Construction of the zero range process and a deposition model with superlinear growth rates

Márton Balázs (UW-Madison)

Joint work with
Firas Rassoul-Agha (University of Utah),
Timo Seppäläinen (UW-Madison)
and
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Markov Processes and Related Topics

July 13, 2006

In Honor of Tom Kurtz on His 65th Birthday

1. The zero range process and the bricklayers' process

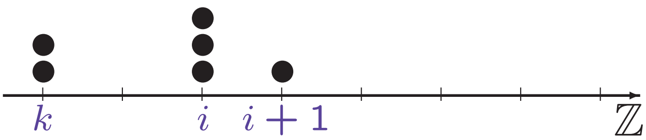
2. Construction materials

3. Transferring the estimates

4. Results

1. The zero range process (ZR):

$\omega_i \in \mathbb{Z}^+$



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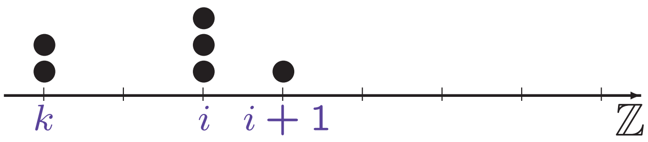

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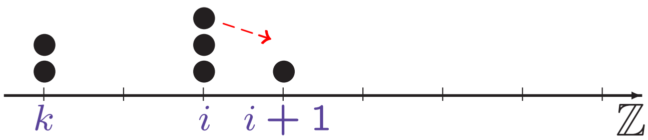

With rate $r(\omega_i)$, $\qquad \begin{pmatrix} \omega_i \\ \omega_{i+1} \end{pmatrix} \longrightarrow \begin{pmatrix} \omega_i - 1 \\ \omega_{i+1} + 1 \end{pmatrix}.$

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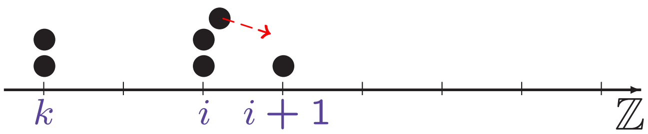

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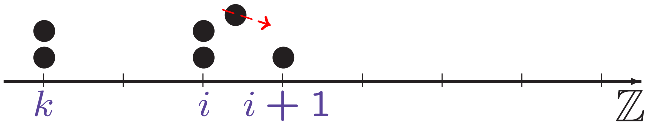

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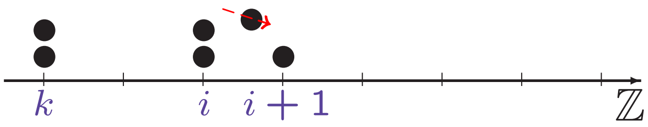

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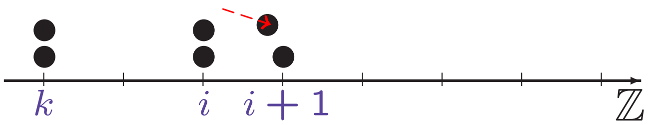

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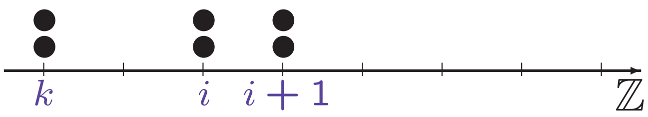

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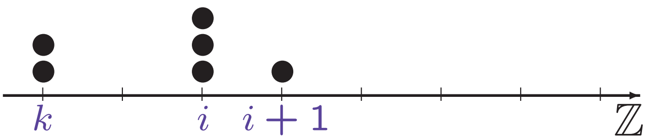

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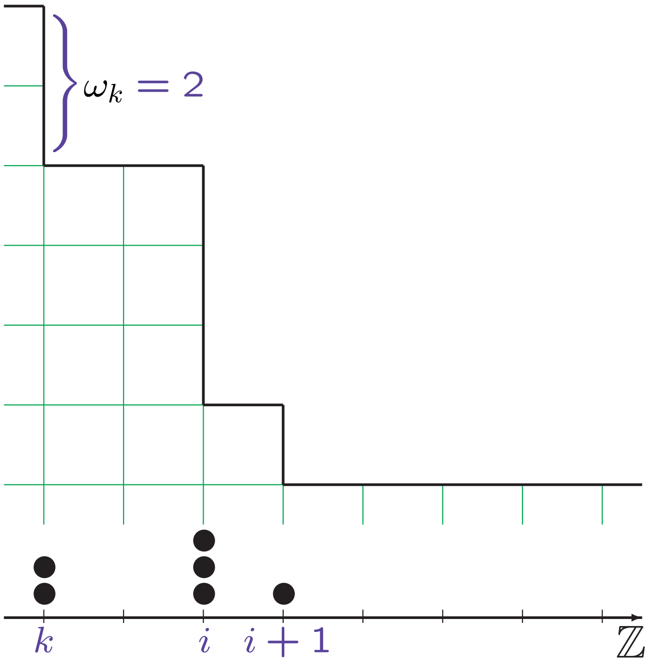

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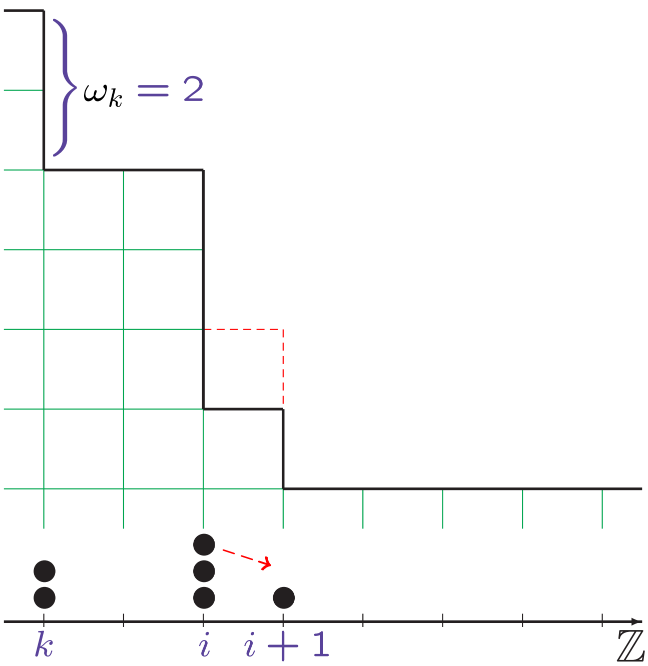

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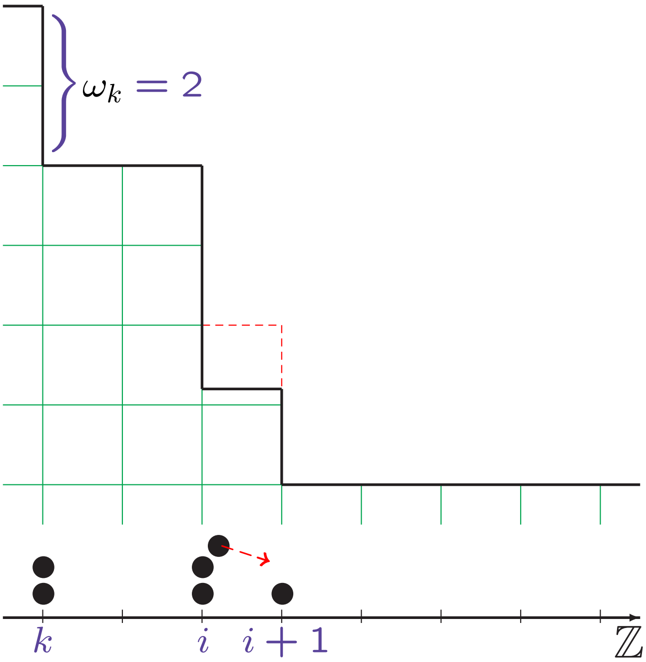

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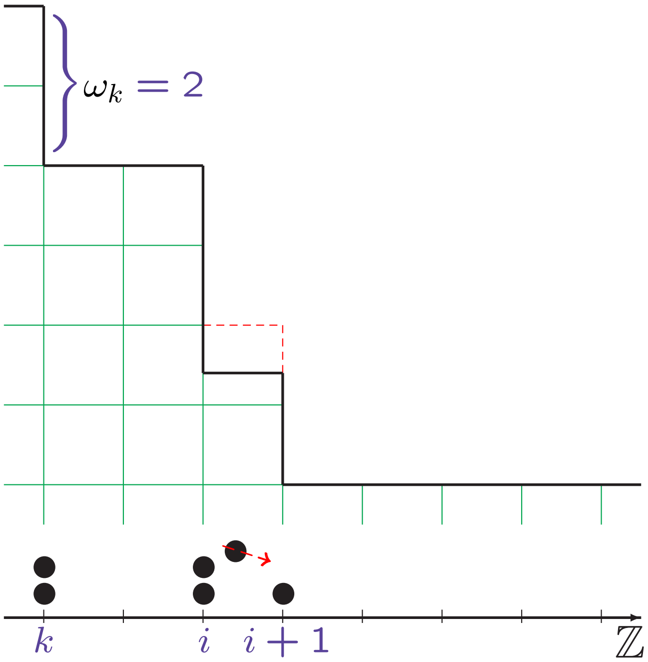

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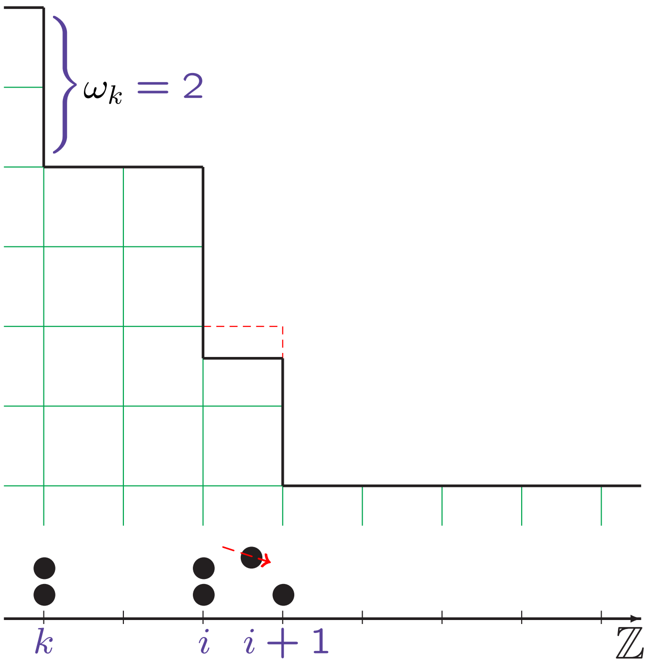

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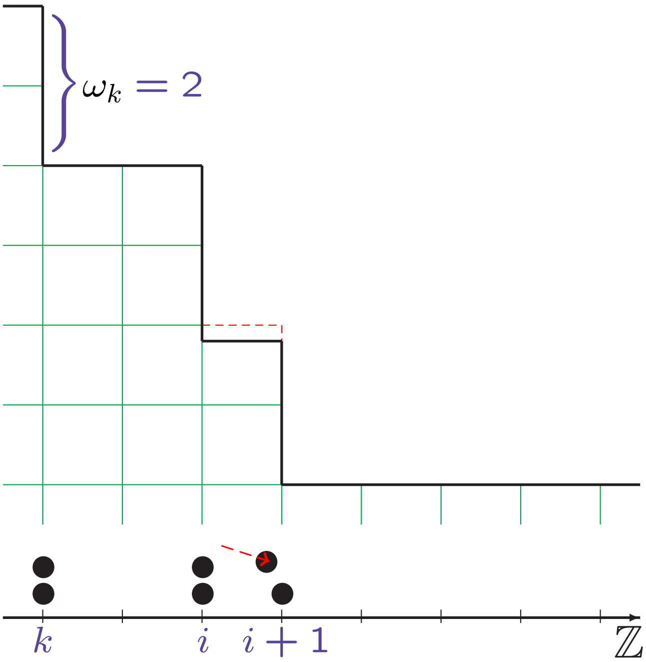

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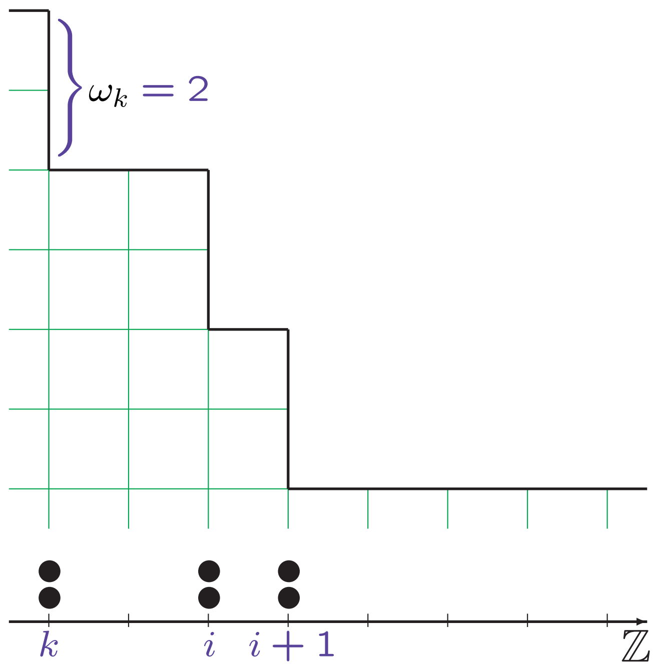

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ω_i = negative discrete gradient

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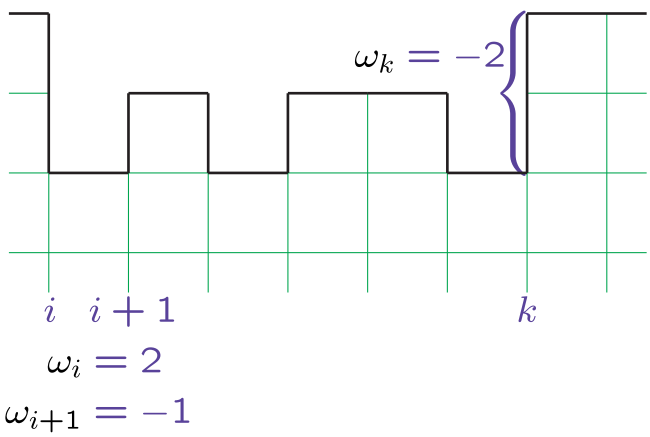

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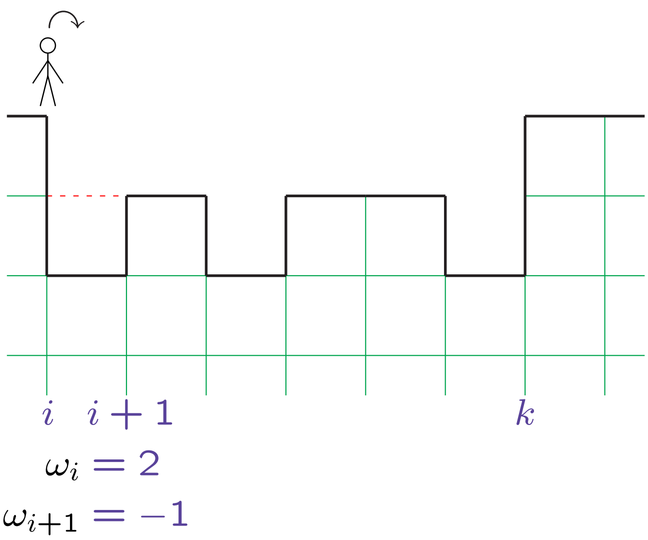

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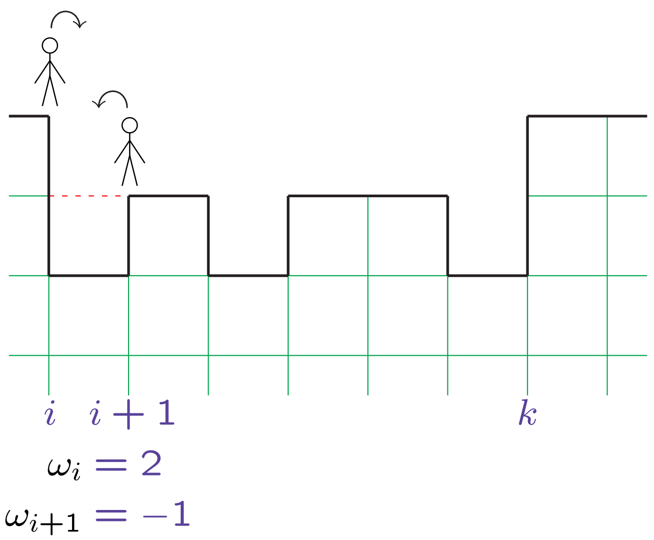

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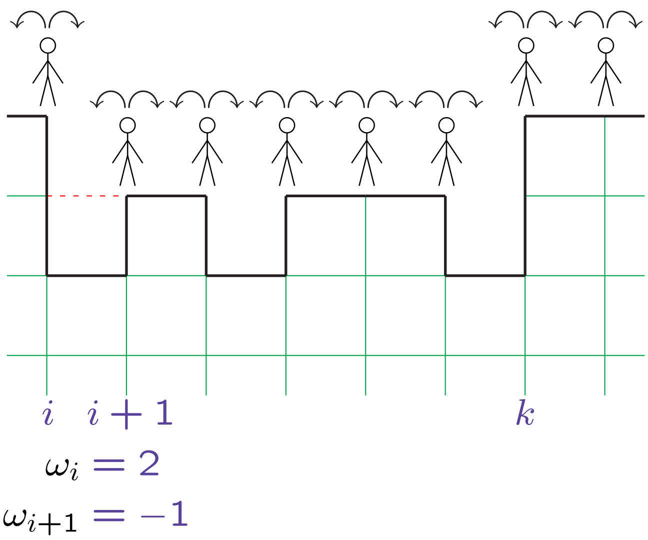

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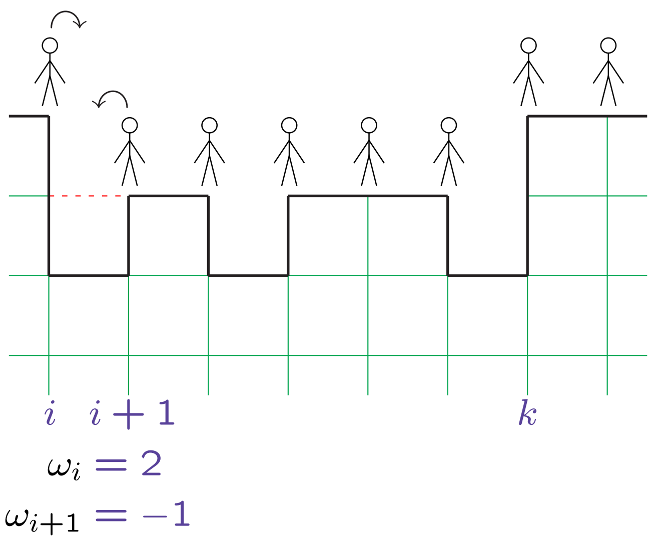

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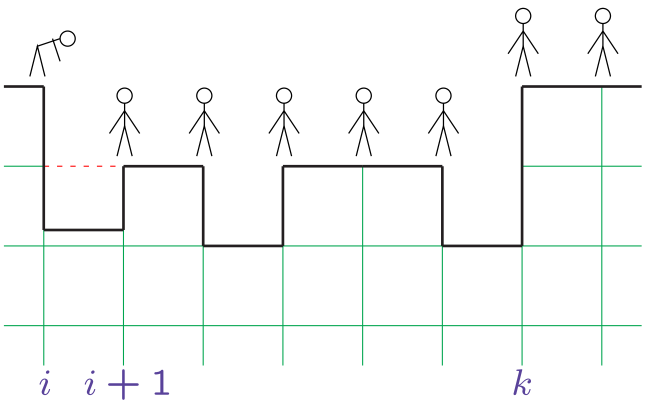

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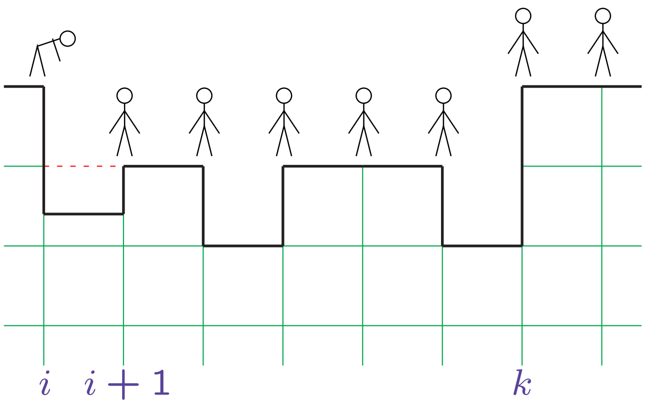

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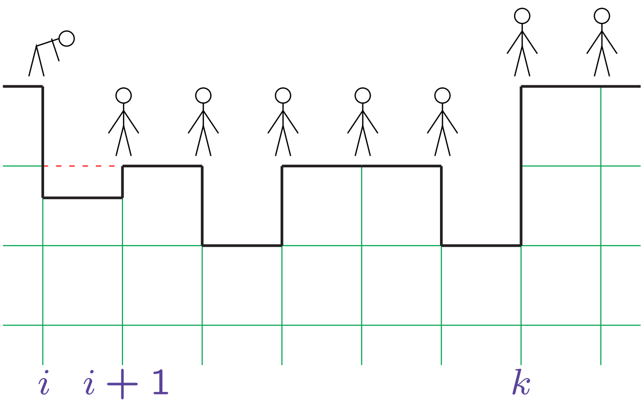

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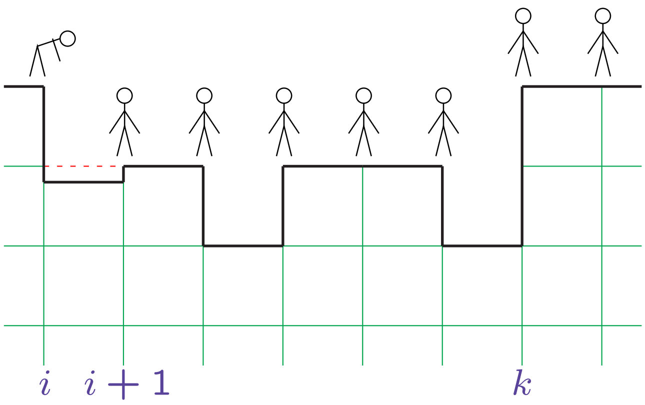

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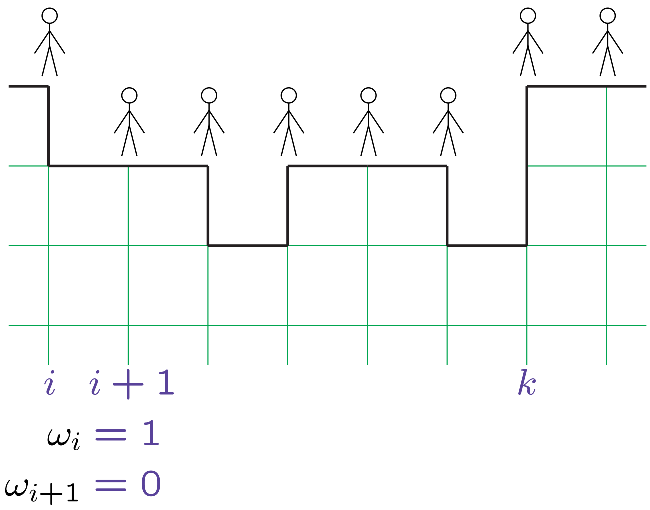

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Estimates used by Andjel do not work.

2. Construction materials

Equilibrium in finite volume

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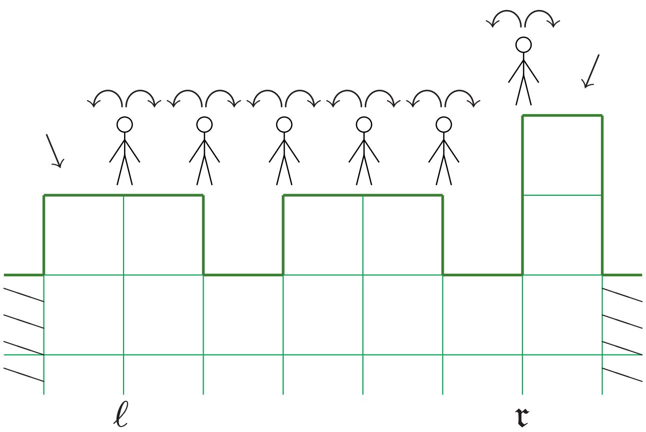

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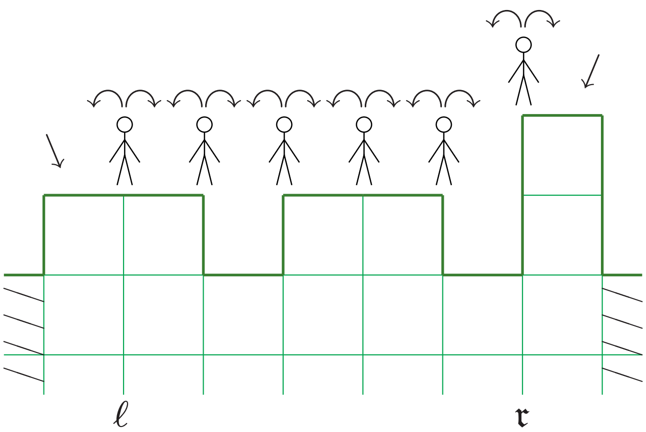

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The monotone process

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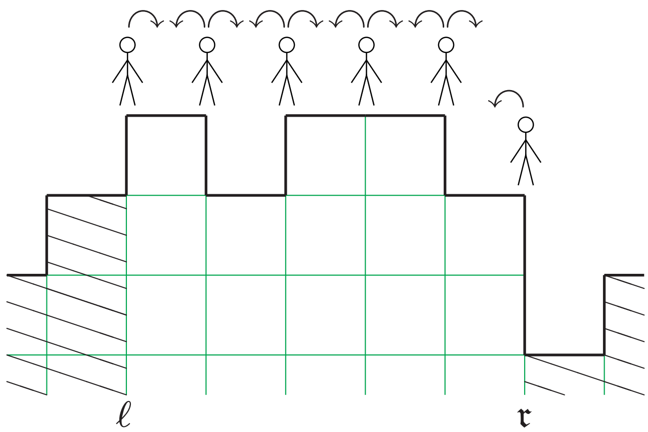

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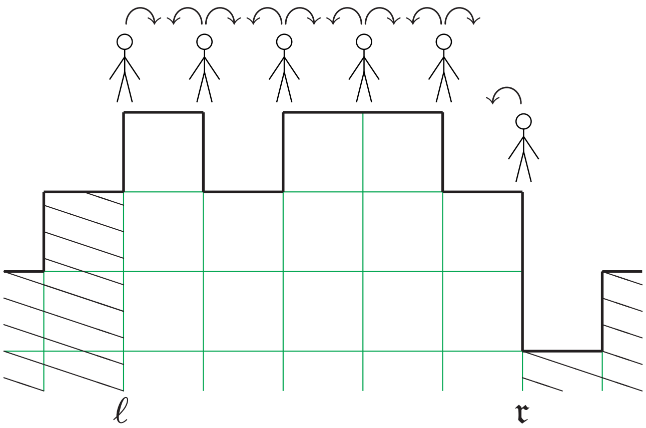

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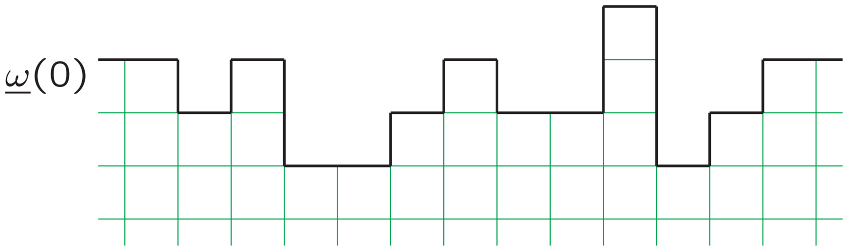
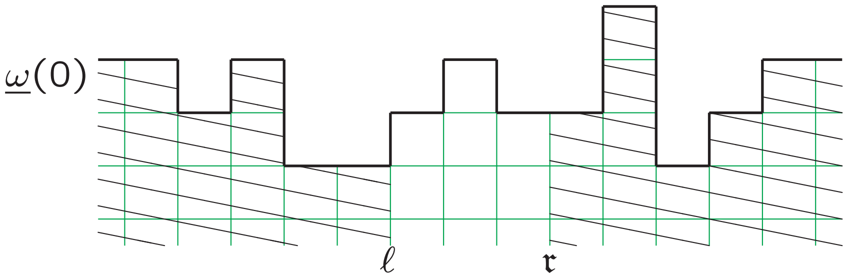
$\leadsto$ This process is far from equilibrium!



- Fix a state $\underline{\omega}(0) \in \widetilde{\Omega}$.



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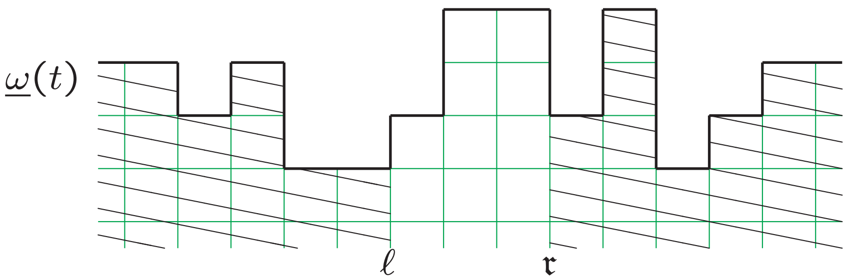

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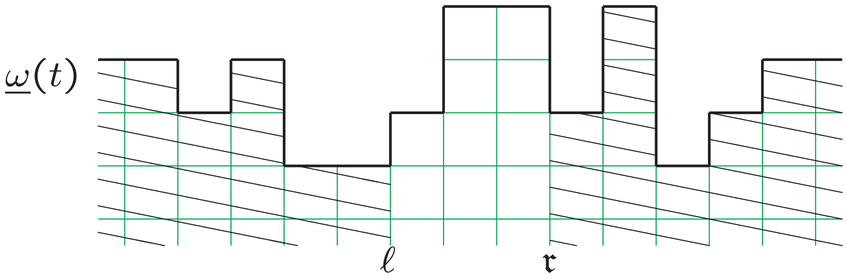

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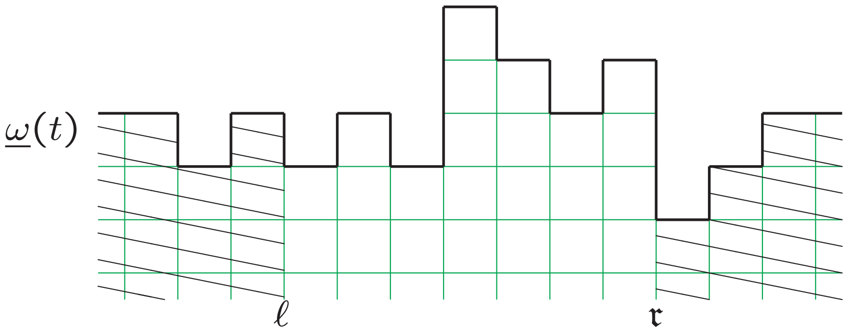

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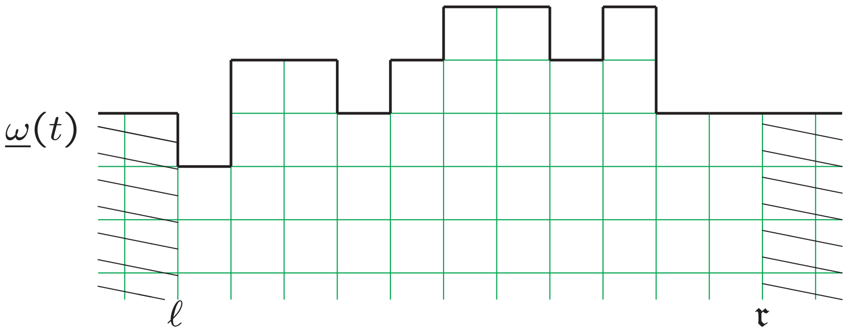

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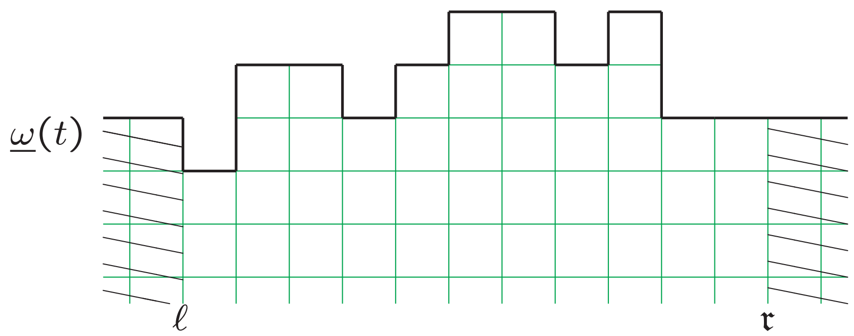

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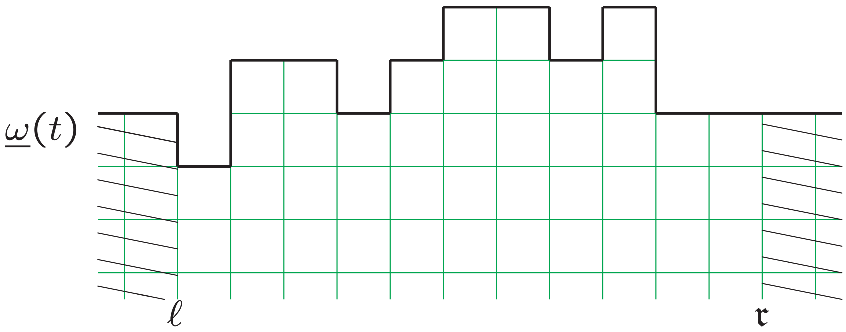

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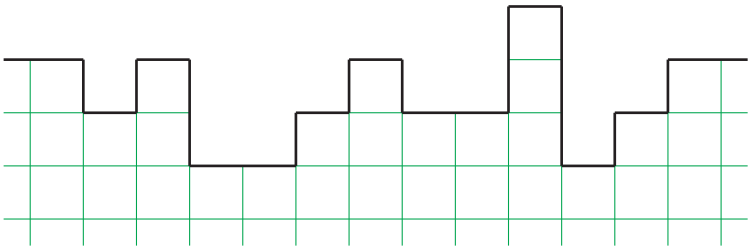
$\underline{\omega}(0)$



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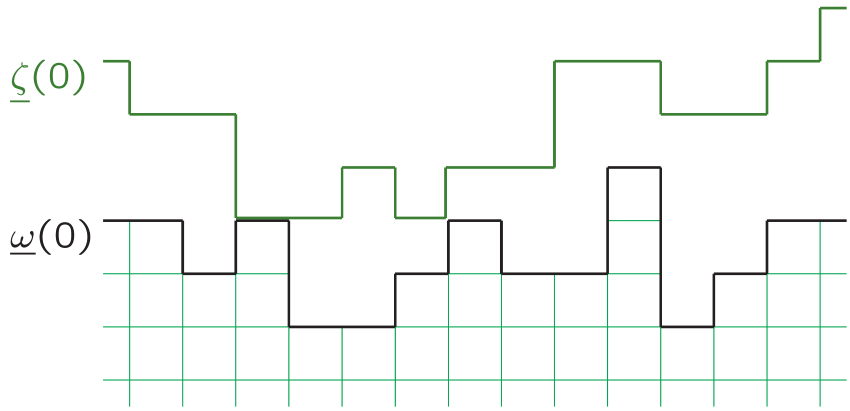

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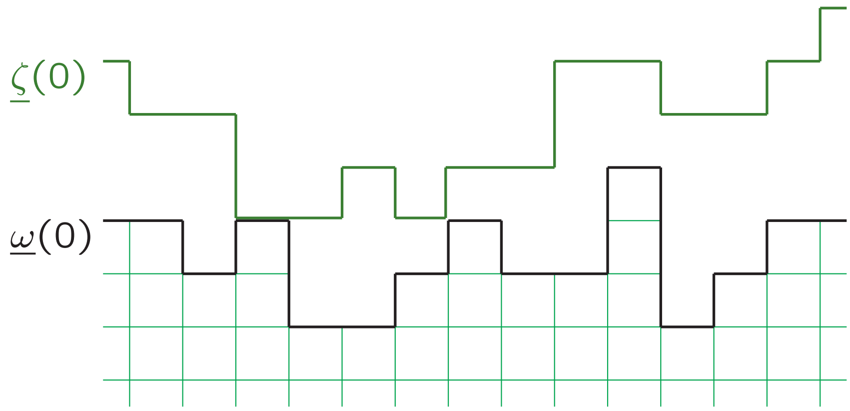

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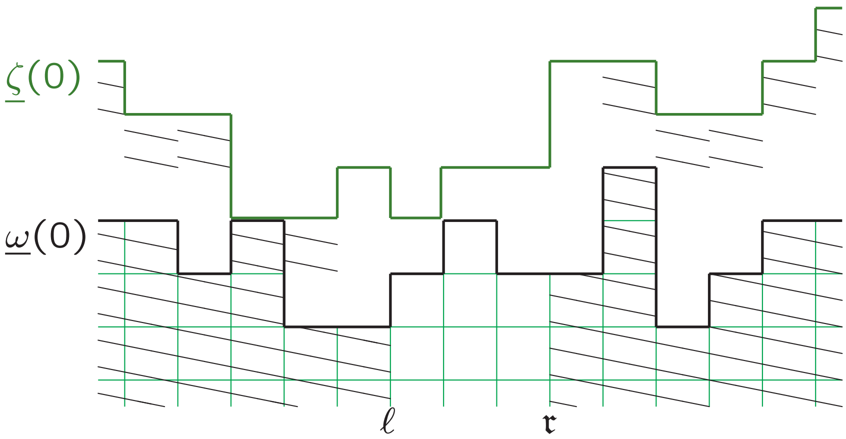

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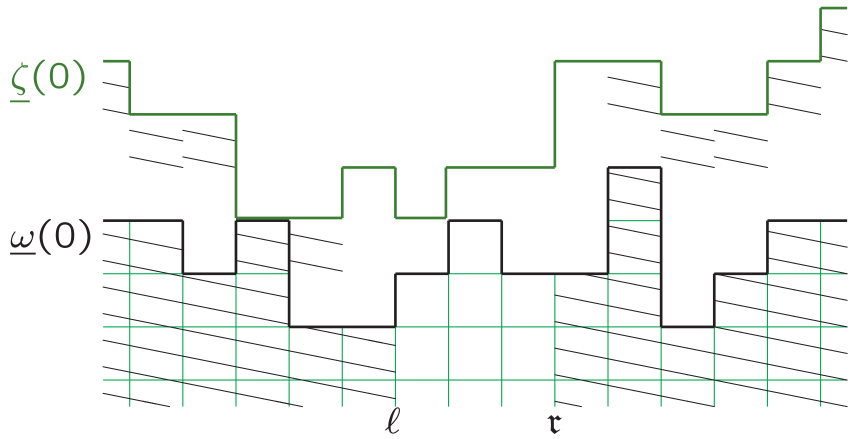

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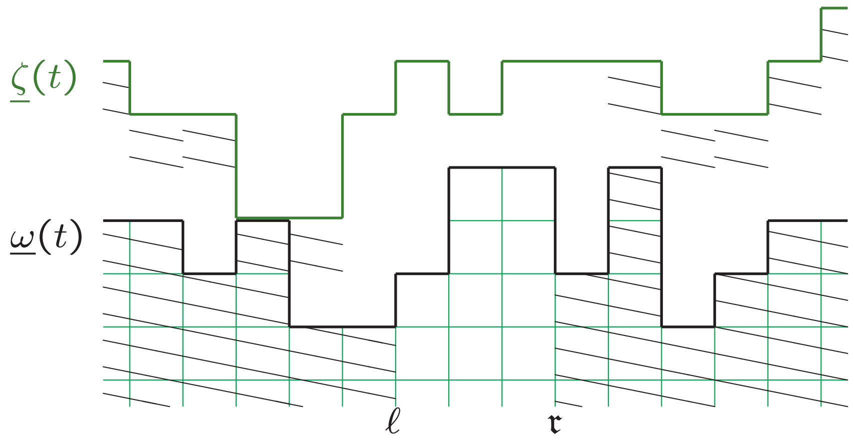

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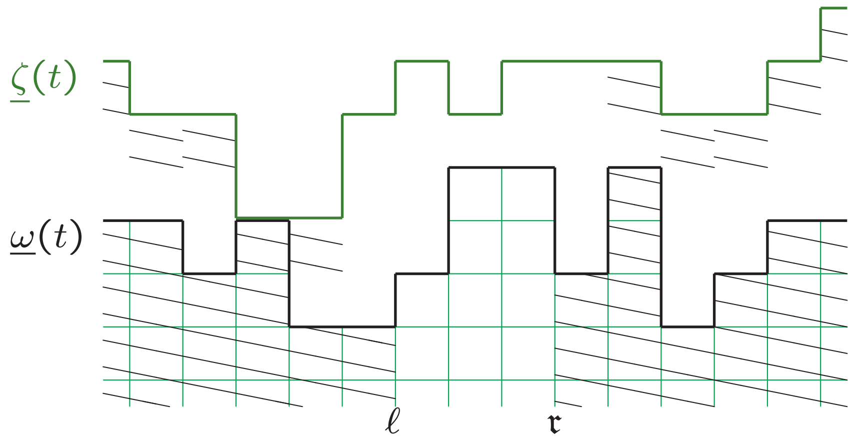

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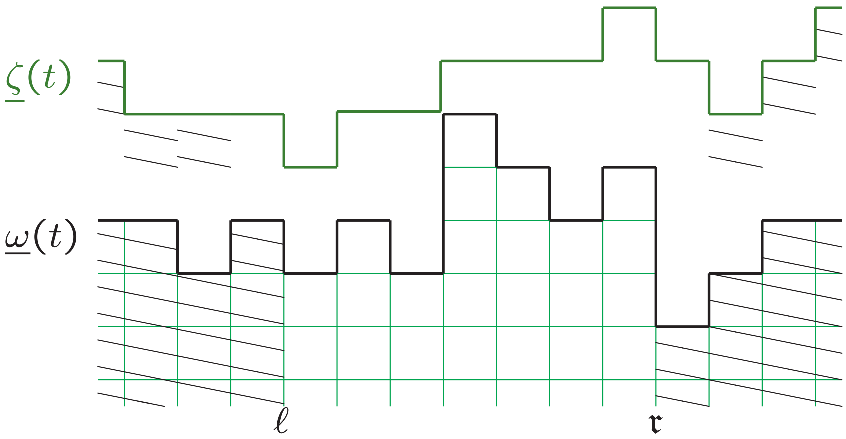

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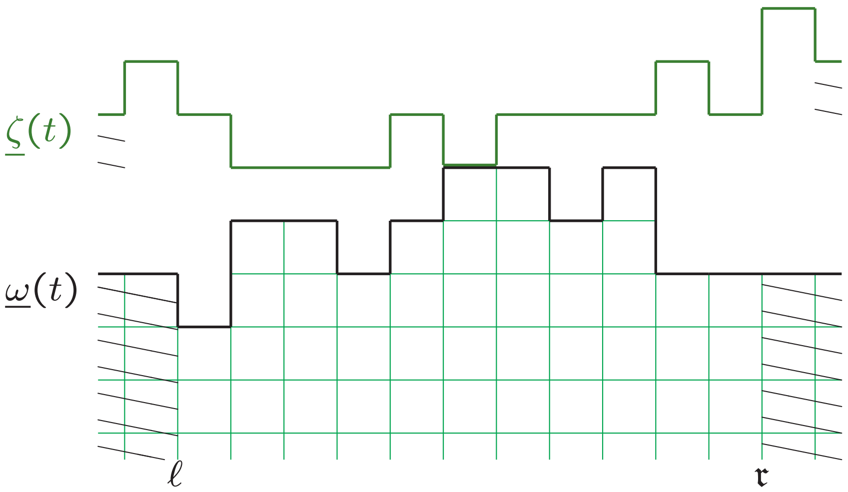

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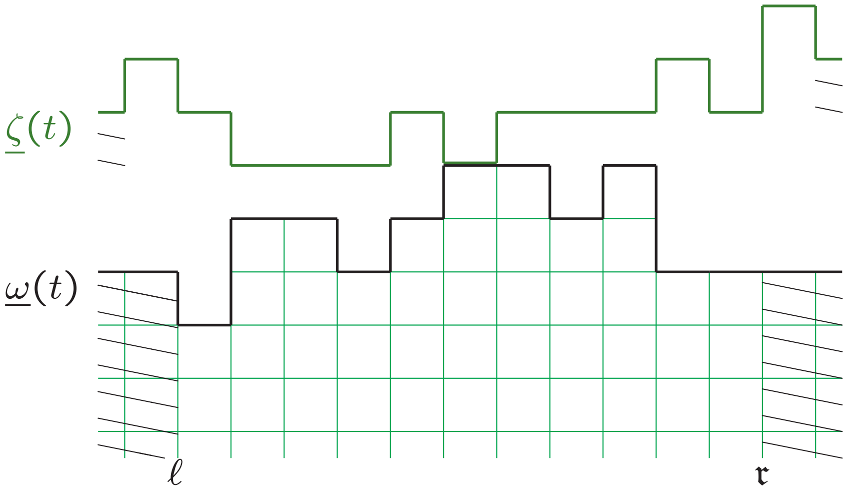

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$\rightsquigarrow$ We have an $S(t)$ semigroup on bounded measurable functions.

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$$S(t)\varphi(\underline{\omega}) = \varphi(\underline{\omega}) + \int_0^t S(s)L\varphi(\underline{\omega})\,\mathrm{d}s$$

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$$\frac{\mathrm{d}}{\mathrm{d}t} S(t)\varphi(\underline{\omega})\Big|_{t=0} = L\varphi(\underline{\omega})$$

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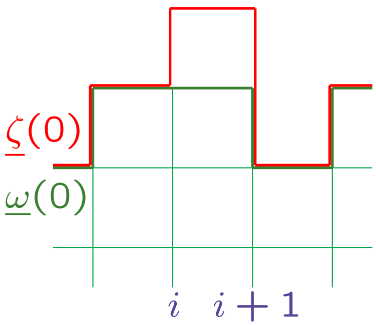

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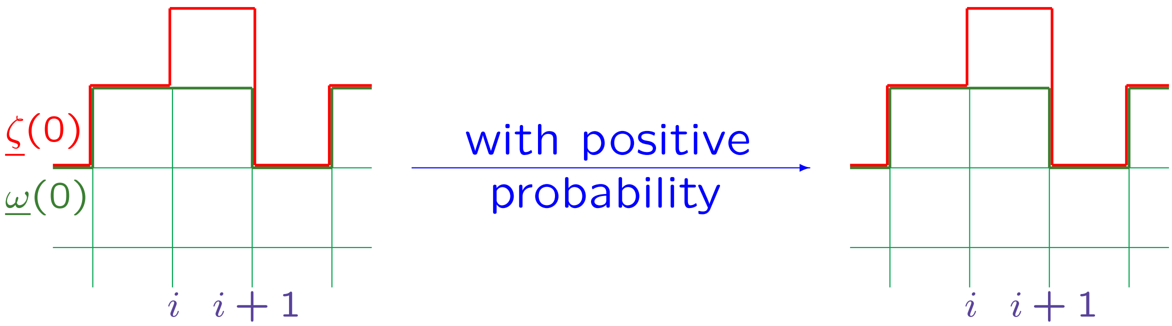

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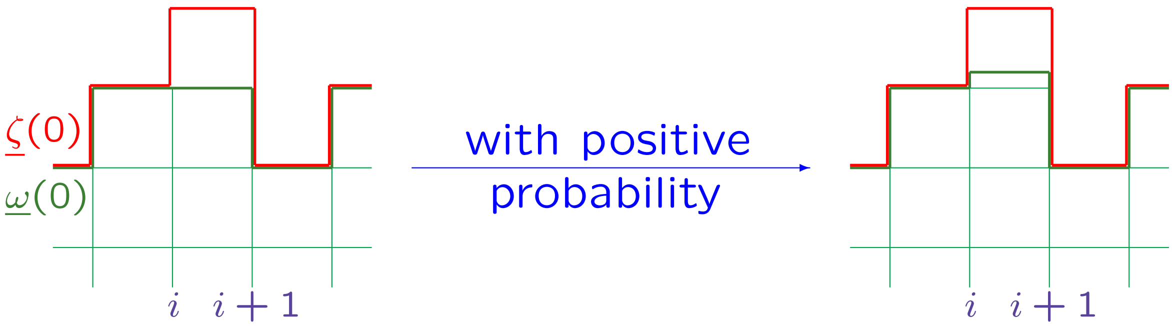

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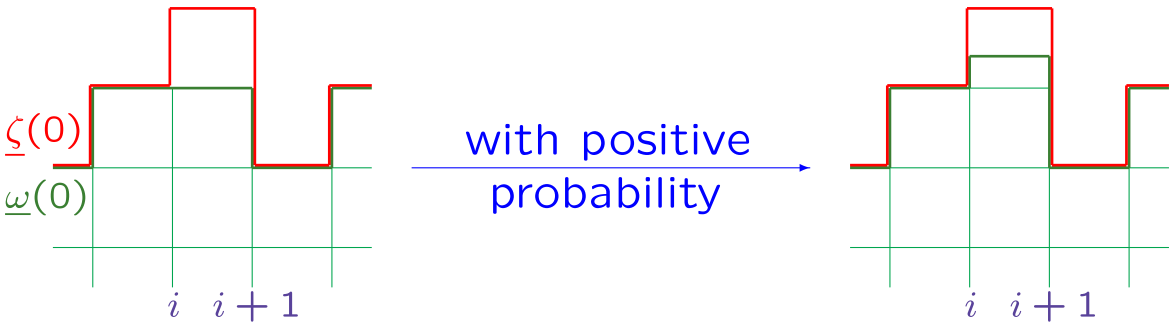

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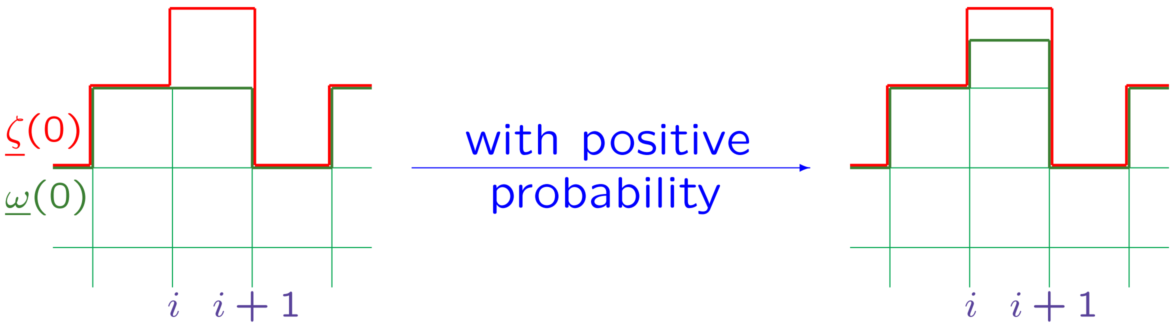

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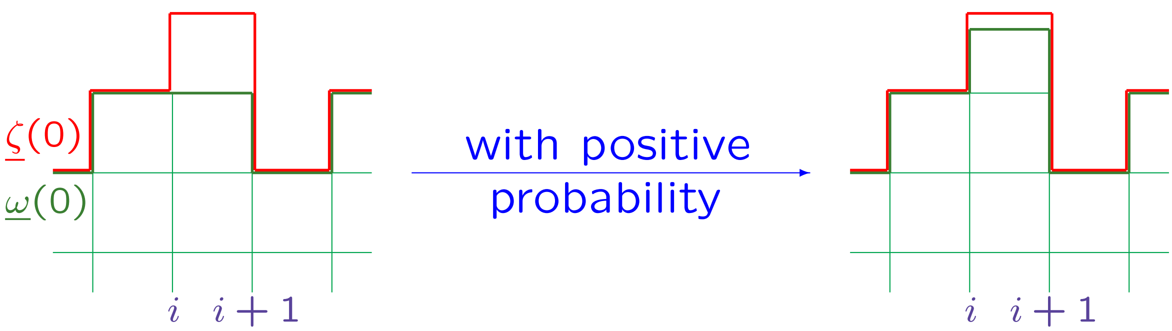

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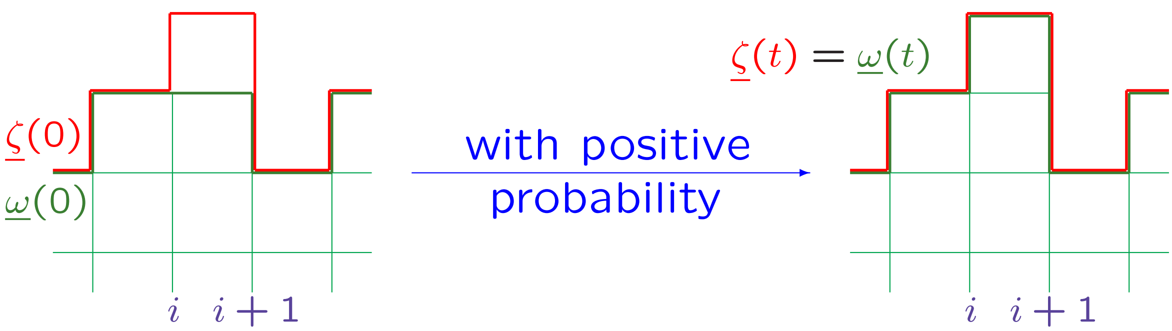

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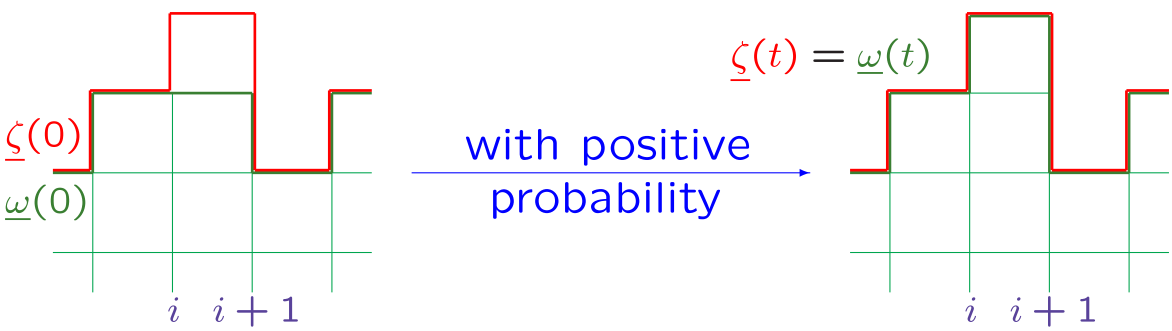

Thus $\psi(\underline{\omega}(0)) = \psi(\underline{\omega}(t)) = \psi(\underline{\zeta}(t)) = \psi(\underline{\zeta}(0))$.

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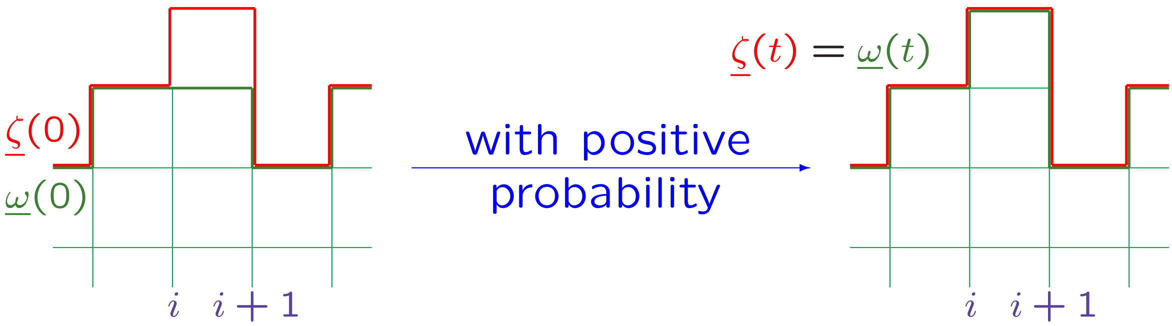

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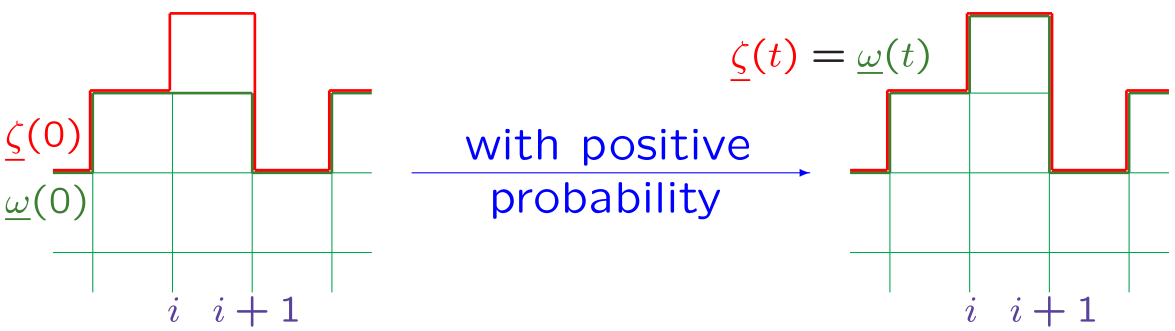

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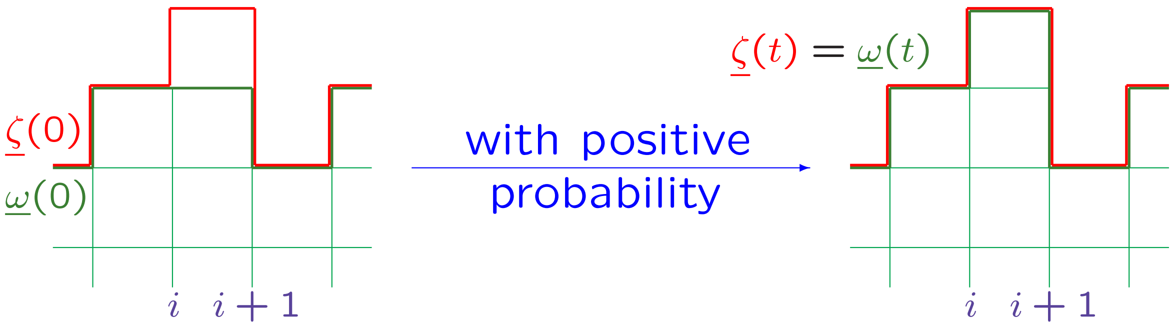

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$\rightsquigarrow$ ψ is $\underline{\mu}^\theta$-a.s. constant (Hewitt-Savage 1-0 Law).

Thank you.