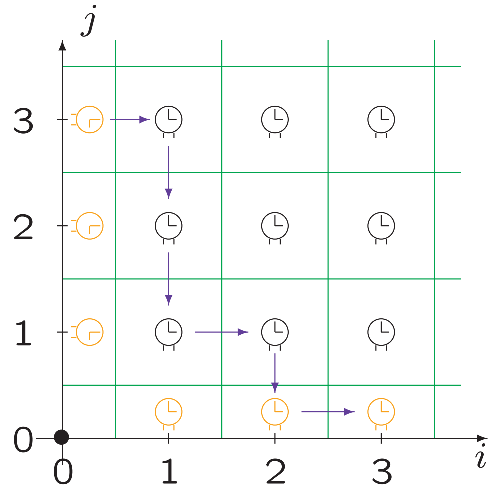

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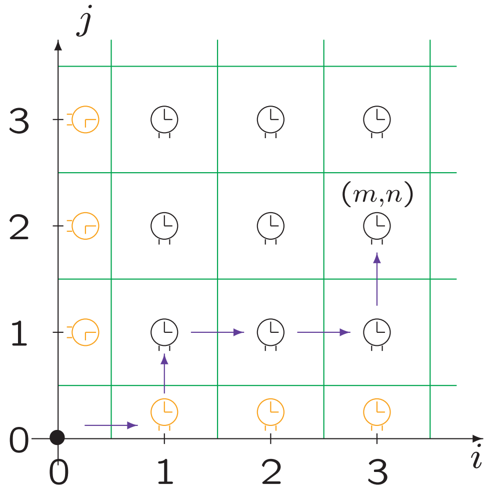
Of course, this doesn't help directly with G_{mn}.

5. Upper bound



G^ϱ: weight collected by the longest path.

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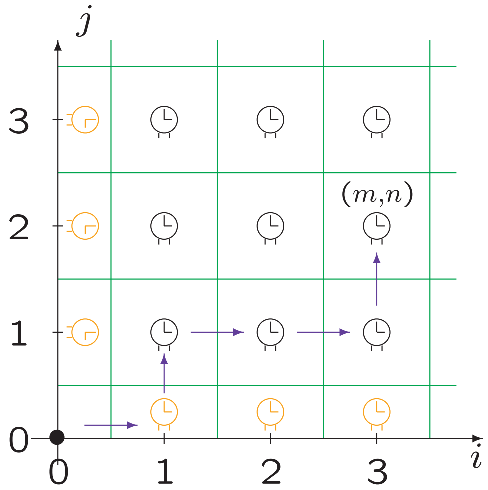

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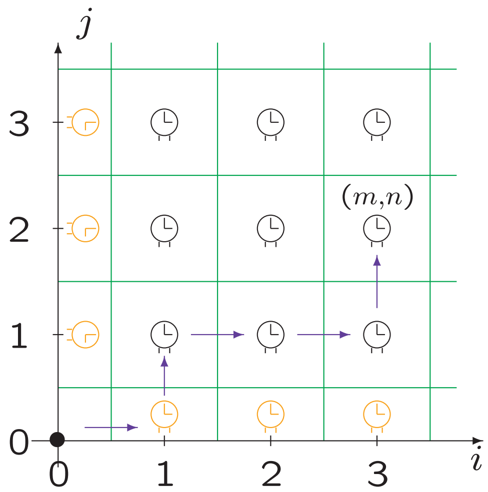

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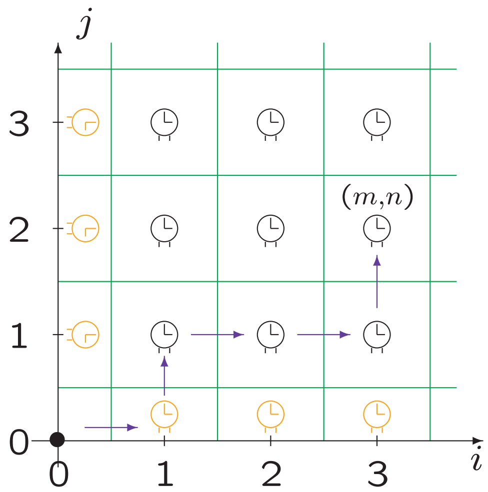

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Step 1:

$$U_z^{\lambda} + A_z \leq G^{\lambda}$$

for any z, any $0 < \lambda < 1$.

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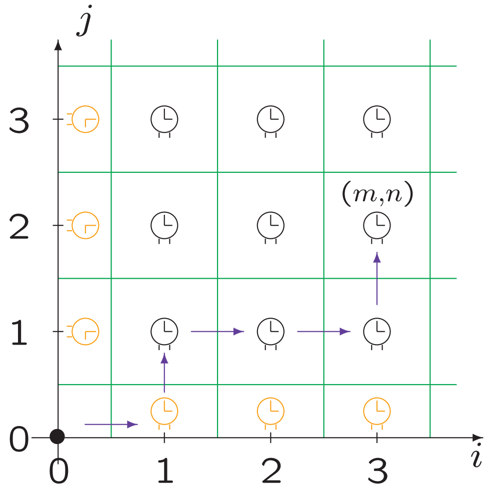

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$$\mathbf{P}\{Z^\varrho > u\} = \mathbf{P}\{\exists z > u \, : \, U_z^\varrho + A_z(t) = G^\varrho\}$$

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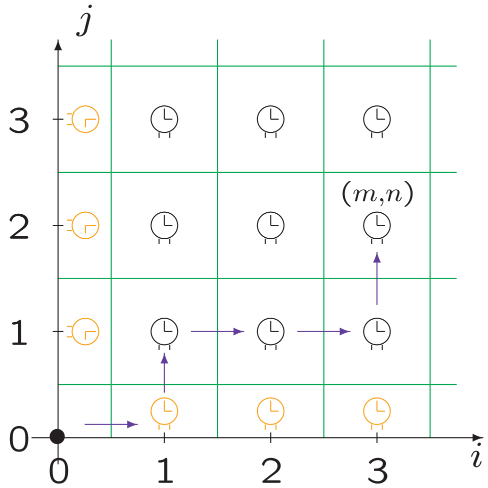

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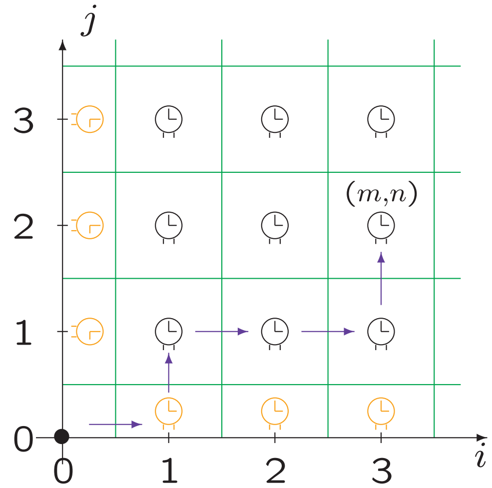

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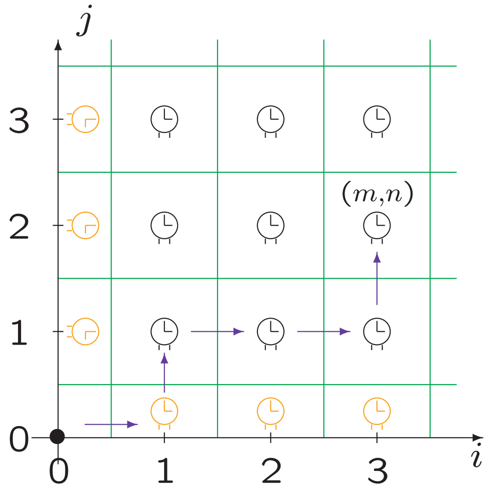

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\end{aligned}$$

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Optimize λ so that $\mathbf{E}(U_u^\lambda - G^\lambda)$ be maximal. (The equilibrium makes it possible to compute the expectation.) This makes the estimate sharp.

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Step 5:

A large deviation estimate connects $\mathbf{P}\{Z^\varrho > y\}$ and $\mathbf{P}\{U^\varrho_{Z^\varrho+} > y\}$.

$$\rightsquigarrow \mathbf{P}\{U^\varrho_{Z+} > y\} \le C\Big(\frac{t^2}{y^4} \cdot \mathbf{E}(U^\varrho_{Z^\varrho+}) + \frac{t^2}{y^3}\Big)$$

$$\mathbf{P}\{Z^{\varrho} > u\} \le \mathbf{P}\{U_u^{\lambda} - U_u^{\varrho} \le G^{\lambda} - G^{\varrho}\}.$$

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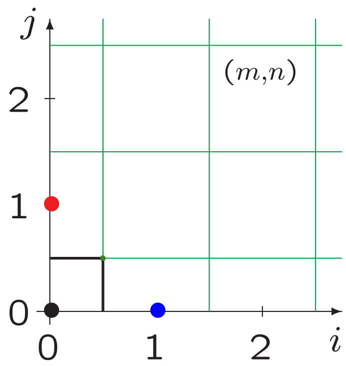
Conclude

$$\limsup_{t\to\infty} \frac{\mathbf{E}(U^{\varrho}_{Z^{\varrho}+})}{t^{2/3}} < \infty, \qquad \limsup_{t\to\infty} \frac{\mathbf{Var}(G^{\varrho})}{t^{2/3}} < \infty.$$

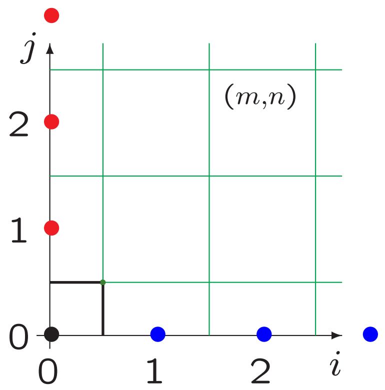
6. The competition interface



Ferrari, Martin, Pimentel (2005)

Which squares are infected via $(1,0)$ and via $(0,1)$?

6. The competition interface



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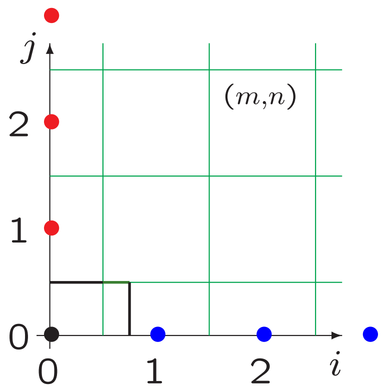

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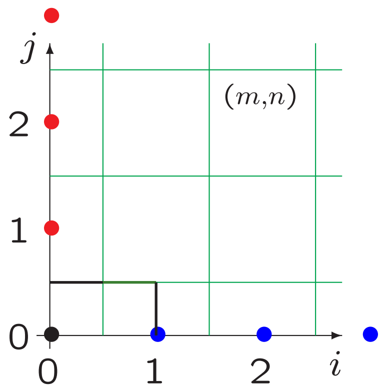

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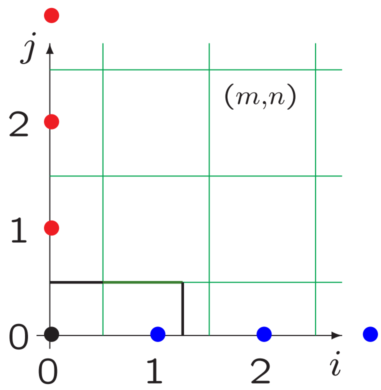

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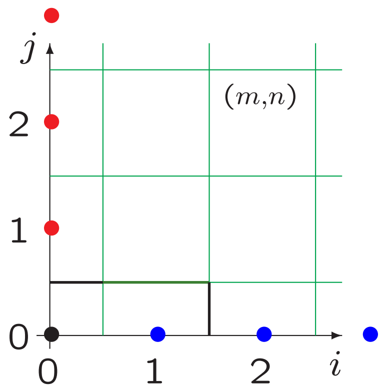

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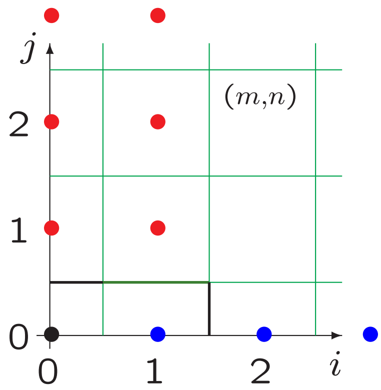

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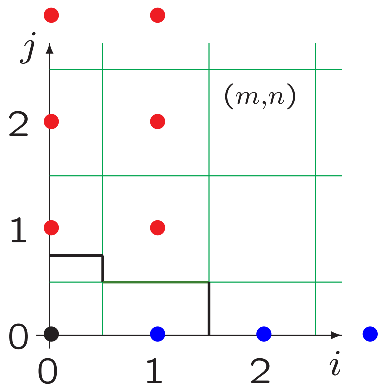

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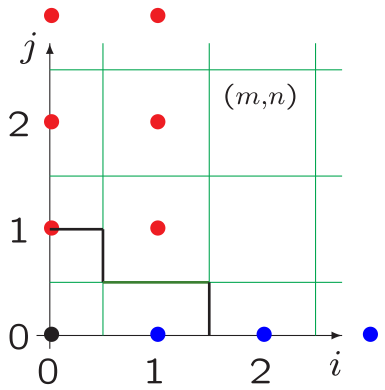

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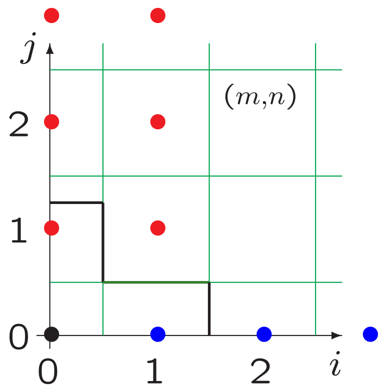

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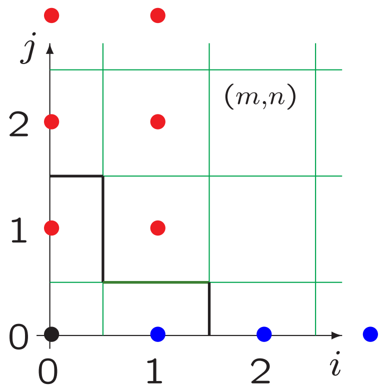

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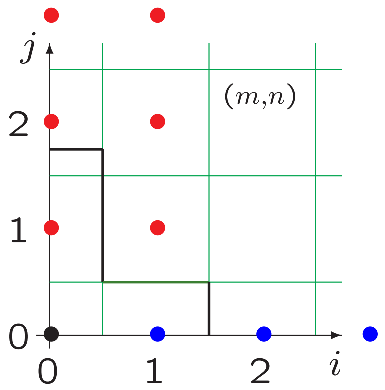

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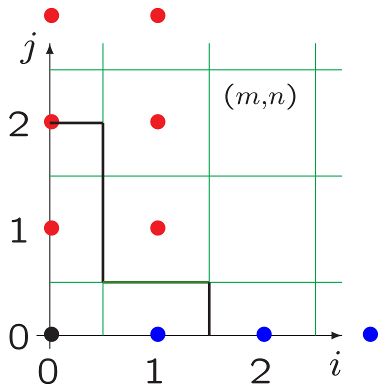

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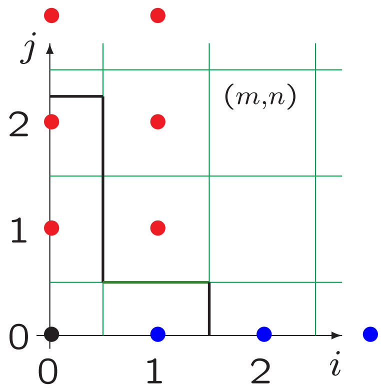

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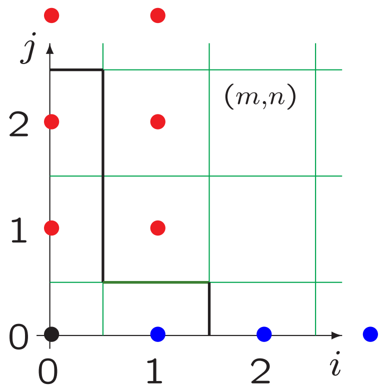

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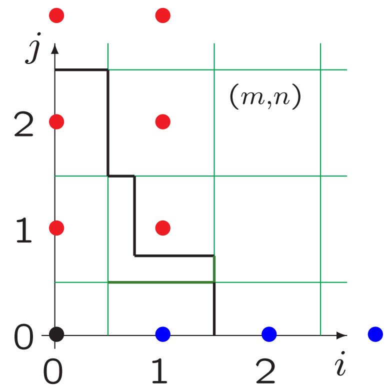

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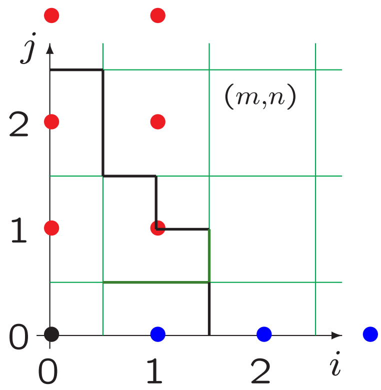

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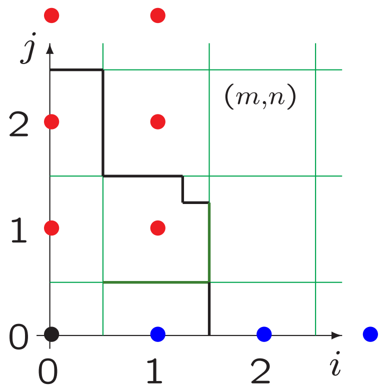

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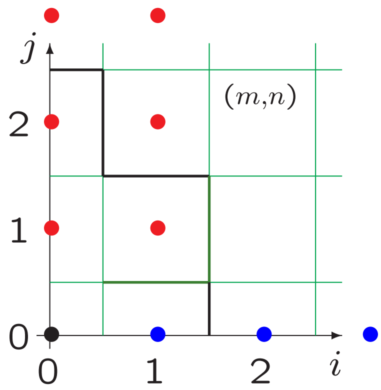

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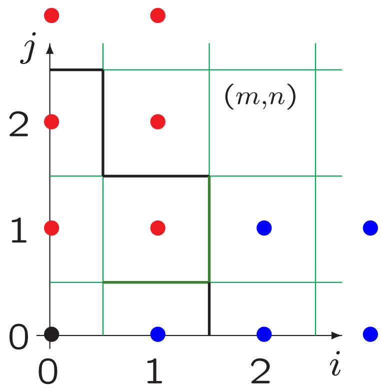

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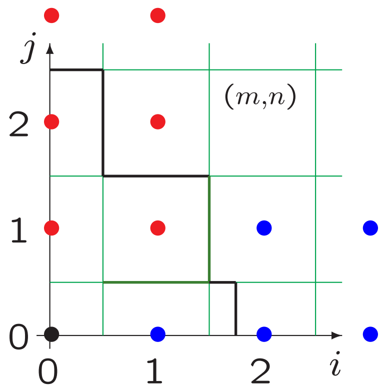

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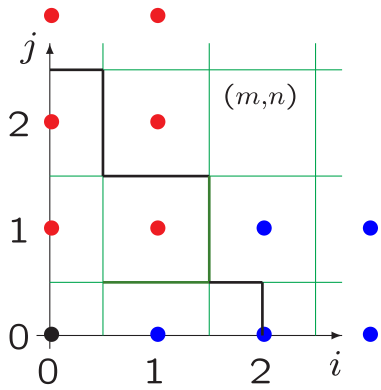

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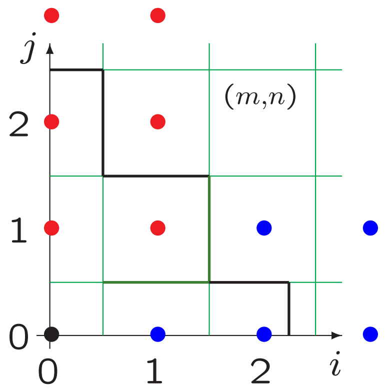

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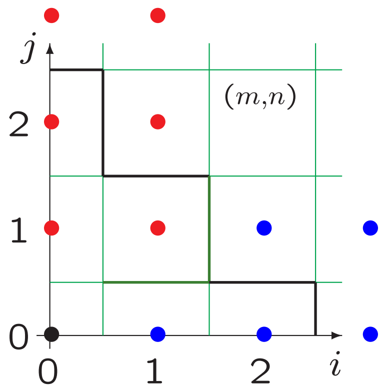

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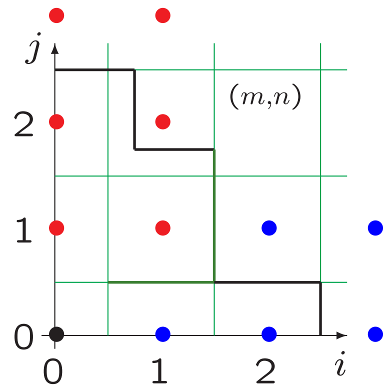

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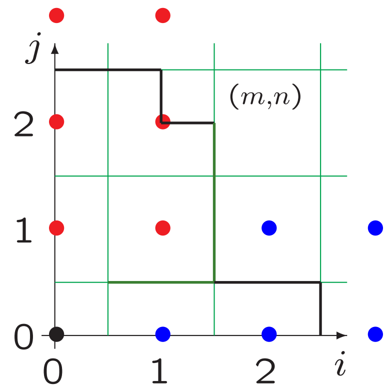

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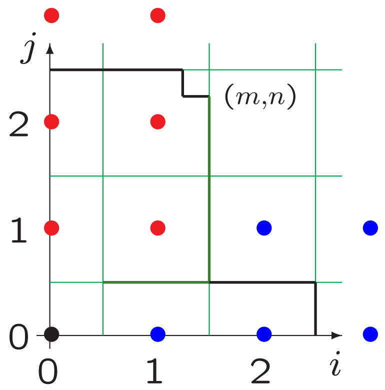

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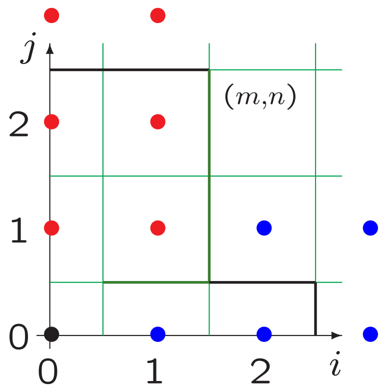

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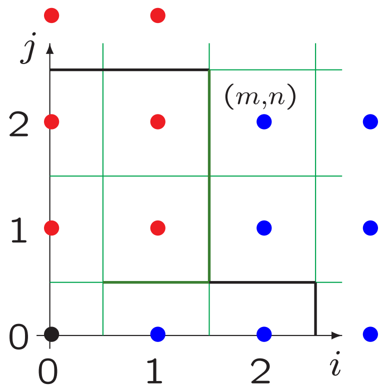

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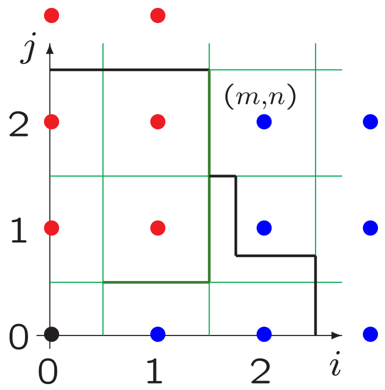

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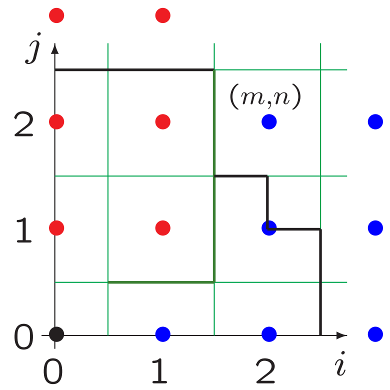

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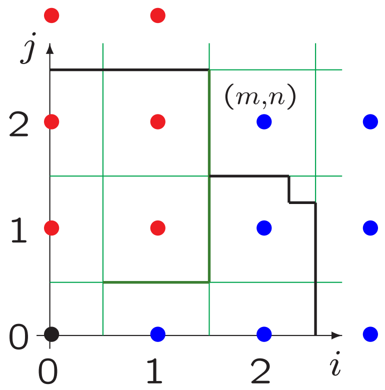

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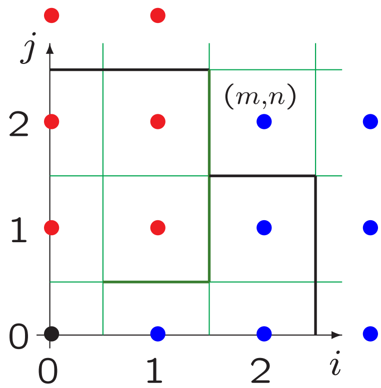

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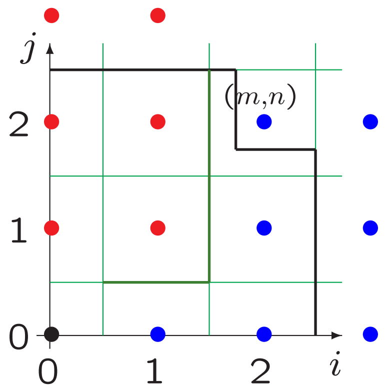

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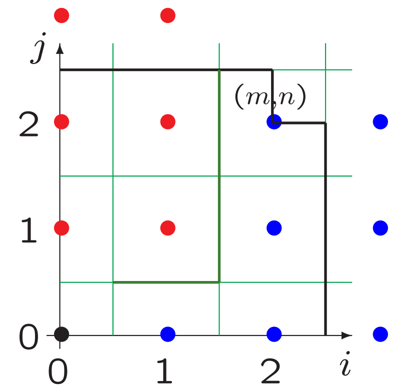

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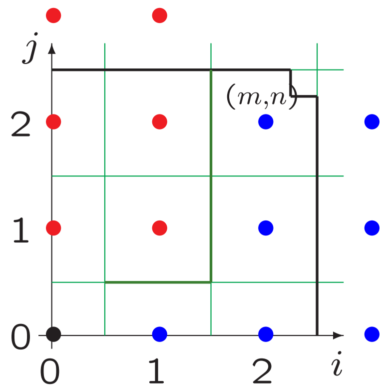

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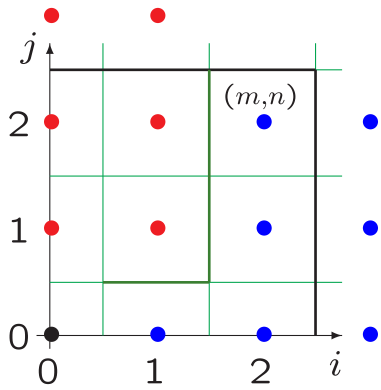

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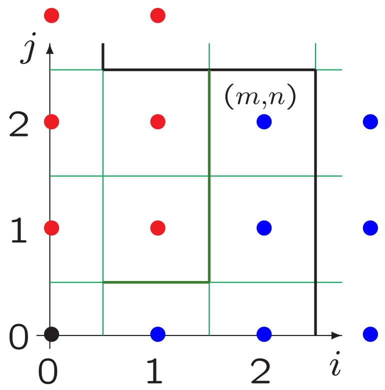

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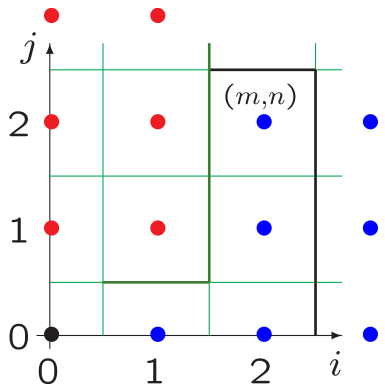

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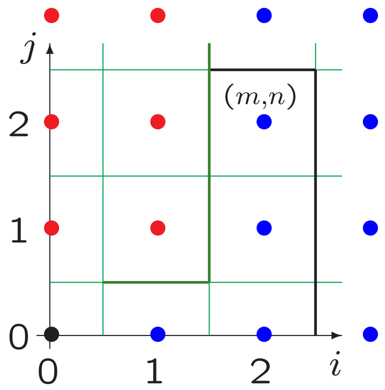

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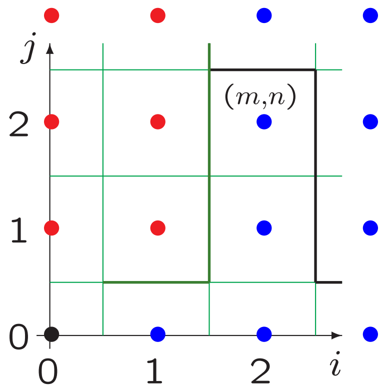

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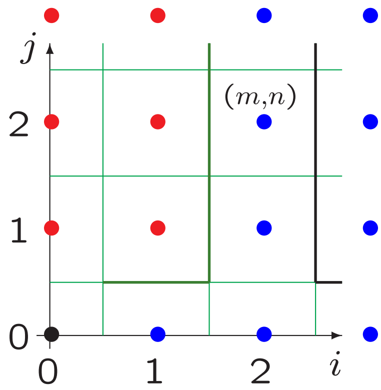

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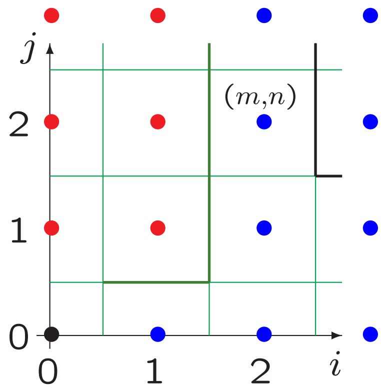

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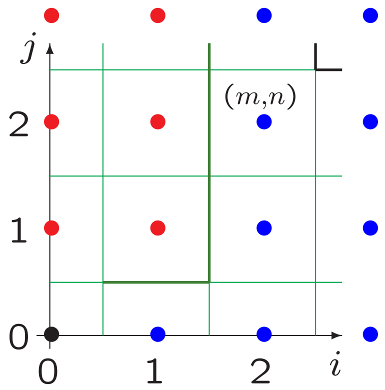

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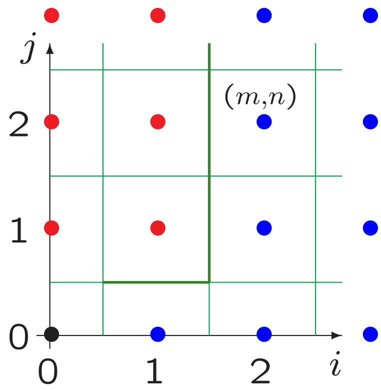

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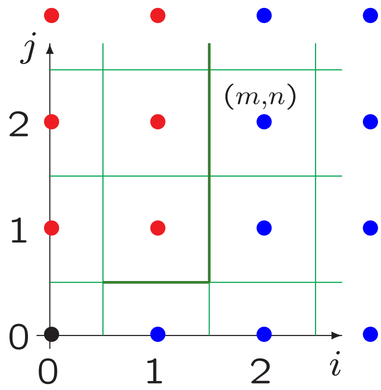

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The competition interface follows the same rules as the *second class particle* of simple exclusion.

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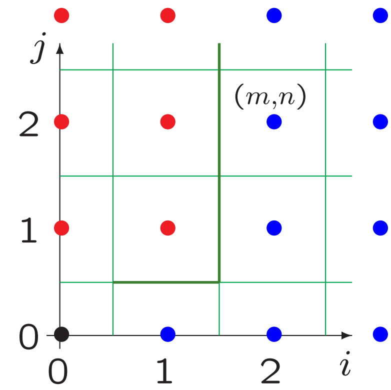

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If it passes left of (m, n), then G_{mn} is not sensitive to decreasing the weights on the j-axis.

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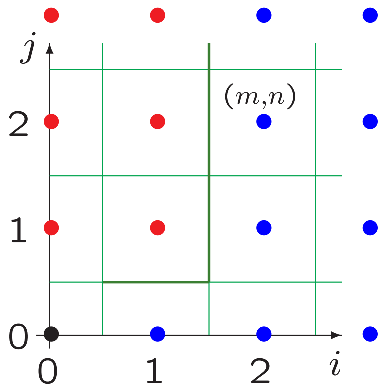

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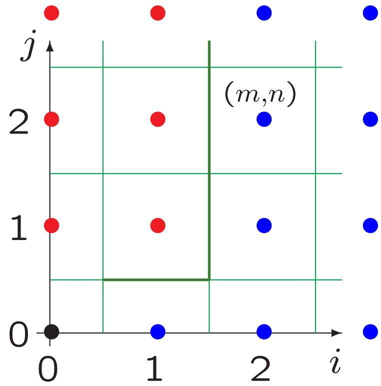
$\rightsquigarrow$ One bounds Z-probabilities differently in these cases.

7. Time-reversal and the lower bound



⇝ Z-probabilities are connected to competition interface-probabilities.

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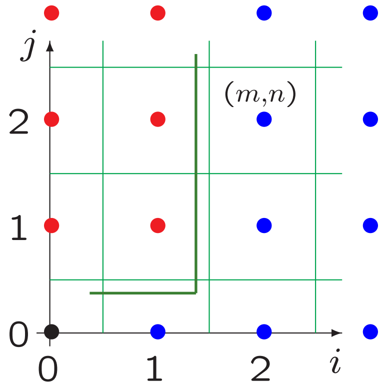

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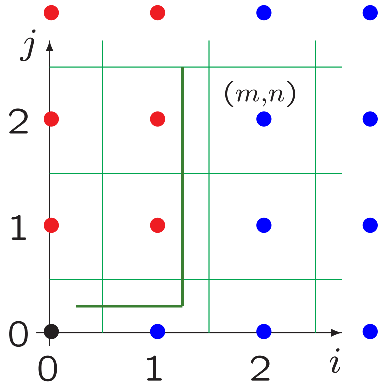

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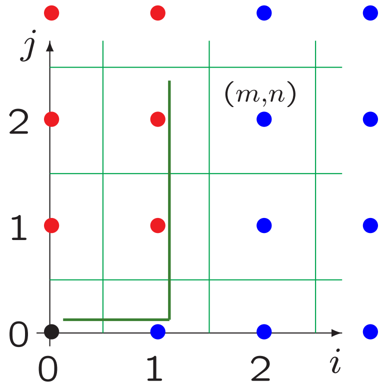

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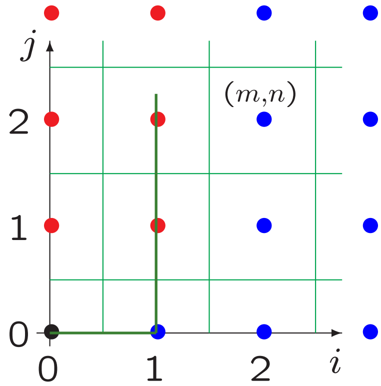

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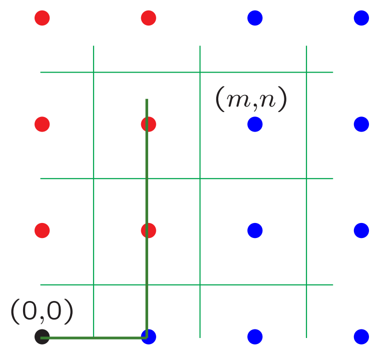

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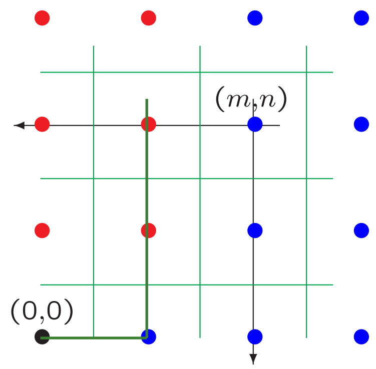

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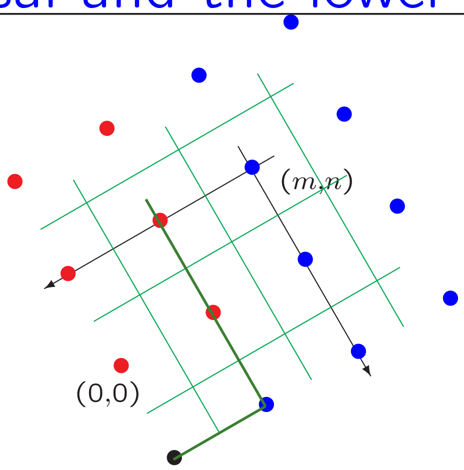

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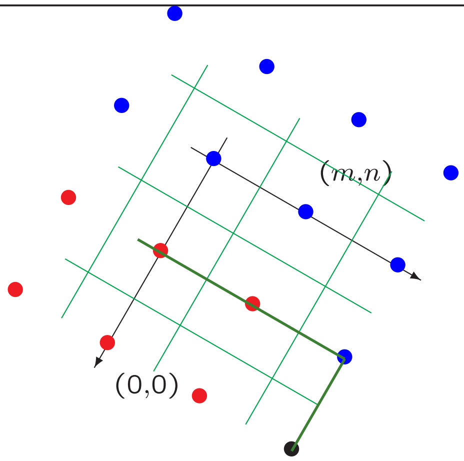

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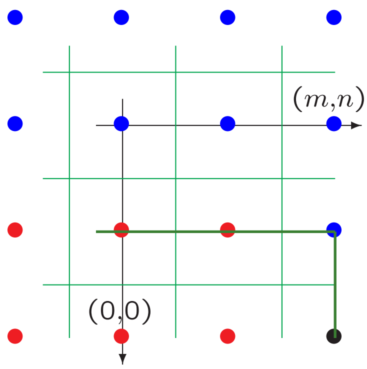

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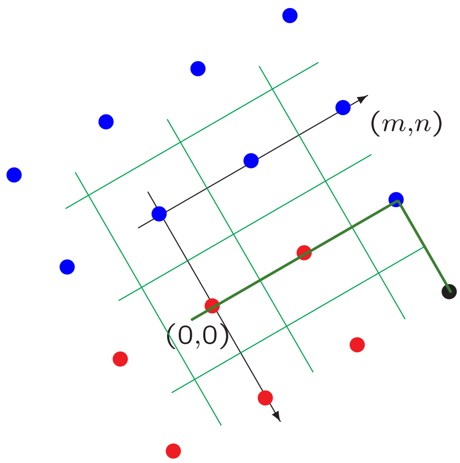

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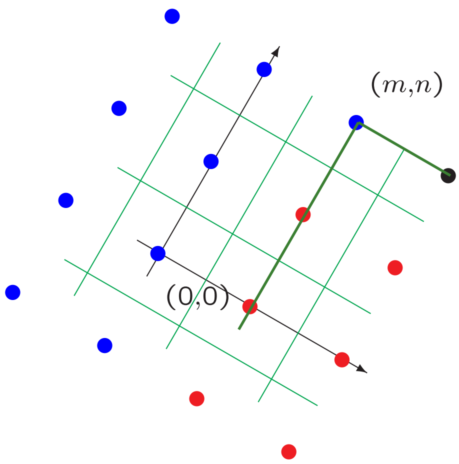

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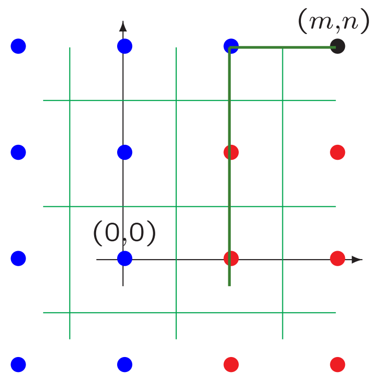

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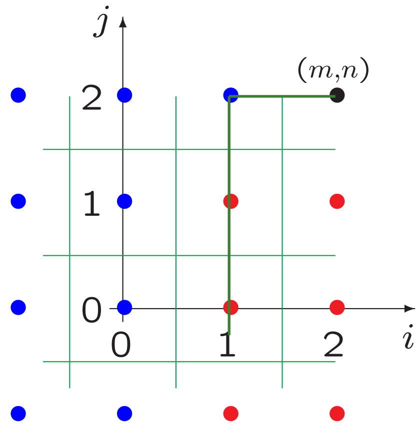

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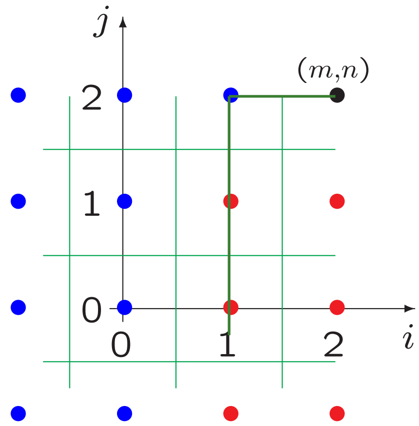

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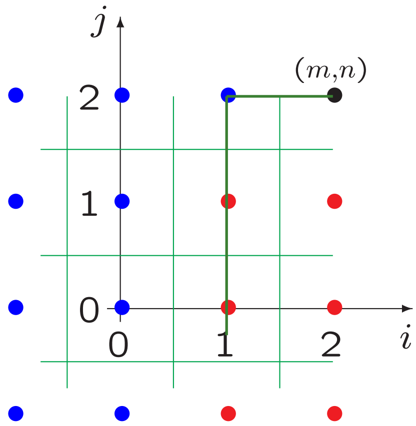

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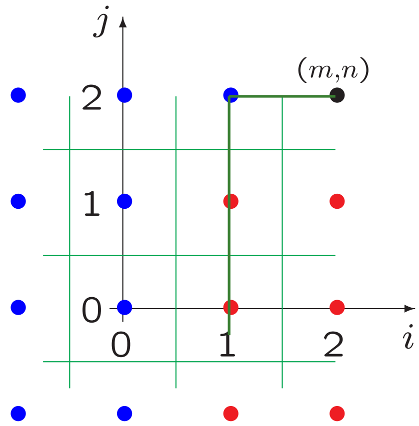

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Conclude

$$\liminf_{t\to\infty} \frac{\mathbf{E}(U^{\varrho}_{Z^{\varrho}+})}{t^{2/3}} > 0, \qquad \liminf_{t\to\infty} \frac{\mathbf{Var}(G^{\varrho})}{t^{2/3}} > 0.$$

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→ Generalize even more: drop the last-passage picture. These methods have the potential to extend to other particle systems directly (zero range, bricklayers', ...?).

Thank you.