Order of current variance in the simple exclusion process

Márton Balázs
(University of Wisconsin - Madison)
(Budapest University of Technology and Economics)

Joint work with

Timo Seppäläinen
(University of Wisconsin - Madison)

Madison, March 15, 2007.

1. ASEP: Interacting particles

2. ASEP: Surface growth

3. Growth fluctuations

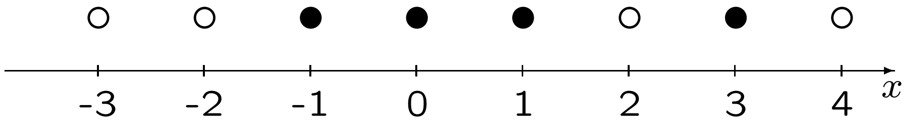
4. The second class particle

5. The upper bound

6. The lower bound

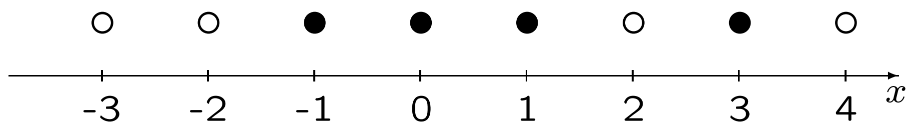
7. Open questions

1. ASEP: Interacting particles



Bernoulli(ϱ) distribution

1. ASEP: Interacting particles



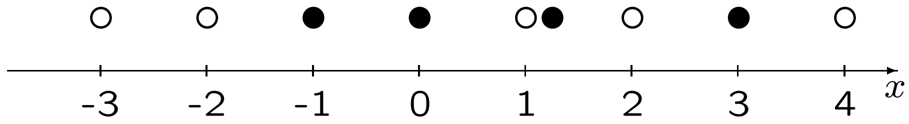
Bernoulli(ϱ) distribution

Particles try to jump

to the right with rate p,
to the left with rate $q = 1 - p < p$.

The jump is suppressed if the destination site is occupied by another particle.

1. ASEP: Interacting particles



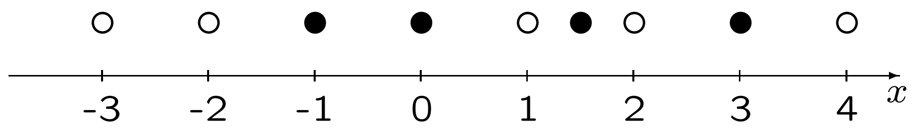
Bernoulli(ϱ) distribution

Particles try to jump

to the right with rate p,
to the left with rate $q = 1 - p < p$.

The jump is suppressed if the destination site is occupied by another particle.

1. ASEP: Interacting particles



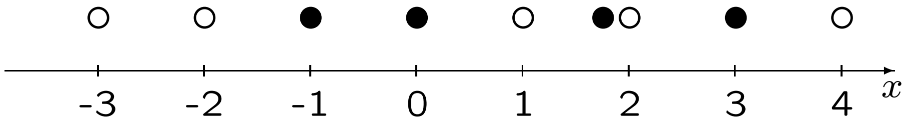
Bernoulli(ϱ) distribution

Particles try to jump

to the right with rate p,
to the left with rate $q = 1 - p < p$.

The jump is suppressed if the destination site is occupied by another particle.

1. ASEP: Interacting particles



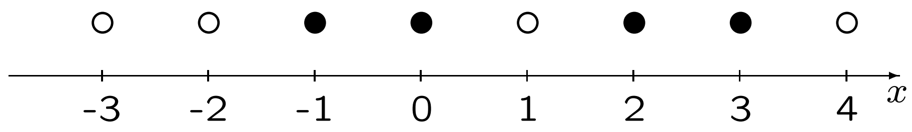
Bernoulli(ϱ) distribution

Particles try to jump

to the right with rate p,
to the left with rate $q = 1 - p < p$.

The jump is suppressed if the destination site is occupied by another particle.

1. ASEP: Interacting particles



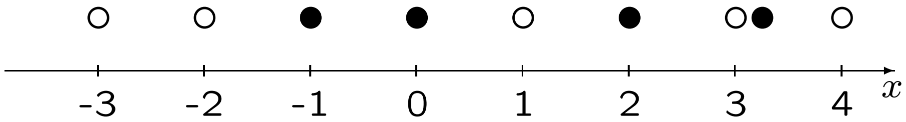
Bernoulli(ϱ) distribution

Particles try to jump

to the right with rate p,
to the left with rate $q = 1 - p < p$.

The jump is suppressed if the destination site is occupied by another particle.

1. ASEP: Interacting particles



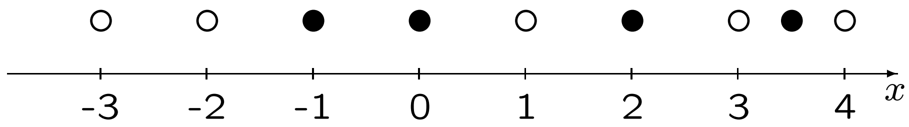
Bernoulli(ϱ) distribution

Particles try to jump

to the right with rate p,
to the left with rate $q = 1 - p < p$.

The jump is suppressed if the destination site is occupied by another particle.

1. ASEP: Interacting particles



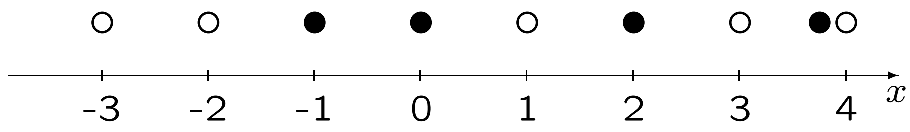
Bernoulli(ϱ) distribution

Particles try to jump

to the right with rate p,
to the left with rate $q = 1 - p < p$.

The jump is suppressed if the destination site is occupied by another particle.

1. ASEP: Interacting particles



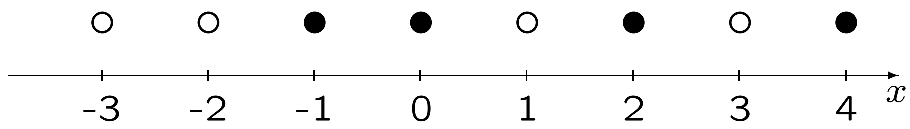
Bernoulli(ϱ) distribution

Particles try to jump

to the right with rate p,
to the left with rate $q = 1 - p < p$.

The jump is suppressed if the destination site is occupied by another particle.

1. ASEP: Interacting particles



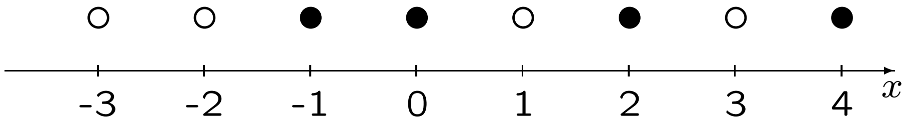
Bernoulli(ϱ) distribution

Particles try to jump

to the right with rate p,
to the left with rate $q = 1 - p < p$.

The jump is suppressed if the destination site is occupied by another particle.

1. ASEP: Interacting particles



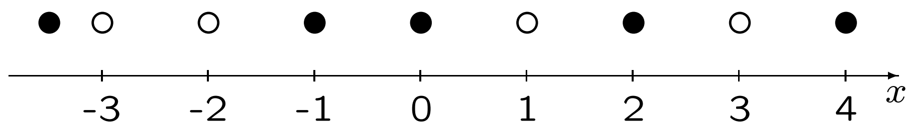
Bernoulli(ϱ) distribution

Particles try to jump

to the right with rate p,
to the left with rate $q = 1 - p < p$.

The jump is suppressed if the destination site is occupied by another particle.

1. ASEP: Interacting particles



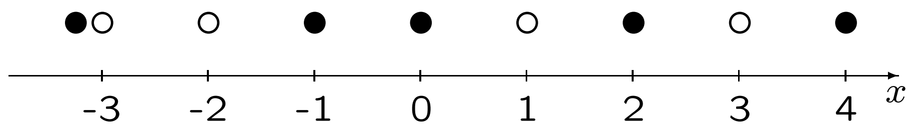
Bernoulli(ϱ) distribution

Particles try to jump

to the right with rate p,
to the left with rate $q = 1 - p < p$.

The jump is suppressed if the destination site is occupied by another particle.

1. ASEP: Interacting particles



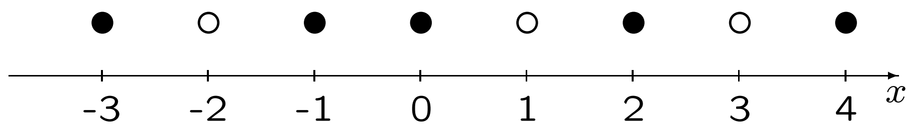
Bernoulli(ϱ) distribution

Particles try to jump

to the right with rate p,
to the left with rate $q = 1 - p < p$.

The jump is suppressed if the destination site is occupied by another particle.

1. ASEP: Interacting particles



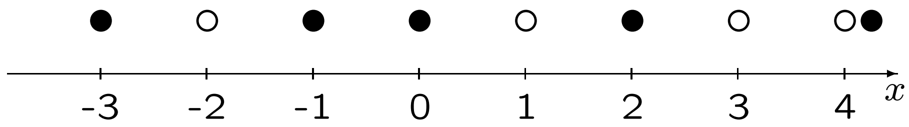
Bernoulli(ϱ) distribution

Particles try to jump

to the right with rate p,
to the left with rate $q = 1 - p < p$.

The jump is suppressed if the destination site is occupied by another particle.

1. ASEP: Interacting particles



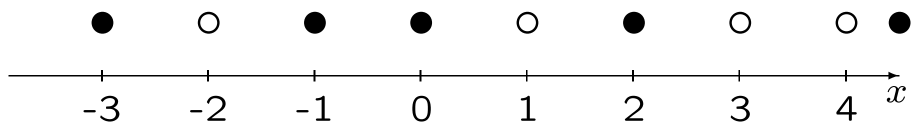
Bernoulli(ϱ) distribution

Particles try to jump

to the right with rate p,
to the left with rate $q = 1 - p < p$.

The jump is suppressed if the destination site is occupied by another particle.

1. ASEP: Interacting particles



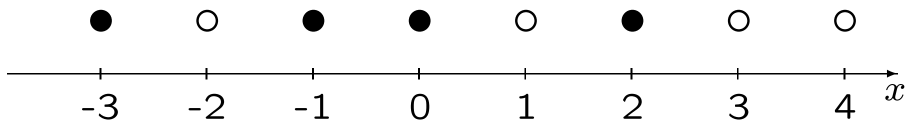
Bernoulli(ϱ) distribution

Particles try to jump

to the right with rate p,
to the left with rate $q = 1 - p < p$.

The jump is suppressed if the destination site is occupied by another particle.

1. ASEP: Interacting particles



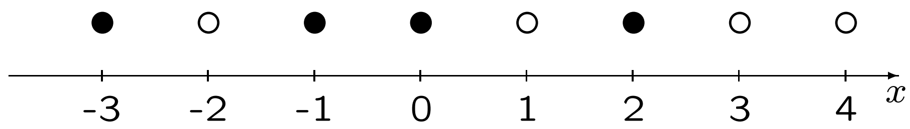
Bernoulli(ϱ) distribution

Particles try to jump

to the right with rate p,
to the left with rate $q = 1 - p < p$.

The jump is suppressed if the destination site is occupied by another particle.

1. ASEP: Interacting particles



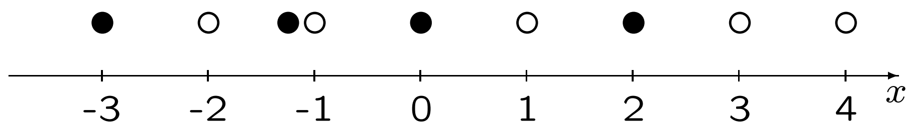
Bernoulli(ϱ) distribution

Particles try to jump

to the right with rate p,
to the left with rate $q = 1 - p < p$.

The jump is suppressed if the destination site is occupied by another particle.

1. ASEP: Interacting particles



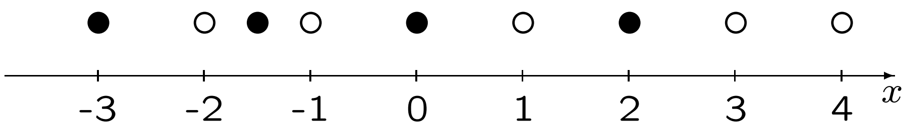
Bernoulli(ϱ) distribution

Particles try to jump

to the right with rate p,
to the left with rate $q = 1 - p < p$.

The jump is suppressed if the destination site is occupied by another particle.

1. ASEP: Interacting particles



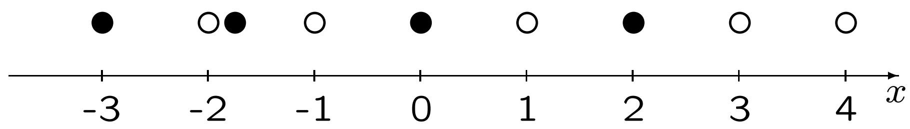
Bernoulli(ϱ) distribution

Particles try to jump

to the right with rate p,
to the left with rate $q = 1 - p < p$.

The jump is suppressed if the destination site is occupied by another particle.

1. ASEP: Interacting particles



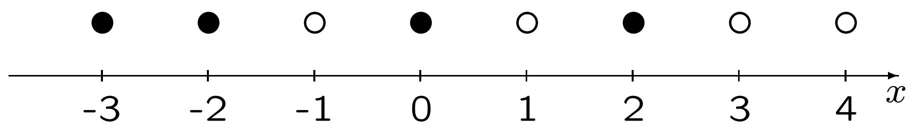
Bernoulli(ϱ) distribution

Particles try to jump

to the right with rate p,
to the left with rate $q = 1 - p < p$.

The jump is suppressed if the destination site is occupied by another particle.

1. ASEP: Interacting particles



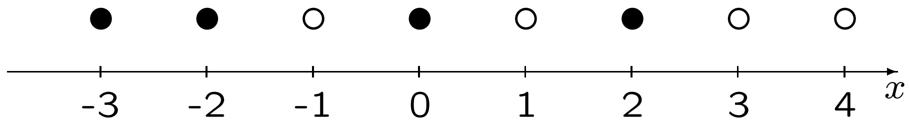
Bernoulli(ϱ) distribution

Particles try to jump

to the right with rate p,
to the left with rate $q = 1 - p < p$.

The jump is suppressed if the destination site is occupied by another particle.

1. ASEP: Interacting particles



Bernoulli(ϱ) distribution

Particles try to jump

to the right with rate p,
to the left with rate $q = 1 - p < p$.

The jump is suppressed if the destination site is occupied by another particle.

The Bernoulli(ϱ) distribution is time-stationary for any ($0 \le \varrho \le 1$).

Any translation-invariant stationary distribution is a mixture of Bernoullis.

Hydrodynamics (briefly)

Let T and X be some large-scale time and space parameters.

Hydrodynamics (briefly)

Let T and X be some large-scale time and space parameters.
$\rightsquigarrow$ Set initially $\varrho = \varrho(T = 0, X)$ to be the density at position $x = X/\varepsilon$.
(Changes on the large scale.)

Hydrodynamics (briefly)

Let T and X be some large-scale time and space parameters.
$\rightsquigarrow$ Set initially $\varrho = \varrho(T = 0, X)$ to be the density at position $x = X/\varepsilon$.
(Changes on the large scale.)
$\rightsquigarrow$ $\varrho(T, X)$ is the density of particles after a long time $t = T/\varepsilon$ at position $x = X/\varepsilon$.

Hydrodynamics (briefly)

Let T and X be some large-scale time and space parameters.
$\rightsquigarrow$ Set initially $\varrho = \varrho(T = 0, X)$ to be the density at position $x = X/\varepsilon$.
(Changes on the large scale.)
$\rightsquigarrow$ $\varrho(T, X)$ is the density of particles after a long time $t = T/\varepsilon$ at position $x = X/\varepsilon$. It satisfies, with $a := p - q$,

$$\frac{\partial}{\partial T}\varrho + \frac{\partial}{\partial X}a[\varrho(1-\varrho)] = 0 \quad \text{(inviscid Burgers)}$$

Hydrodynamics (briefly)

Let T and X be some large-scale time and space parameters.
$\rightsquigarrow$ Set initially $\varrho = \varrho(T = 0, X)$ to be the density at position $x = X/\varepsilon$.
(Changes on the large scale.)
$\rightsquigarrow$ $\varrho(T, X)$ is the density of particles after a long time $t = T/\varepsilon$ at position $x = X/\varepsilon$. It satisfies, with $a := p - q$,

$$\frac{\partial}{\partial T}\varrho + \frac{\partial}{\partial X} a[\varrho(1-\varrho)] = 0 \qquad \text{(inviscid Burgers)}$$

$$\frac{\partial}{\partial T}\varrho + a[1-2\varrho] \cdot \frac{\partial}{\partial X}\varrho = 0 \qquad \text{(while smooth)}$$

Hydrodynamics (briefly)

Let T and X be some large-scale time and space parameters.
$\rightsquigarrow$ Set initially $\varrho = \varrho(T = 0, X)$ to be the density at position $x = X/\varepsilon$.
(Changes on the large scale.)
$\rightsquigarrow$ $\varrho(T, X)$ is the density of particles after a long time $t = T/\varepsilon$ at position $x = X/\varepsilon$. It satisfies, with $a := p - q$,

$$\frac{\partial}{\partial T}\varrho + \frac{\partial}{\partial X}a[\varrho(1-\varrho)] = 0 \qquad \text{(inviscid Burgers)}$$

$$\frac{\partial}{\partial T}\varrho + a[1-2\varrho]\cdot\frac{\partial}{\partial X}\varrho = 0 \qquad \text{(while smooth)}$$

$$\frac{\mathrm{d}}{\mathrm{d}T}\varrho(T, X(T)) = 0$$

Hydrodynamics (briefly)

Let T and X be some large-scale time and space parameters.
$\rightsquigarrow$ Set initially $\varrho = \varrho(T = 0, X)$ to be the density at position $x = X/\varepsilon$.
(Changes on the large scale.)
$\rightsquigarrow$ $\varrho(T, X)$ is the density of particles after a long time $t = T/\varepsilon$ at position $x = X/\varepsilon$. It satisfies, with $a := p - q$,

$$\frac{\partial}{\partial T}\varrho + \frac{\partial}{\partial X}a[\varrho(1-\varrho)] = 0 \qquad \text{(inviscid Burgers)}$$

$$\frac{\partial}{\partial T}\varrho + a[1-2\varrho]\cdot\frac{\partial}{\partial X}\varrho = 0 \qquad \text{(while smooth)}$$

$$\frac{\partial}{\partial T}\varrho + \frac{\mathrm{d}X(T)}{\mathrm{d}T}\cdot\frac{\partial}{\partial X}\varrho = \frac{\mathrm{d}}{\mathrm{d}T}\varrho(T, X(T)) = 0$$

Hydrodynamics (briefly)

Let T and X be some large-scale time and space parameters.
$\rightsquigarrow$ Set initially $\varrho = \varrho(T = 0, X)$ to be the density at position $x = X/\varepsilon$.
(Changes on the large scale.)
$\rightsquigarrow$ $\varrho(T, X)$ is the density of particles after a long time $t = T/\varepsilon$ at position $x = X/\varepsilon$. It satisfies, with $a := p - q$,

$$\frac{\partial}{\partial T}\varrho + \frac{\partial}{\partial X}a[\varrho(1-\varrho)] = 0 \qquad \text{(inviscid Burgers)}$$

$$\frac{\partial}{\partial T}\varrho + a[1-2\varrho]\cdot\frac{\partial}{\partial X}\varrho = 0 \qquad \text{(while smooth)}$$

$$\frac{\partial}{\partial T}\varrho + \frac{\mathrm{d}X(T)}{\mathrm{d}T}\cdot\frac{\partial}{\partial X}\varrho = \frac{\mathrm{d}}{\mathrm{d}T}\varrho(T, X(T)) = 0$$

Hydrodynamics (briefly)

Let T and X be some large-scale time and space parameters.
$\rightsquigarrow$ Set initially $\varrho = \varrho(T = 0, X)$ to be the density at position $x = X/\varepsilon$.
(Changes on the large scale.)
$\rightsquigarrow$ $\varrho(T, X)$ is the density of particles after a long time $t = T/\varepsilon$ at position $x = X/\varepsilon$. It satisfies, with $a := p - q$,

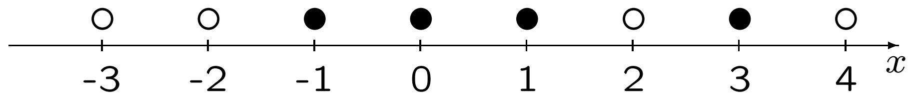
$$\frac{\partial}{\partial T}\varrho + \frac{\partial}{\partial X} a[\varrho(1-\varrho)] = 0 \qquad \text{(inviscid Burgers)}$$

$$\frac{\partial}{\partial T}\varrho + a[1-2\varrho]\cdot\frac{\partial}{\partial X}\varrho = 0 \qquad \text{(while smooth)}$$

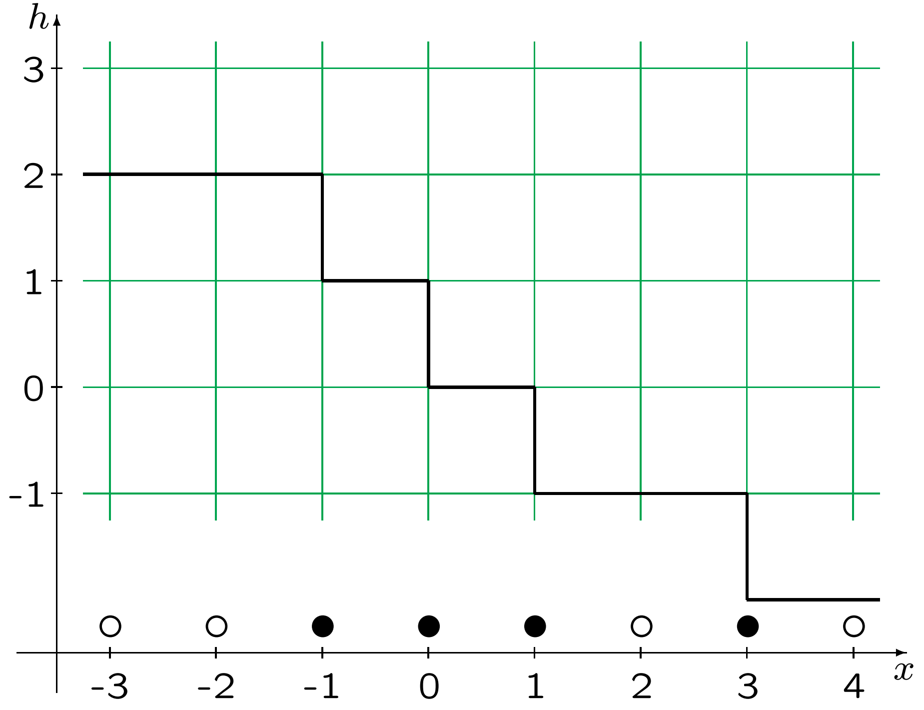
$$\frac{\partial}{\partial T}\varrho + \frac{\mathrm{d}X(T)}{\mathrm{d}T}\cdot\frac{\partial}{\partial X}\varrho = \frac{\mathrm{d}}{\mathrm{d}T}\varrho(T, X(T)) = 0$$

$\rightsquigarrow$ The characteristic speed $C(\varrho) := a[1-2\varrho]$.
(ϱ is constant along $\dot{X}(T) = C(\varrho)$.)

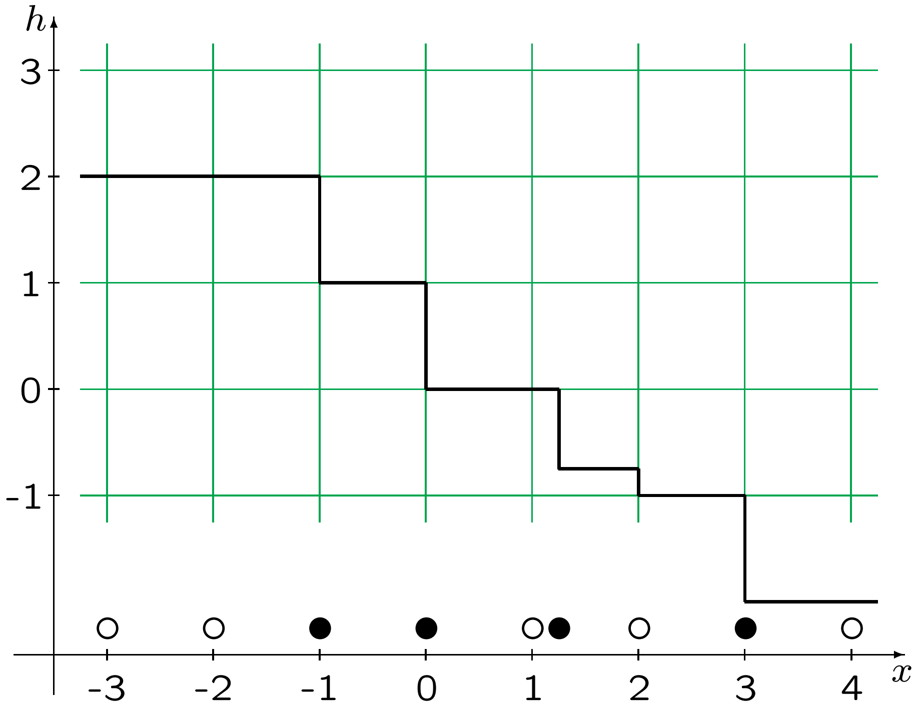
2. ASEP: Surface growth



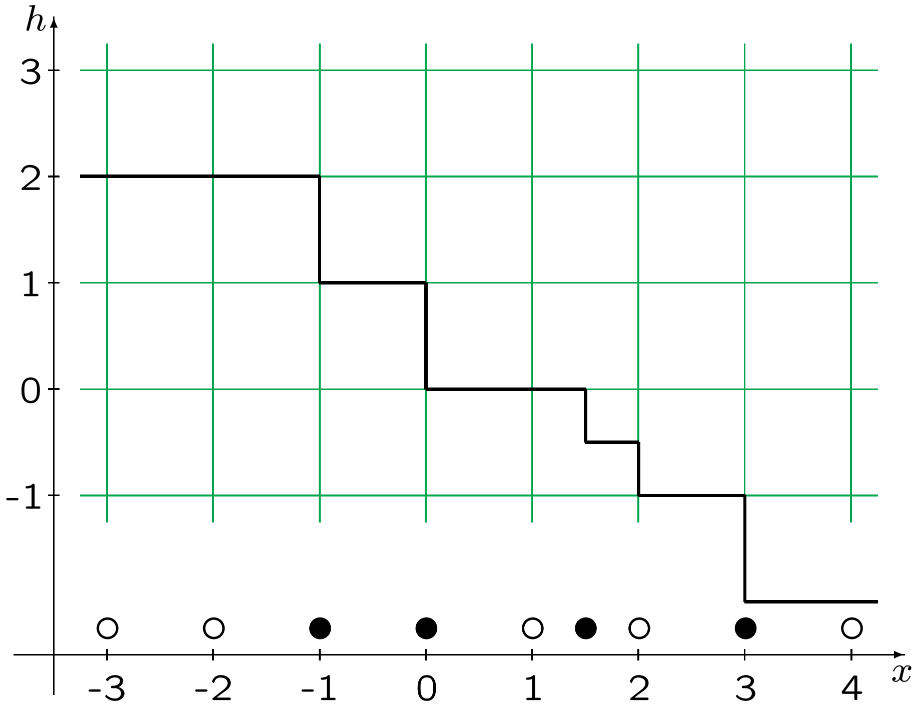
2. ASEP: Surface growth



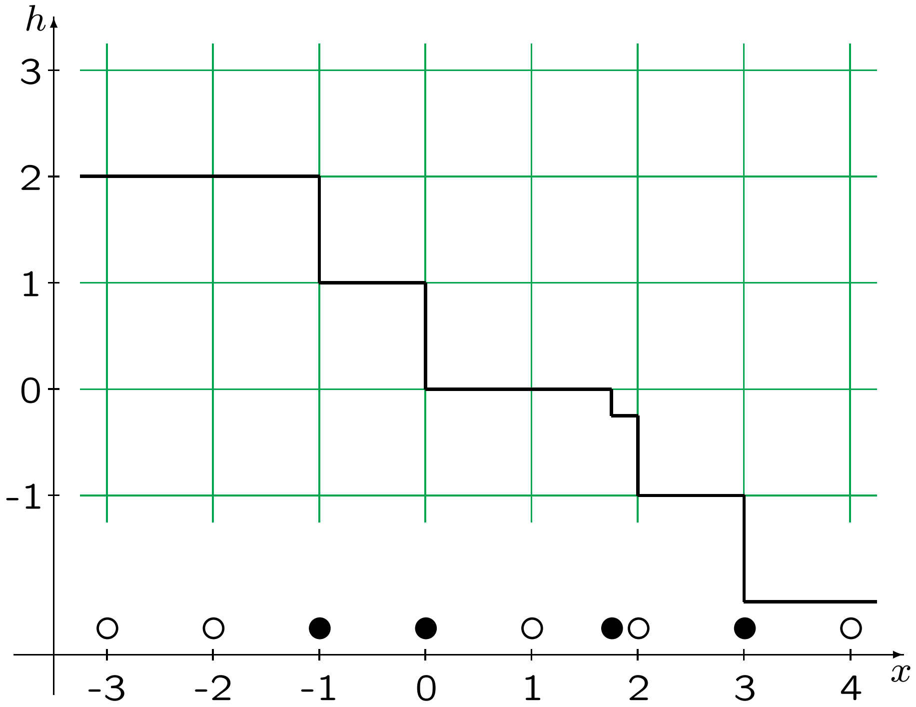
2. ASEP: Surface growth



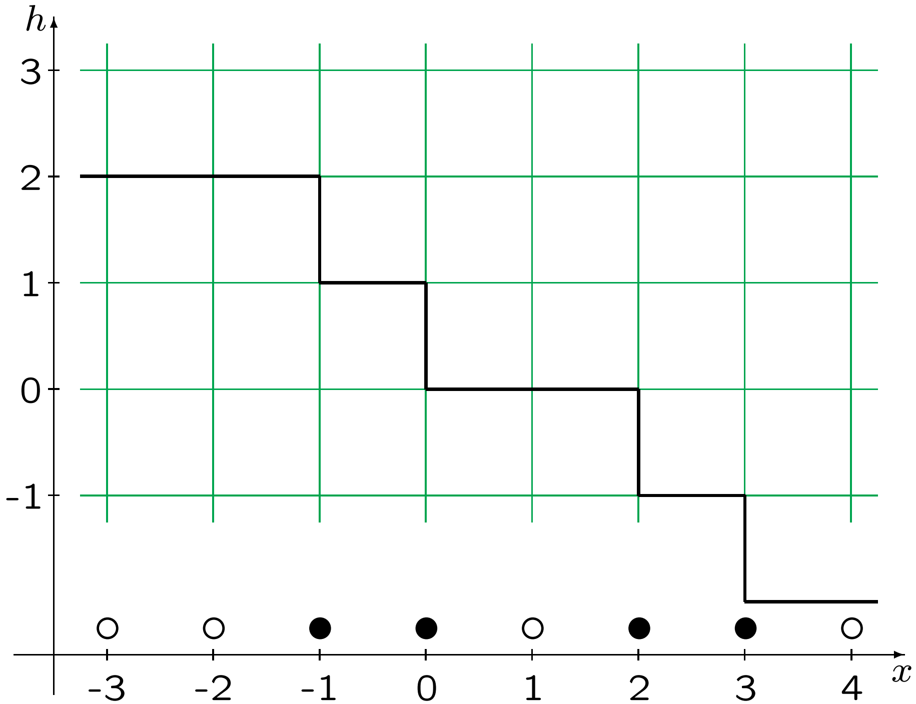
2. ASEP: Surface growth



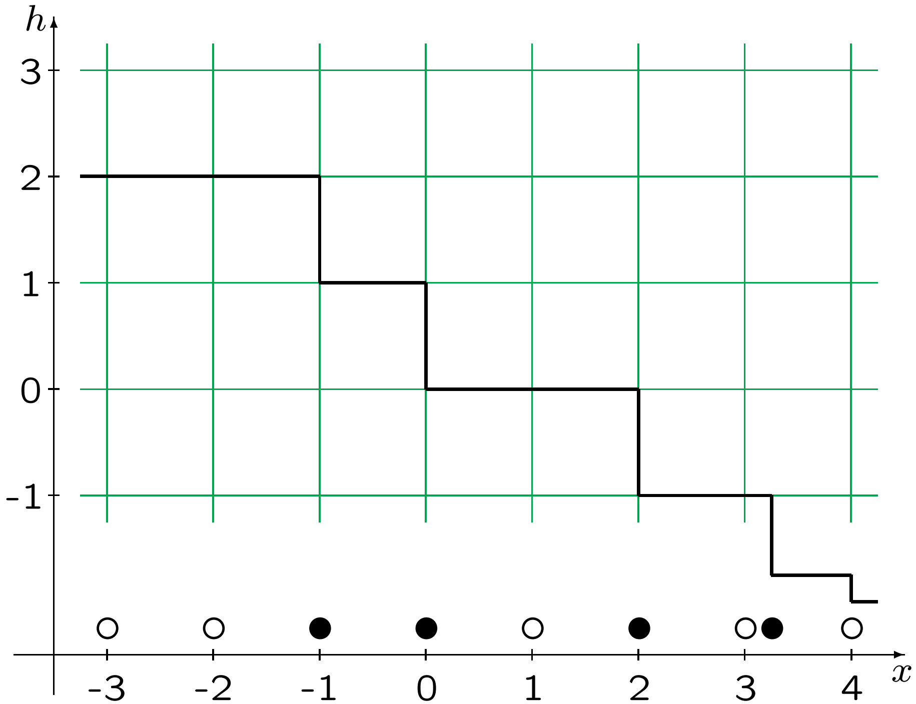
2. ASEP: Surface growth



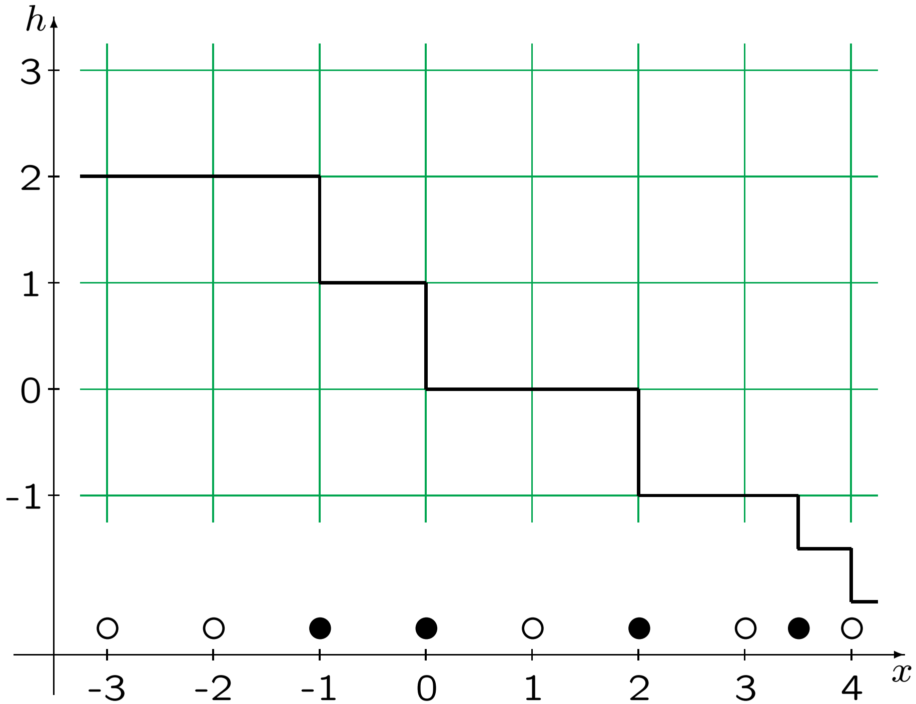
2. ASEP: Surface growth



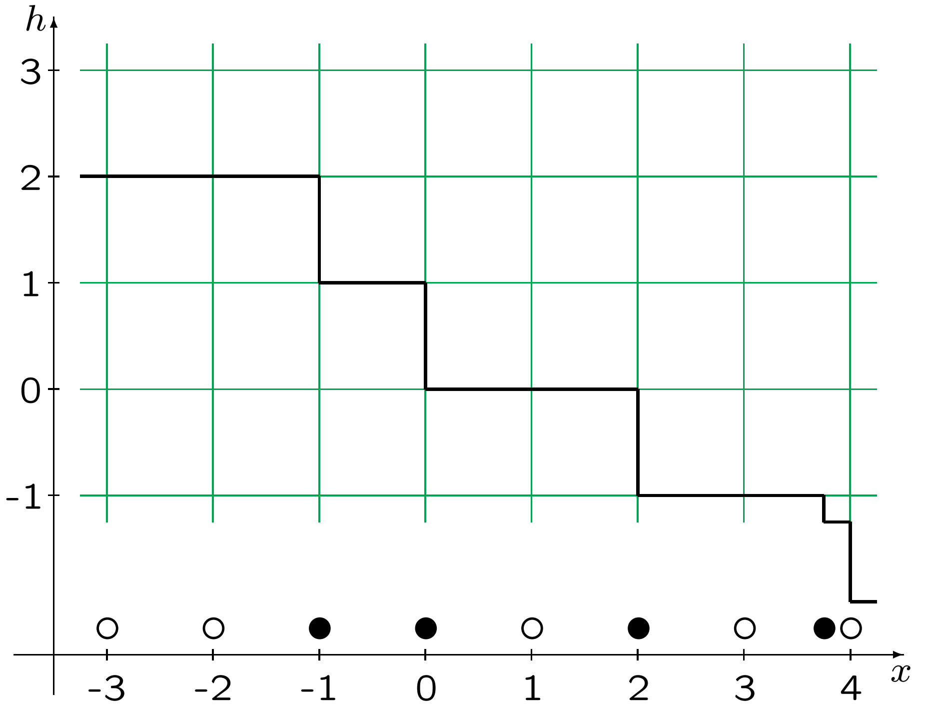
2. ASEP: Surface growth



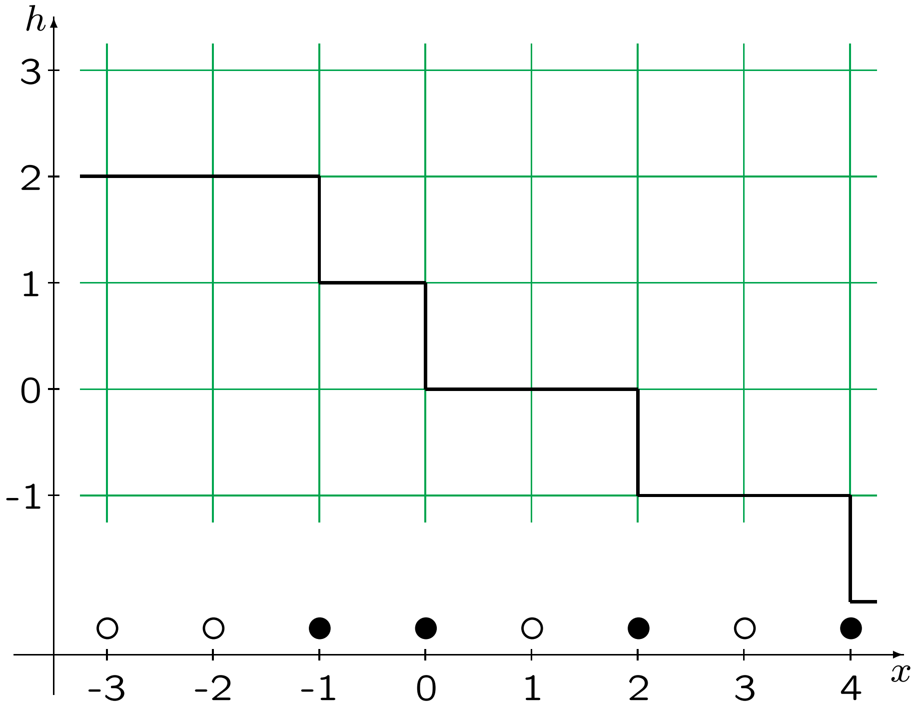
2. ASEP: Surface growth



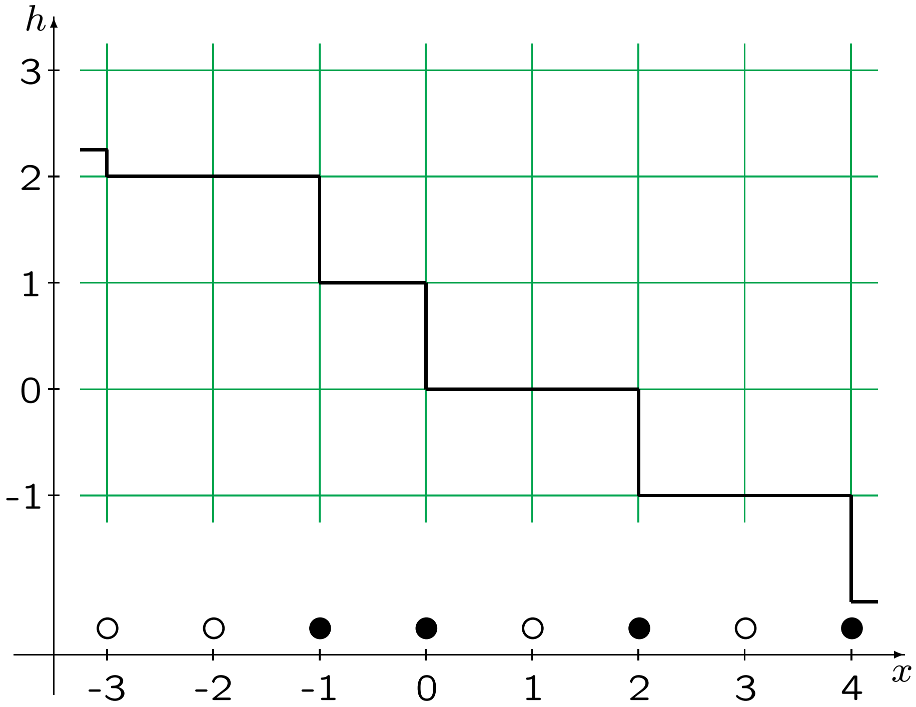
2. ASEP: Surface growth



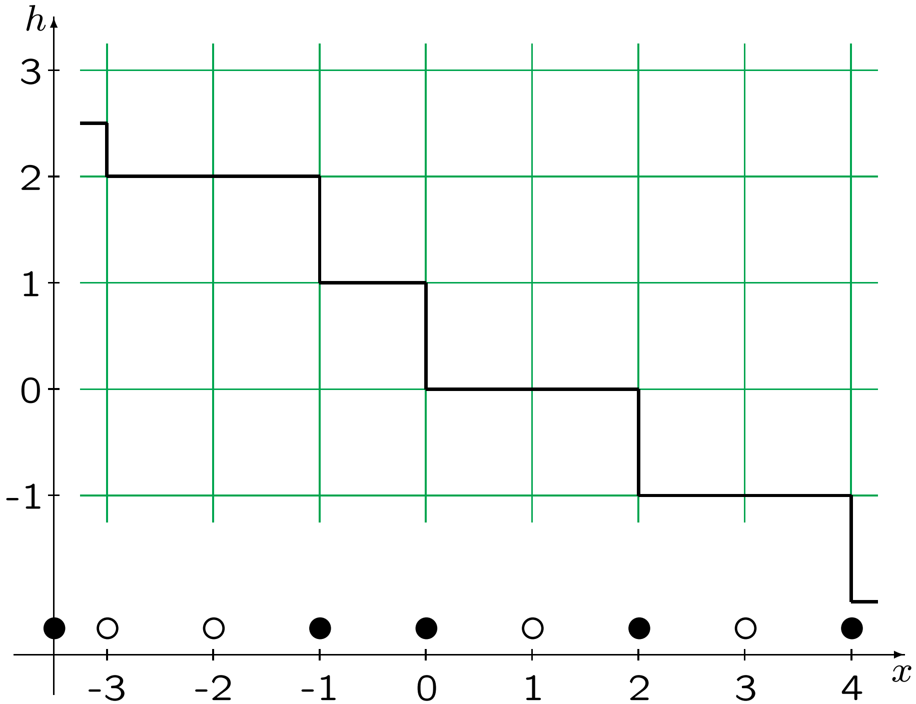
2. ASEP: Surface growth



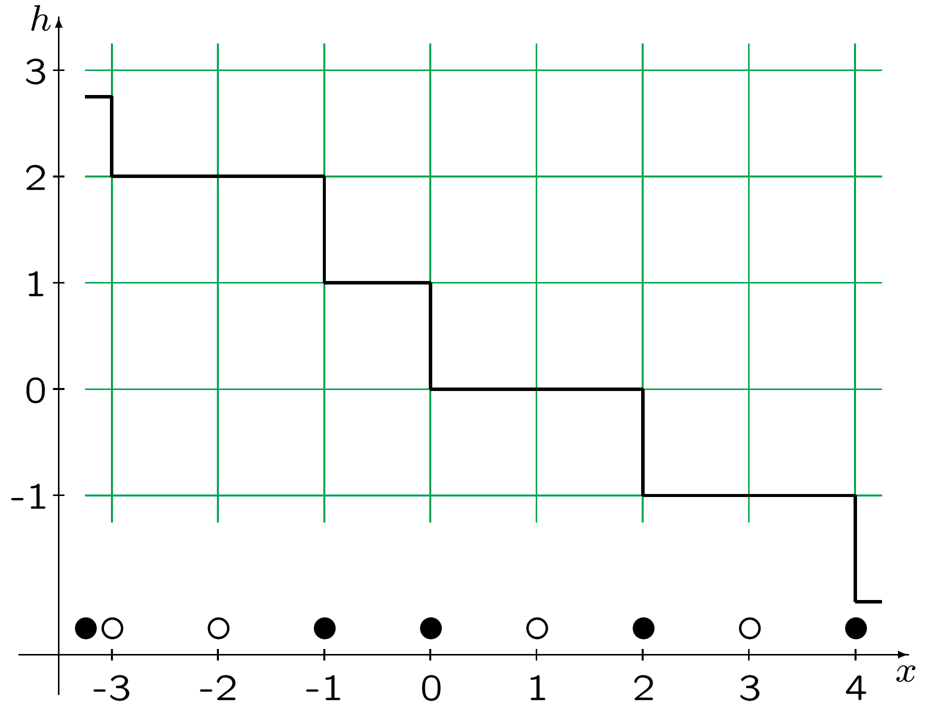
2. ASEP: Surface growth



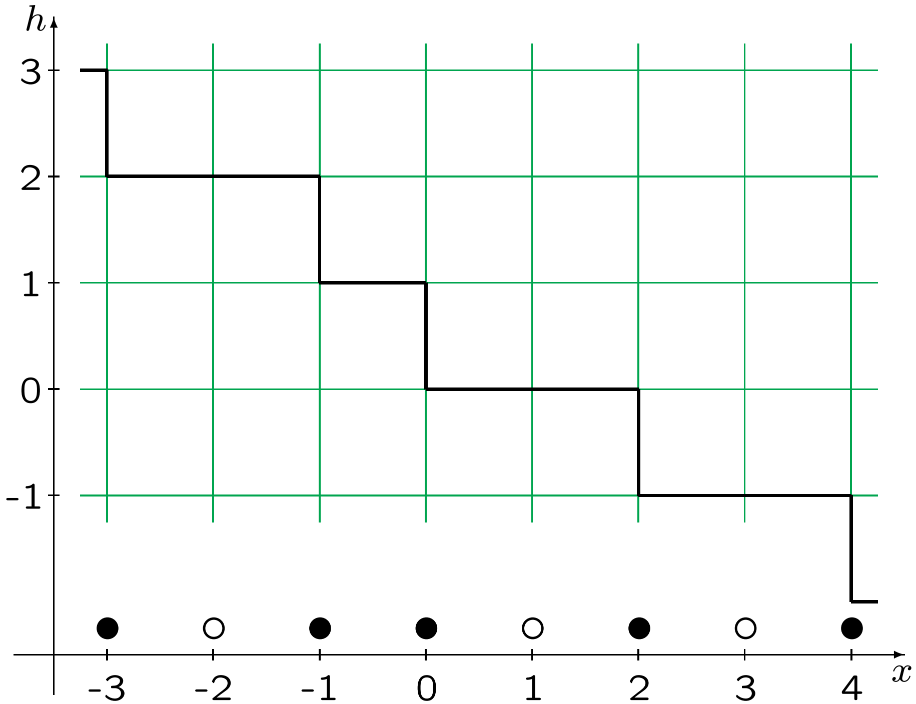
2. ASEP: Surface growth



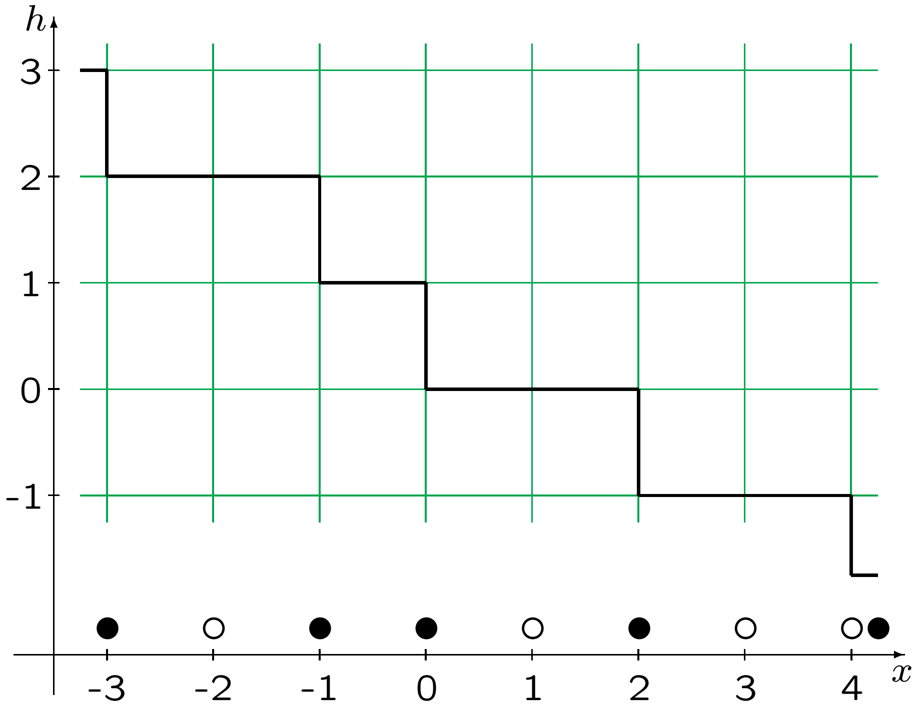
2. ASEP: Surface growth



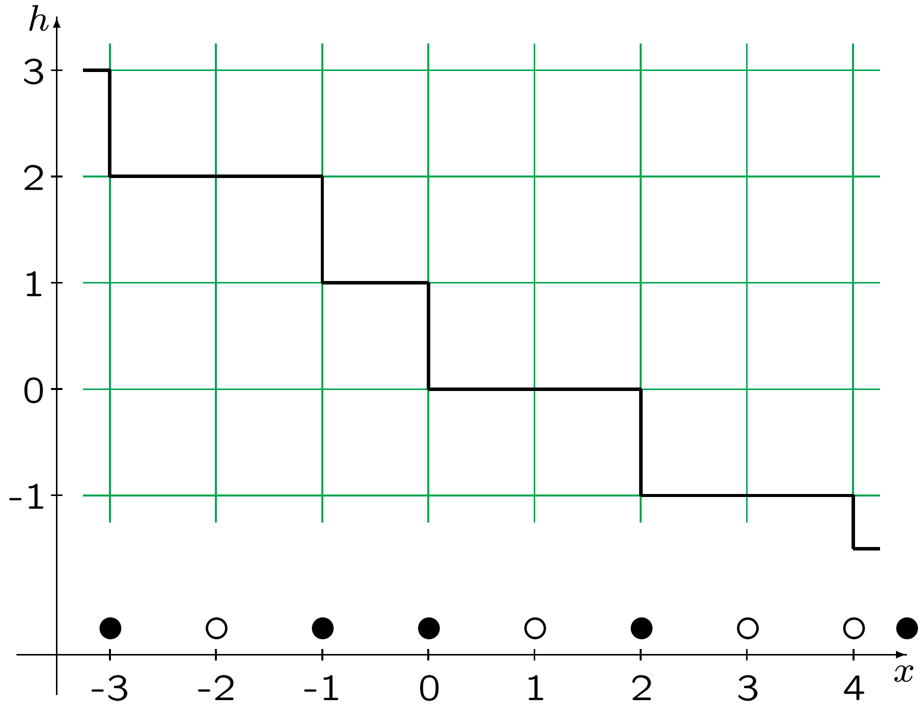
2. ASEP: Surface growth



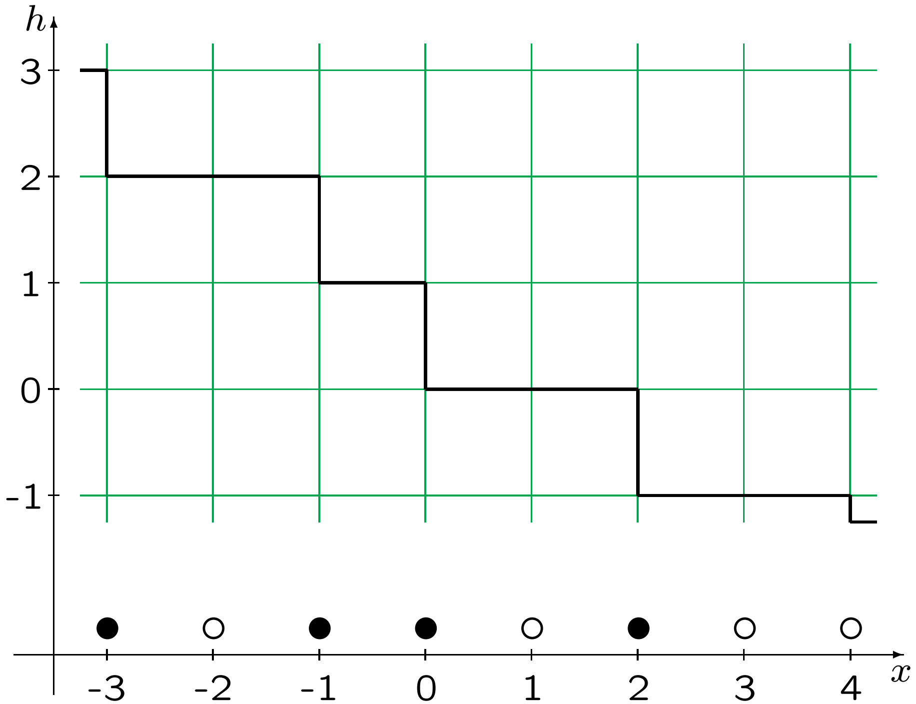
2. ASEP: Surface growth



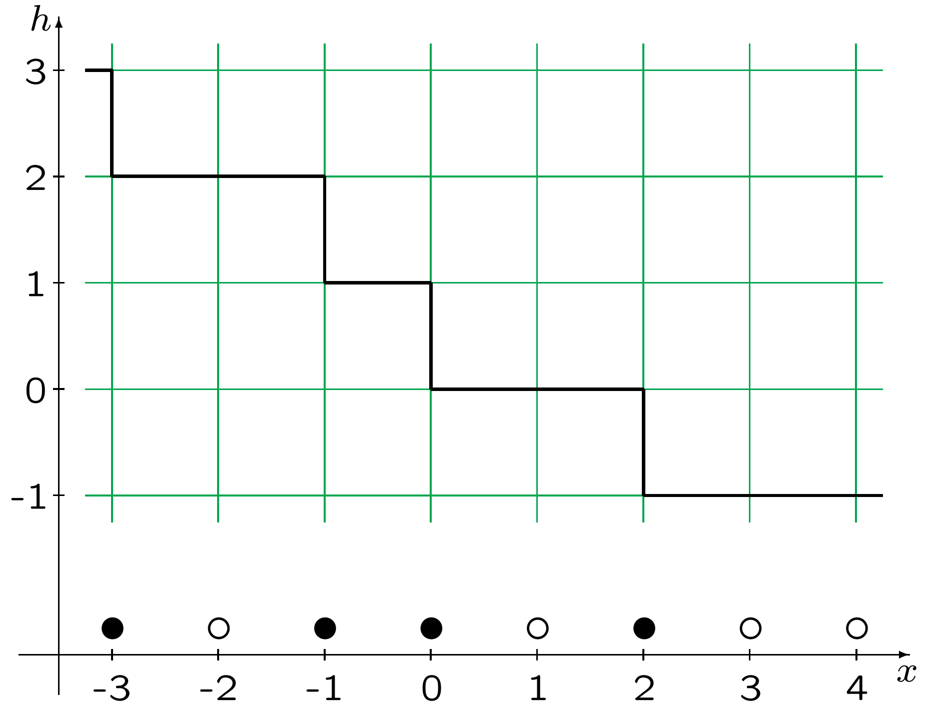
2. ASEP: Surface growth



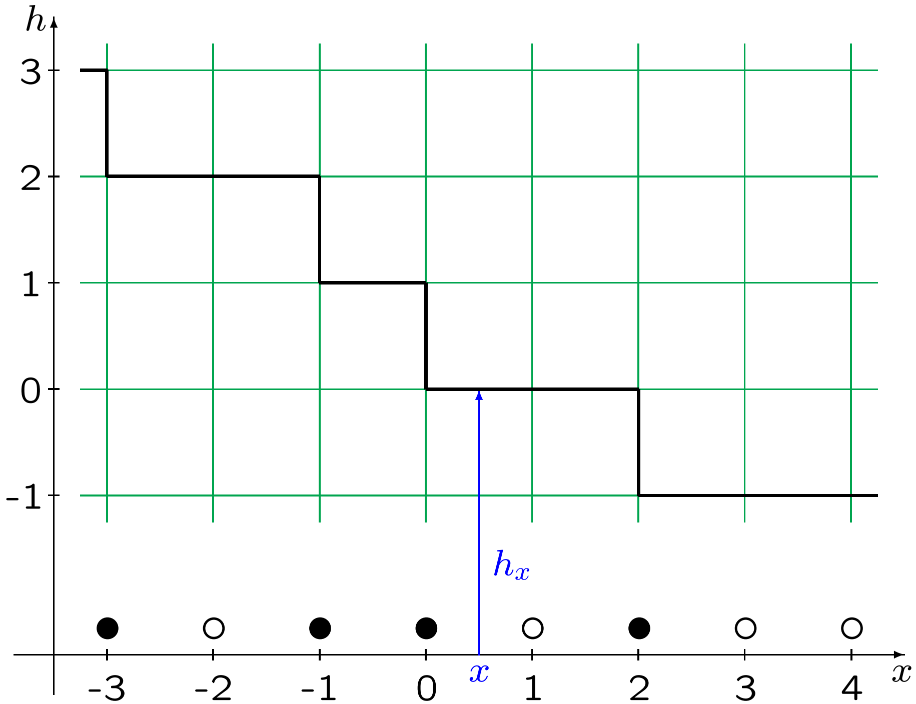
2. ASEP: Surface growth



2. ASEP: Surface growth

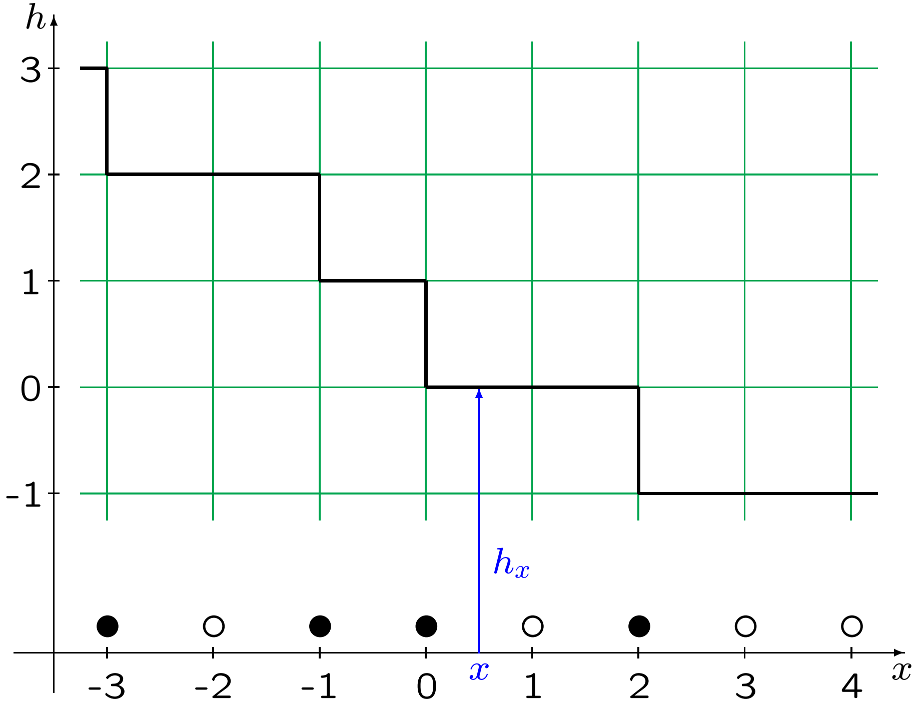


2. ASEP: Surface growth



$h_x(t)$ = height of the surface above x.

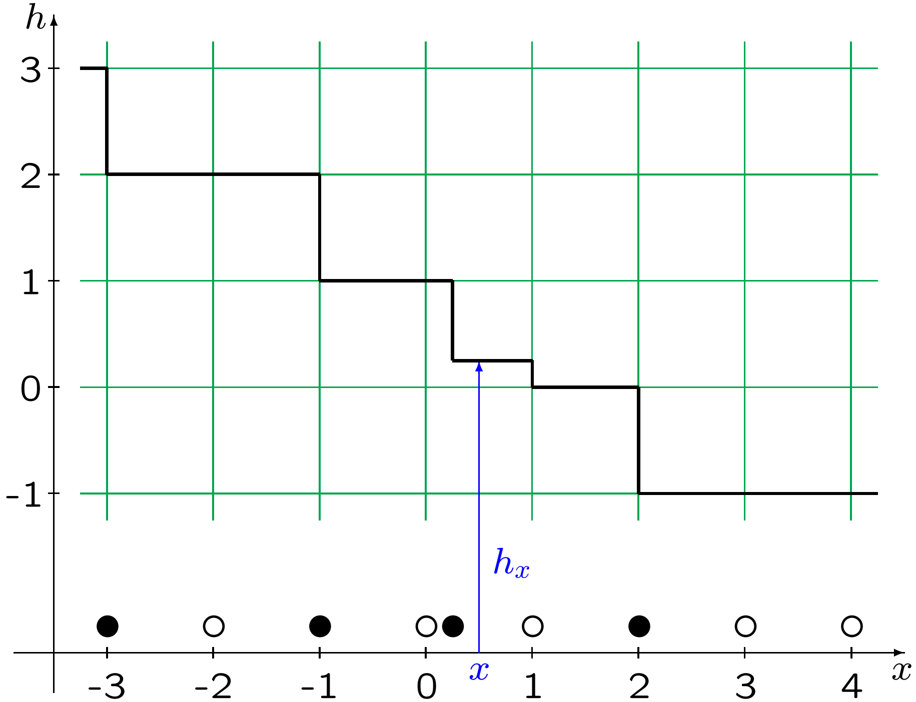
2. ASEP: Surface growth



$h_x(t)$ = height of the surface above x.

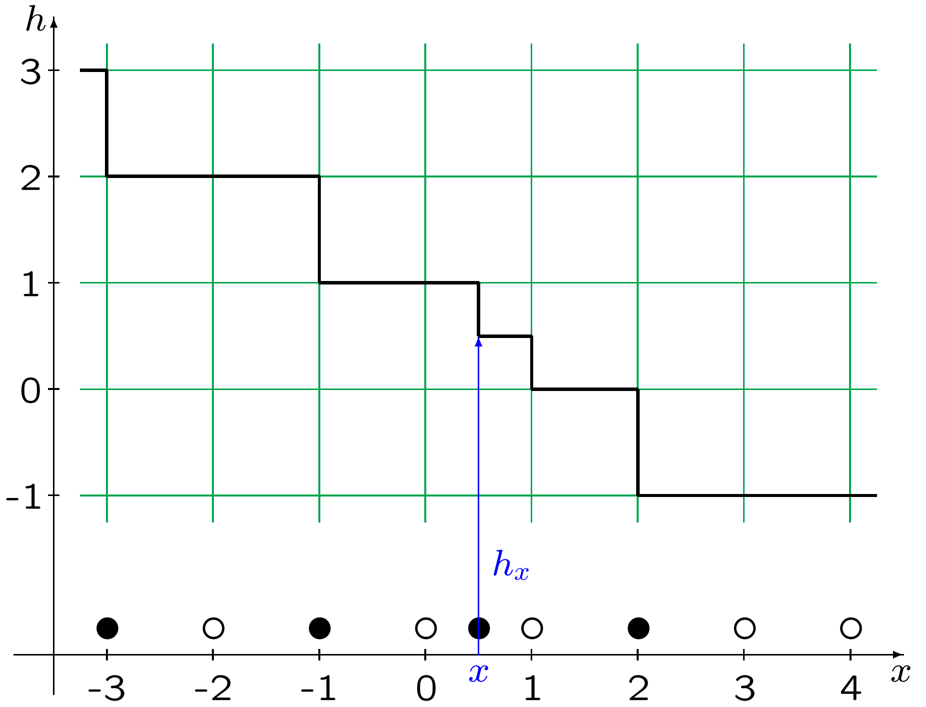
$h_x(t) - h_x(0)$ = net number of particles passed above x.

2. ASEP: Surface growth



$h_x(t)$ = height of the surface above x.
$h_x(t) - h_x(0)$ = net number of particles passed above x.

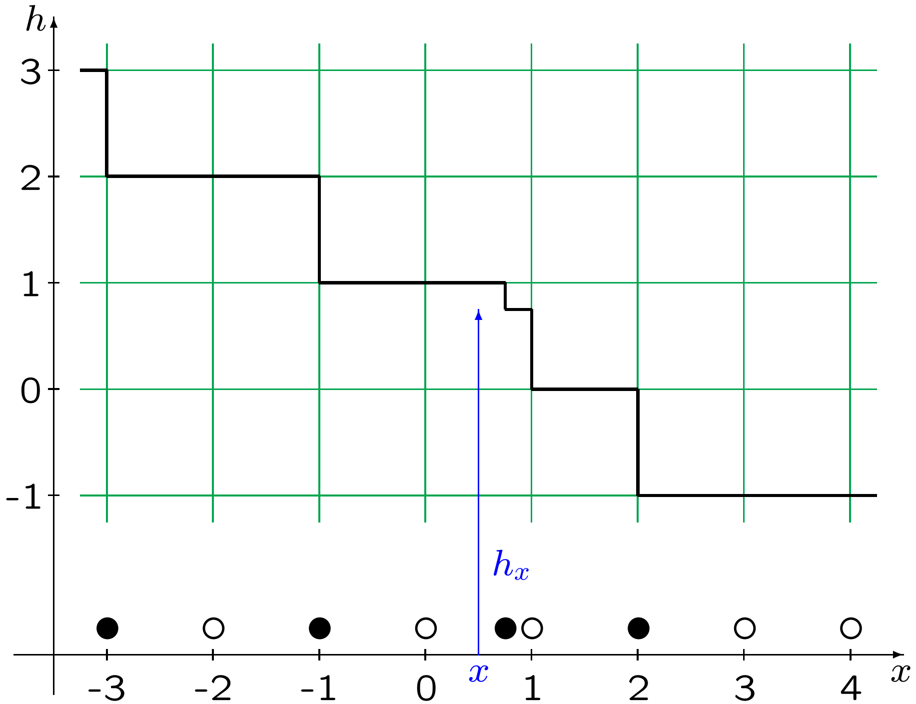
2. ASEP: Surface growth



$h_x(t)$ = height of the surface above x.

$h_x(t) - h_x(0)$ = net number of particles passed above x.

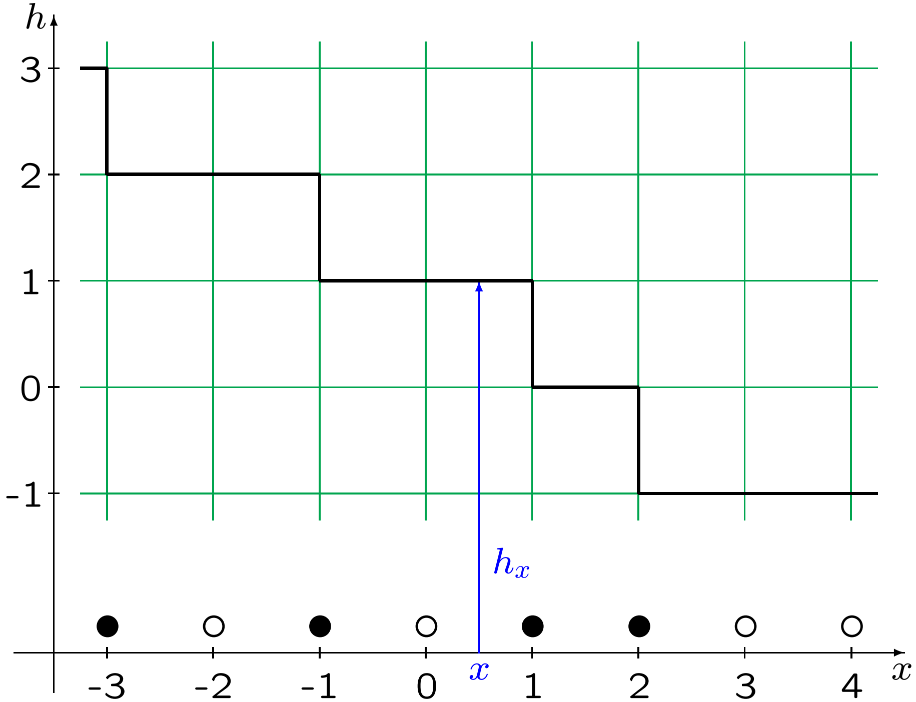
2. ASEP: Surface growth



$h_x(t)$ = height of the surface above x.

$h_x(t) - h_x(0)$ = net number of particles passed above x.

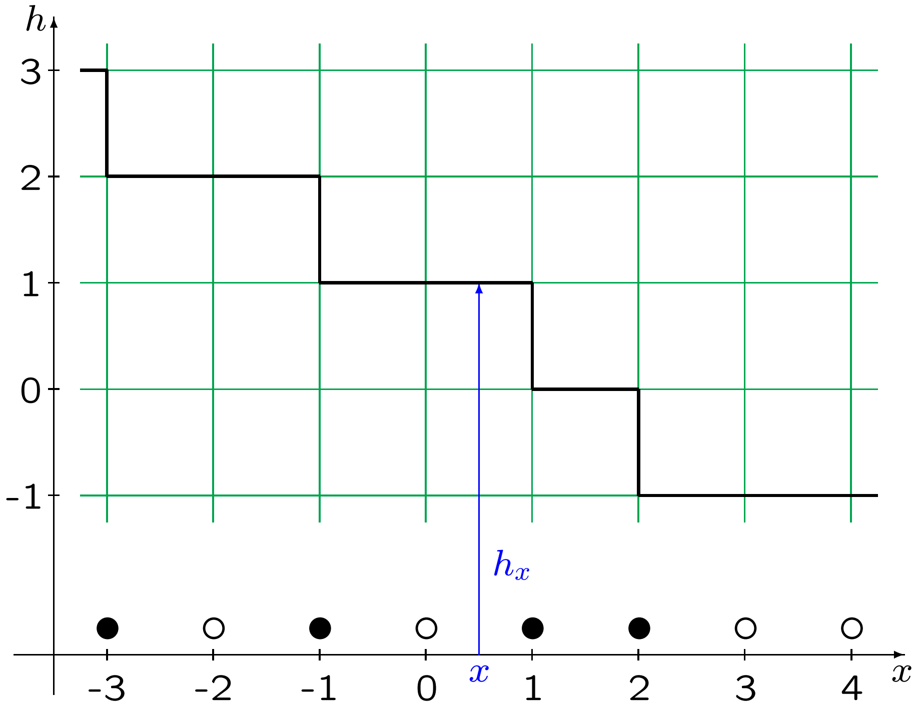
2. ASEP: Surface growth



$h_x(t)$ = height of the surface above x.

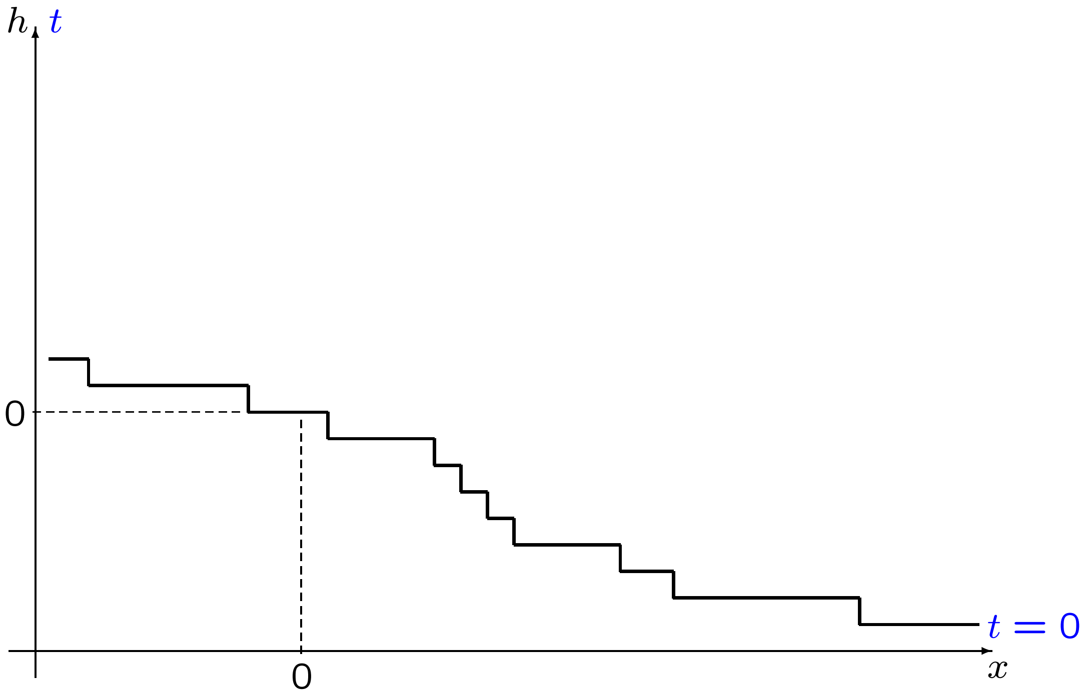
$h_x(t) - h_x(0)$ = net number of particles passed above x.

2. ASEP: Surface growth

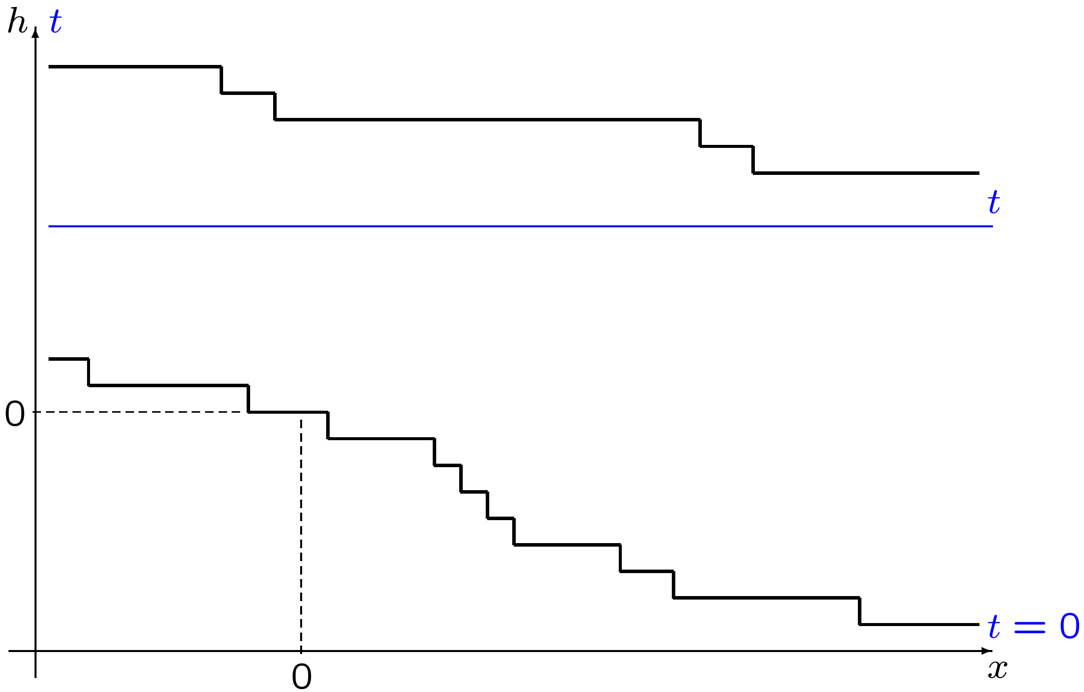


$h_x(t)$ = height of the surface above x.
$h_x(t) - h_x(0)$ = net number of particles passed above x.
$h_{Vt}(t)$ = net number of particles passed through the moving window at Vt $(V \in \mathbb{R})$.

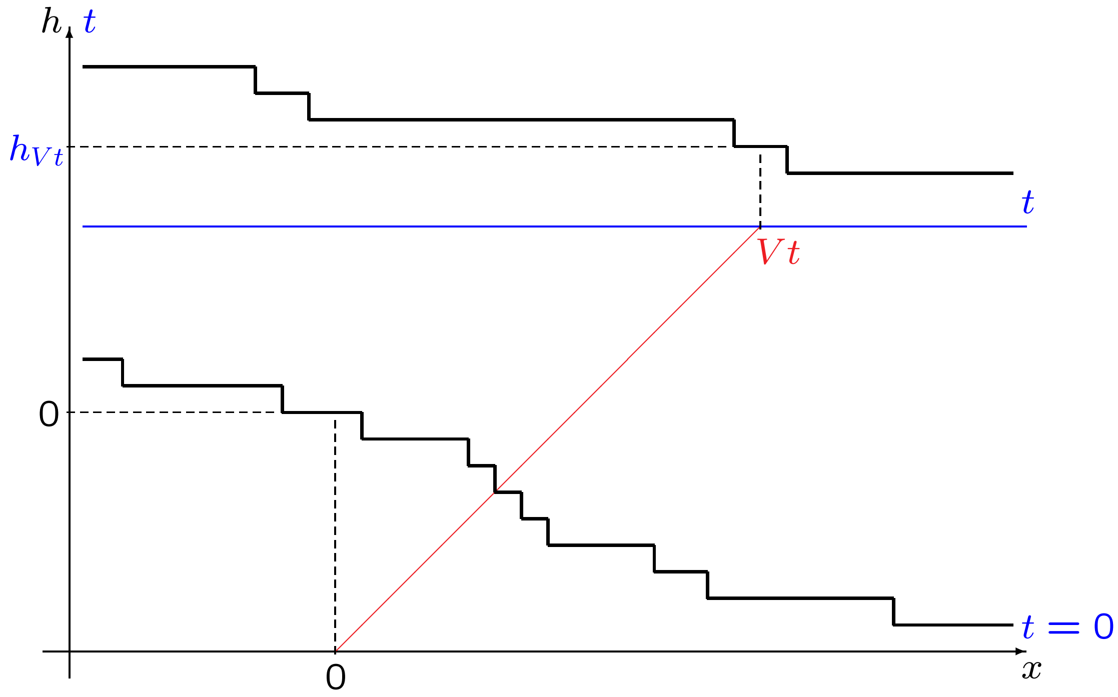
3. Growth fluctuations



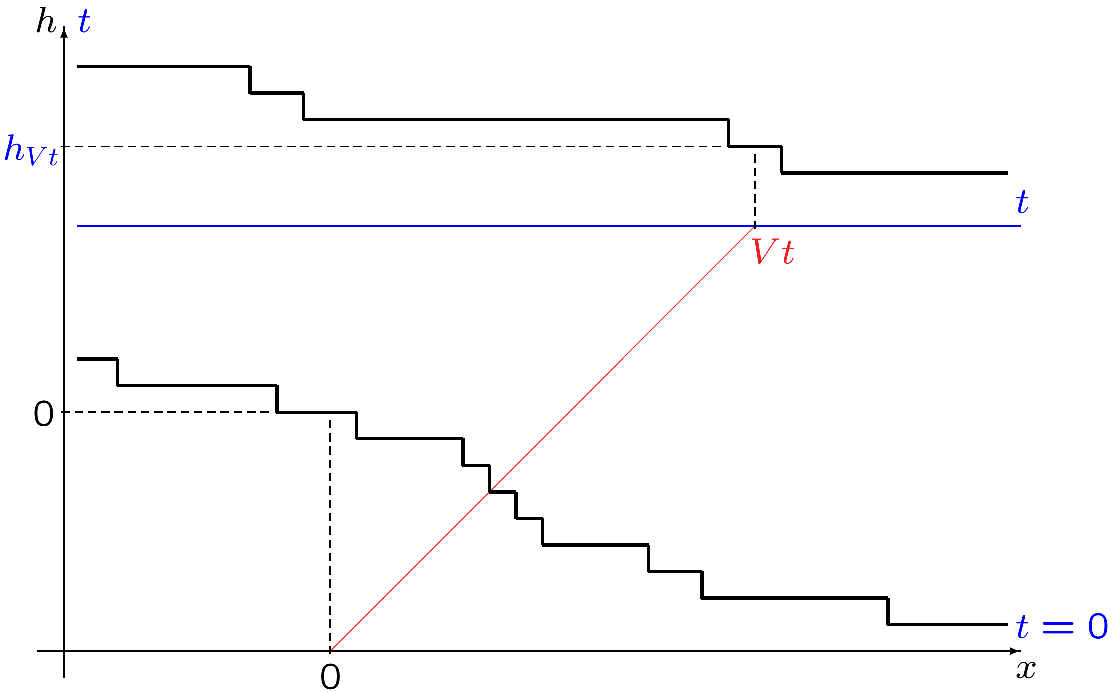
3. Growth fluctuations



3. Growth fluctuations

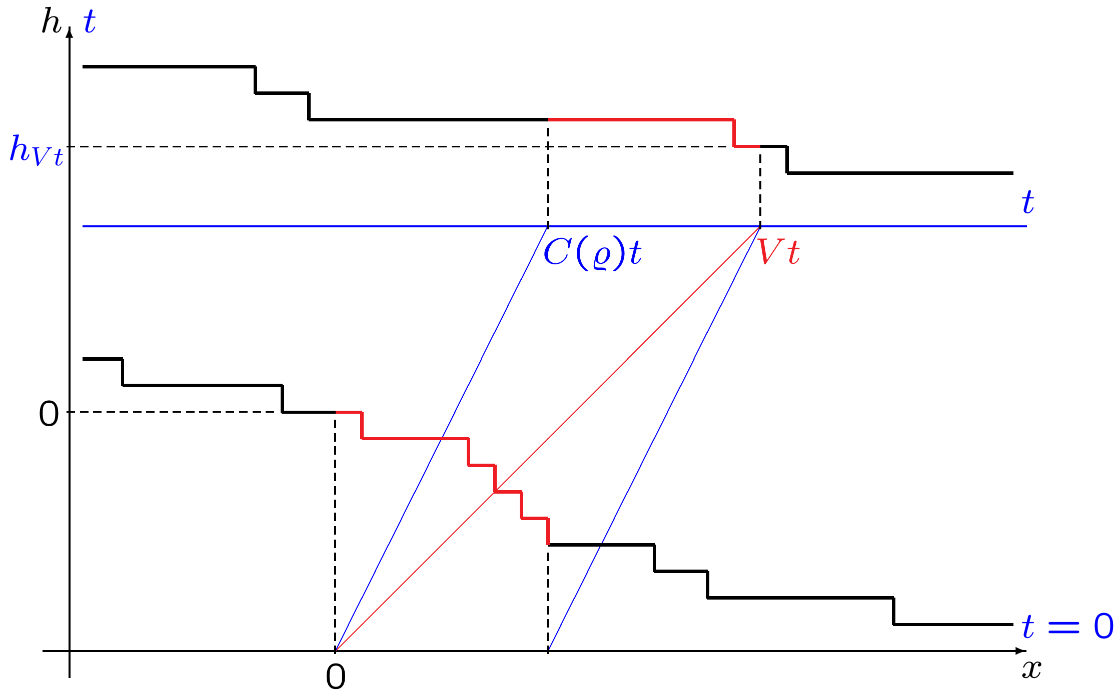


3. Growth fluctuations



Ferrari - Fontes 1994: $\lim_{t \to \infty} \frac{\mathbf{Var}(h_{Vt}(t))}{t} = \text{const} \cdot |V - C(\varrho)|$

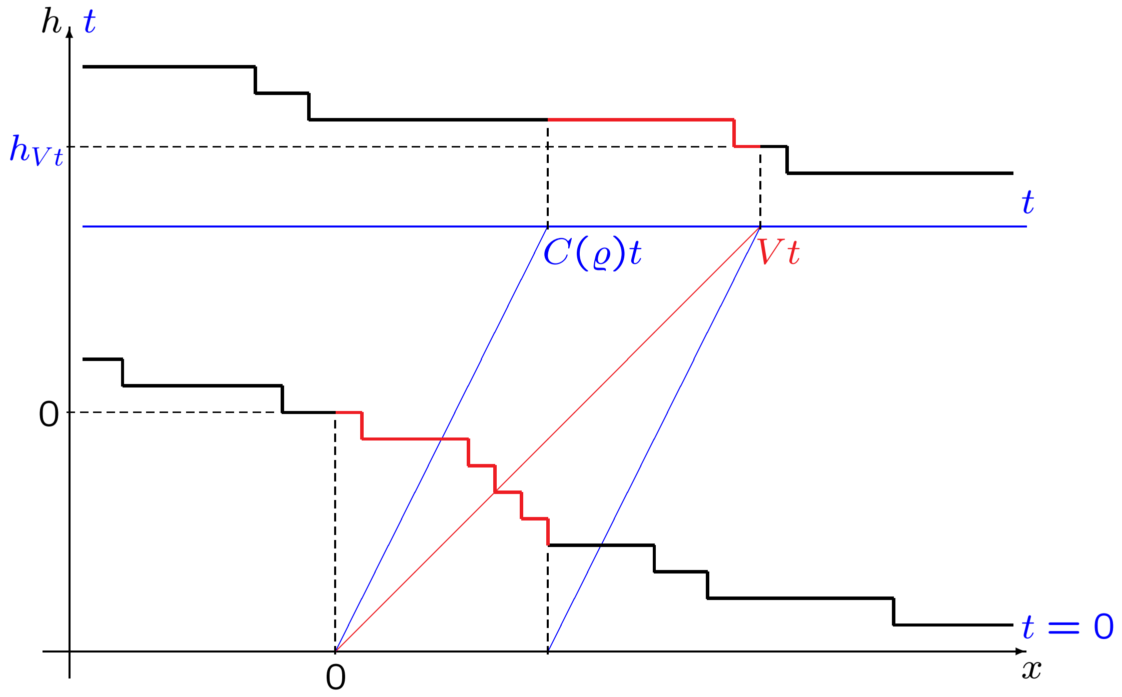
3. Growth fluctuations



Ferrari - Fontes 1994: $\lim_{t\to\infty} \frac{\mathbf{Var}(h_{Vt}(t))}{t} = \text{const} \cdot |V - C(\varrho)|$

$\rightsquigarrow$ Initial fluctuations are transported along the characteristics.

3. Growth fluctuations



Ferrari - Fontes 1994: $\lim\limits_{t\to\infty} \frac{\textbf{Var}(h_{Vt}(t))}{t} = \text{const} \cdot |V - C(\varrho)|$

$\rightsquigarrow$ Initial fluctuations are transported along the characteristics.

$\rightsquigarrow$ How about $V = C(\varrho)$?

3. Growth fluctuations

Conjecture:

$$\lim_{t\to\infty} \frac{\mathbf{Var}(h_{C(\varrho)t}(t))}{t^{2/3}} = [\text{sg. non trivial}].$$

3. Growth fluctuations

Conjecture:

$$\lim_{t\to\infty} \frac{\mathbf{Var}(h_{C(\varrho)t}(t))}{t^{2/3}} = \text{[sg. non trivial]}.$$

Theorem (B., Seppäläinen): For any $0 < \varrho < 1$, and any $q < p$,

$$0 < \liminf_{t\to\infty} \frac{\mathbf{Var}(h_{C(\varrho)t}(t))}{t^{2/3}} \le \limsup_{t\to\infty} \frac{\mathbf{Var}(h_{C(\varrho)t}(t))}{t^{2/3}} < \infty.$$

3. Growth fluctuations

Conjecture:

$$\lim_{t\to\infty} \frac{\mathbf{Var}(h_{C(\varrho)t}(t))}{t^{2/3}} = \text{[sg. non trivial]}.$$

Theorem (B., Seppäläinen): For any $0 < \varrho < 1$, and any $q < p$,

$$0 < \liminf_{t\to\infty} \frac{\mathbf{Var}(h_{C(\varrho)t}(t))}{t^{2/3}} \le \limsup_{t\to\infty} \frac{\mathbf{Var}(h_{C(\varrho)t}(t))}{t^{2/3}} < \infty.$$

Corollary: The corresponding scaling of the diffusivity is also proved.

3. Growth fluctuations

Limit distributions (not yet controlling the second moment) in terms of the Tracy-Widom distribution (GUE random matrices) were found by Baik, Deift and Johansson 1999, Johansson 2000, and Ferrari and Spohn 2006 for the *totally* asymmetric exclusion (TASEP: $p = 1$, $q = 0$).

3. Growth fluctuations

Limit distributions (not yet controlling the second moment) in terms of the Tracy-Widom distribution (GUE random matrices) were found by Baik, Deift and Johansson 1999, Johansson 2000, and Ferrari and Spohn 2006 for the *totally* asymmetric exclusion (TASEP: $p = 1$, $q = 0$).

Method was: Last passage percolation, heavy combinatorics and asymptotic analysis.

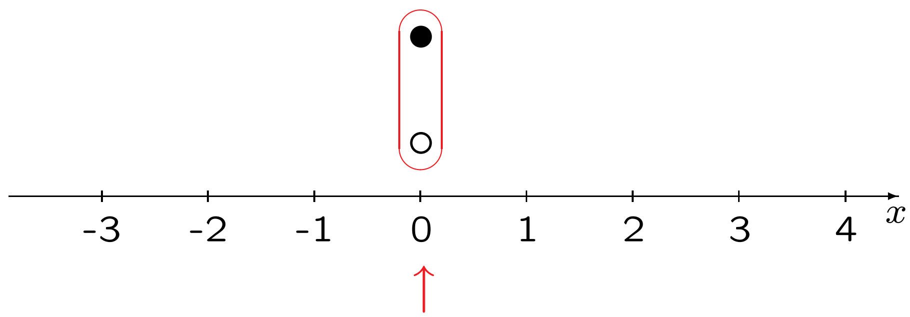
3. Growth fluctuations

Limit distributions (not yet controlling the second moment) in terms of the Tracy-Widom distribution (GUE random matrices) were found by Baik, Deift and Johansson 1999, Johansson 2000, and Ferrari and Spohn 2006 for the *totally* asymmetric exclusion ($TASEP$: $p = 1$, $q = 0$).

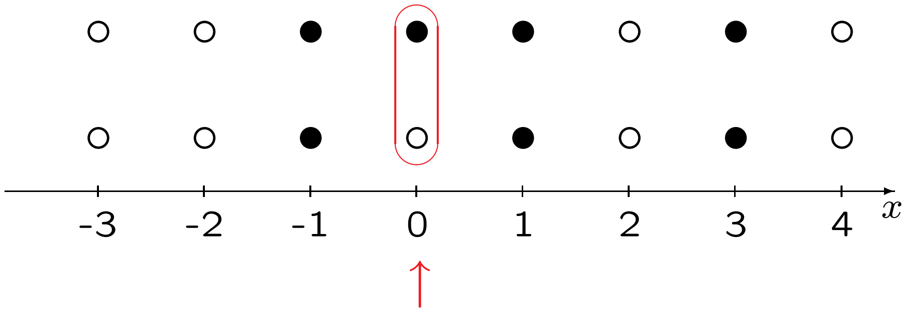
Method was: Last passage percolation, heavy combinatorics and asymptotic analysis.

$\rightsquigarrow$ We needed to get rid of these tools. Premises: Cator and Groeneboom 2006 (Hammersley's process), B., Cator and Seppäläinen 2006 (TASEP, last passage).

4. The second class particle

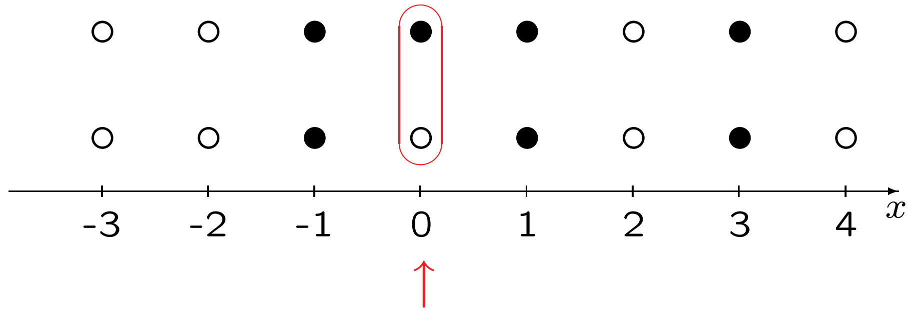


4. The second class particle



Bernoulli(ϱ) distribution except for 0

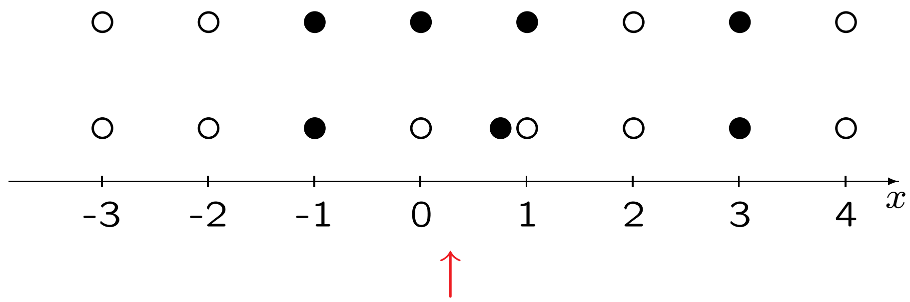
4. The second class particle



Bernoulli(ϱ) distribution except for 0

<u>Coupling:</u> A single discrepancy is always conserved

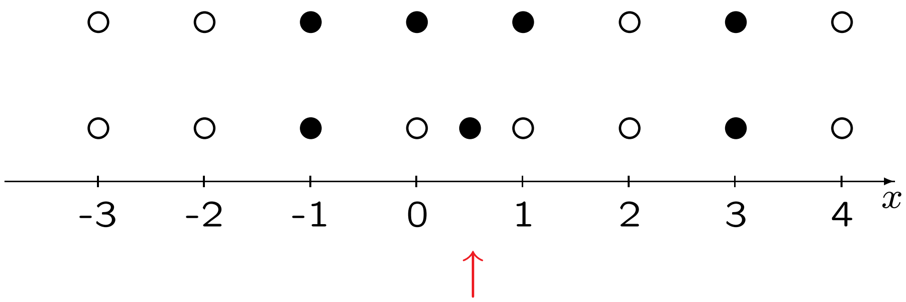
4. The second class particle



Bernoulli(ϱ) distribution except for 0

Coupling: A single discrepancy is always conserved

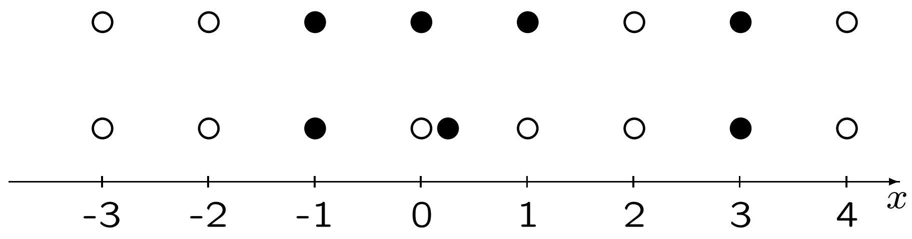
4. The second class particle



Bernoulli(ϱ) distribution except for 0

Coupling: A single discrepancy is always conserved

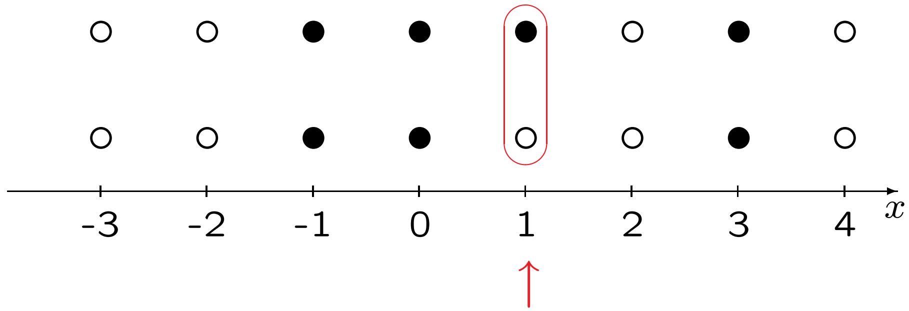
4. The second class particle



Bernoulli(ϱ) distribution except for 0

Coupling: A single discrepancy is always conserved

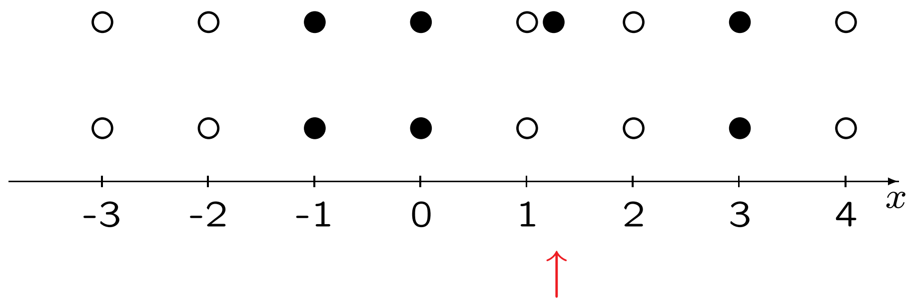
4. The second class particle



Bernoulli(ϱ) distribution except for 0

Coupling: A single discrepancy is always conserved

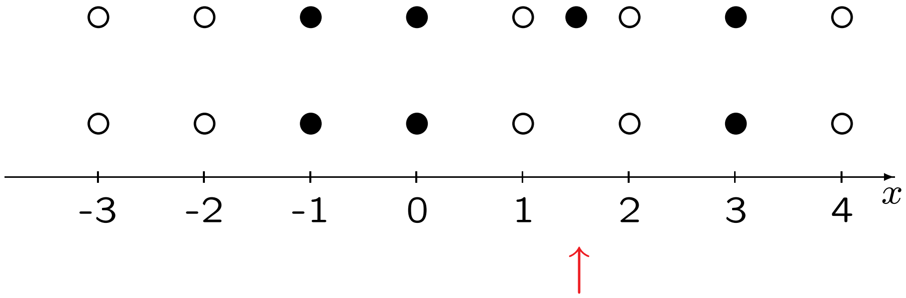
4. The second class particle



Bernoulli(ϱ) distribution except for 0

Coupling: A single discrepancy is always conserved

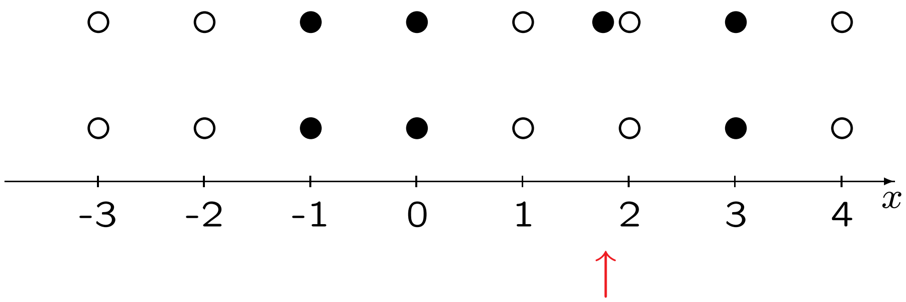
4. The second class particle



Bernoulli(ϱ) distribution except for 0

Coupling: A single discrepancy is always conserved

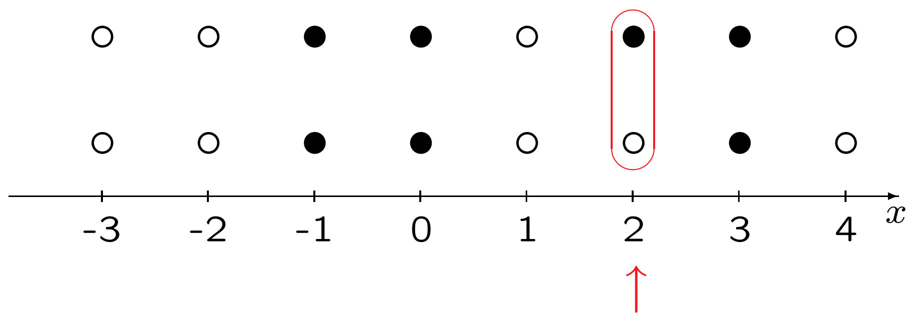
4. The second class particle



Bernoulli(ϱ) distribution except for 0

Coupling: A single discrepancy is always conserved

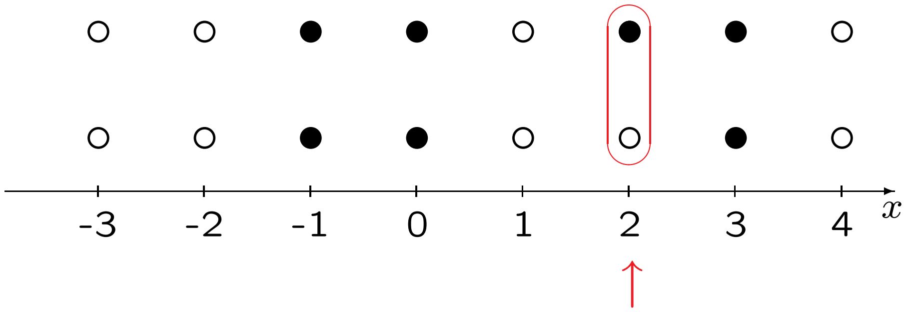
4. The second class particle



Bernoulli(ϱ) distribution except for 0

Coupling: A single discrepancy is always conserved

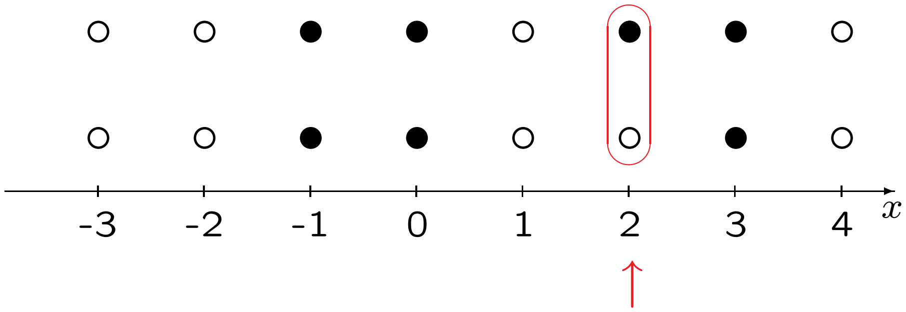
4. The second class particle



Bernoulli(ϱ) distribution except for 0

Coupling: A single discrepancy is always conserved = the second class particle. Its location at time t is $Q(t)$.

4. The second class particle



Bernoulli(ϱ) distribution except for 0

Coupling: A single discrepancy is always conserved = the second class particle. Its location at time t is $Q(t)$.

The second class particle is a highly nontrivial object. For example, the Bernoulli(ϱ) distribution is *not* stationary as seen by the second class particle.

4. The second class particle

Theorem:

$$\mathbf{E}(Q(t)) = C(\varrho)t$$

(characteristic speed),

4. The second class particle

Theorem:

$$\mathbf{E}(Q(t)) = C(\varrho)t$$

(characteristic speed), and

$$\textbf{Var}(h_{Vt}(t)) = \text{const} \cdot \mathbf{E}|Vt - Q(t)|.$$

4. The second class particle

Theorem:

$$\mathbf{E}(Q(t)) = C(\varrho)t$$

(characteristic speed), and

$$\textbf{Var}(h_{Vt}(t)) = \text{const} \cdot \mathbf{E}|Vt - Q(t)|.$$

Method of proof: martingale arguments, time-reversal, and conservation of particles.

4. The second class particle

Theorem:

$$\mathbf{E}(Q(t)) = C(\varrho)t$$

(characteristic speed), and

$$\textbf{Var}(h_{Vt}(t)) = \text{const} \cdot \mathbf{E}|Vt - Q(t)|.$$

Method of proof: martingale arguments, time-reversal, and conservation of particles.

The proof is based on ideas of Bálint Tóth, he said these ideas were standard.

Main idea for prooving $t^{1/3}$ scaling:

$$
\begin{tikzcd}
\boxed{\text{Deviations of } Q(t)} \arrow[r, bend left, "\text{Couplings}"] & \boxed{\text{Deviations of } h_{C(\varrho)}(t)} \arrow[l, bend left, "\text{The above thm.}"]
\end{tikzcd}
$$

The coupling measure

Let $\lambda < \varrho$, and

$$\mu\binom{\circ}{\circ} = 1 - \varrho, \quad \mu\binom{\bullet}{\circ} = \varrho - \lambda, \quad \mu\binom{\bullet}{\bullet} = \lambda.$$

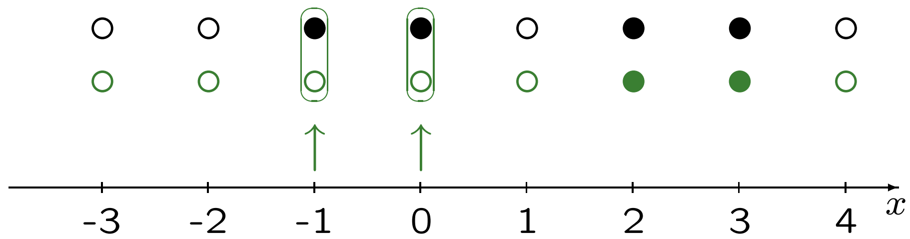
The coupling measure

Let $\lambda < \varrho$, and

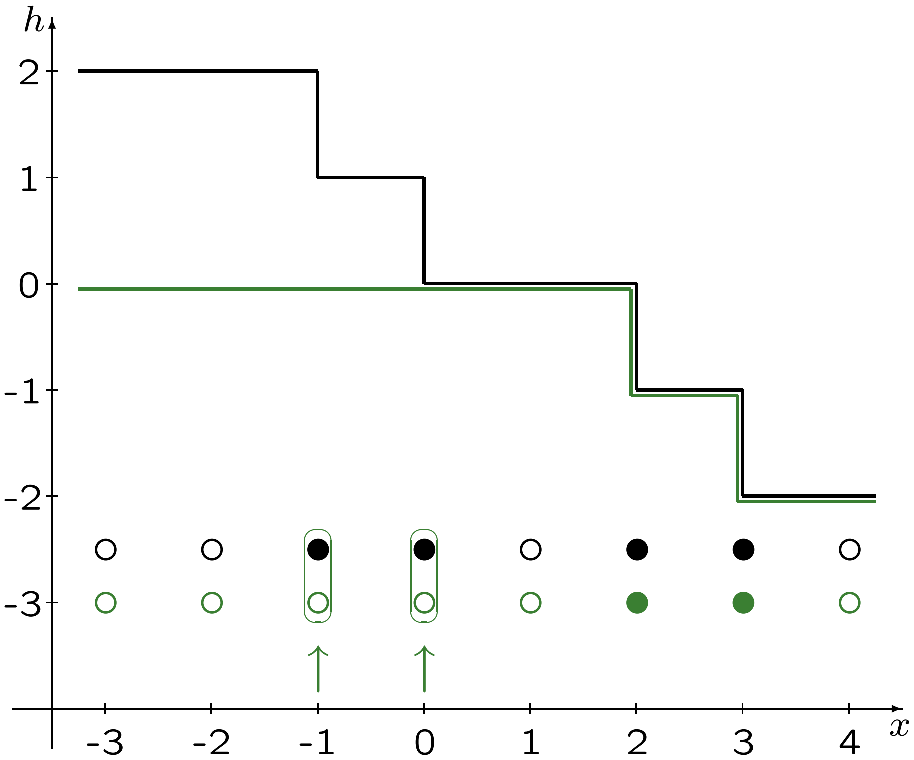
$$\mu\binom{\circ}{\circ} = 1 - \varrho, \quad \mu\binom{\bullet}{\circ} = \varrho - \lambda, \quad \mu\binom{\bullet}{\bullet} = \lambda.$$

Then the "upper" marginal is Bernoulli(ϱ), and the "lower" marginal is Bernoulli(λ).

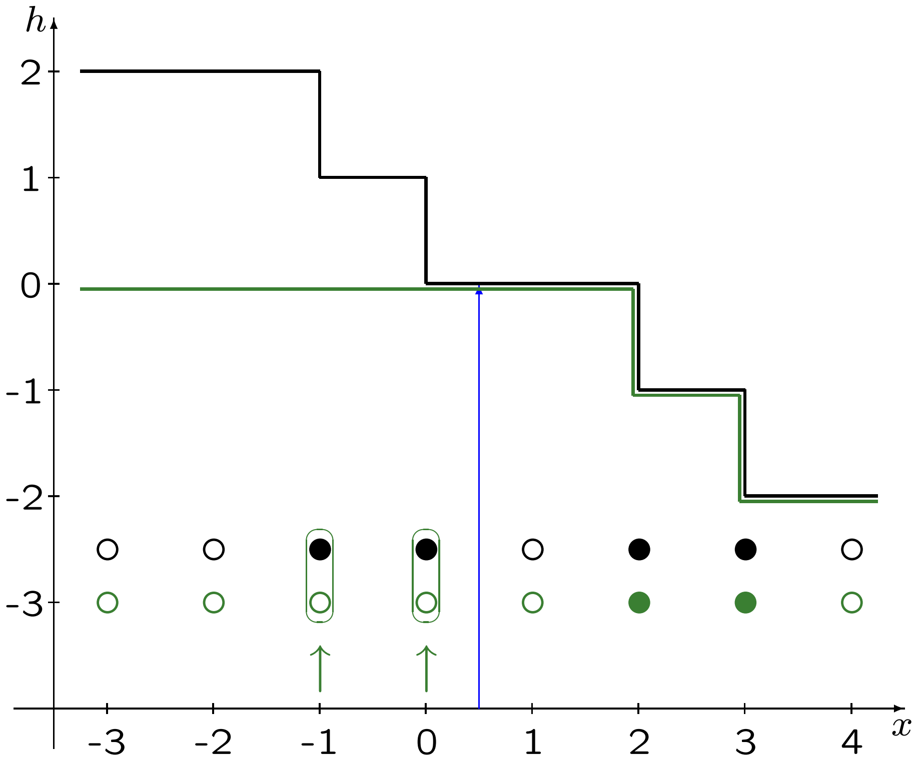
According to the product of μ's:



According to the product of μ's:

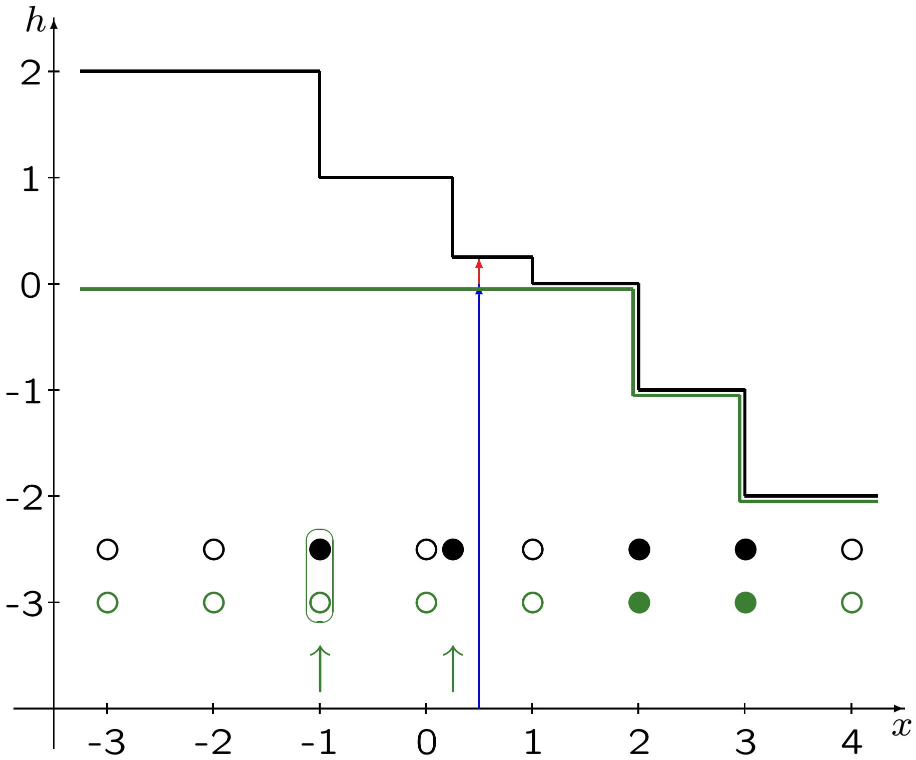


According to the product of μ's:



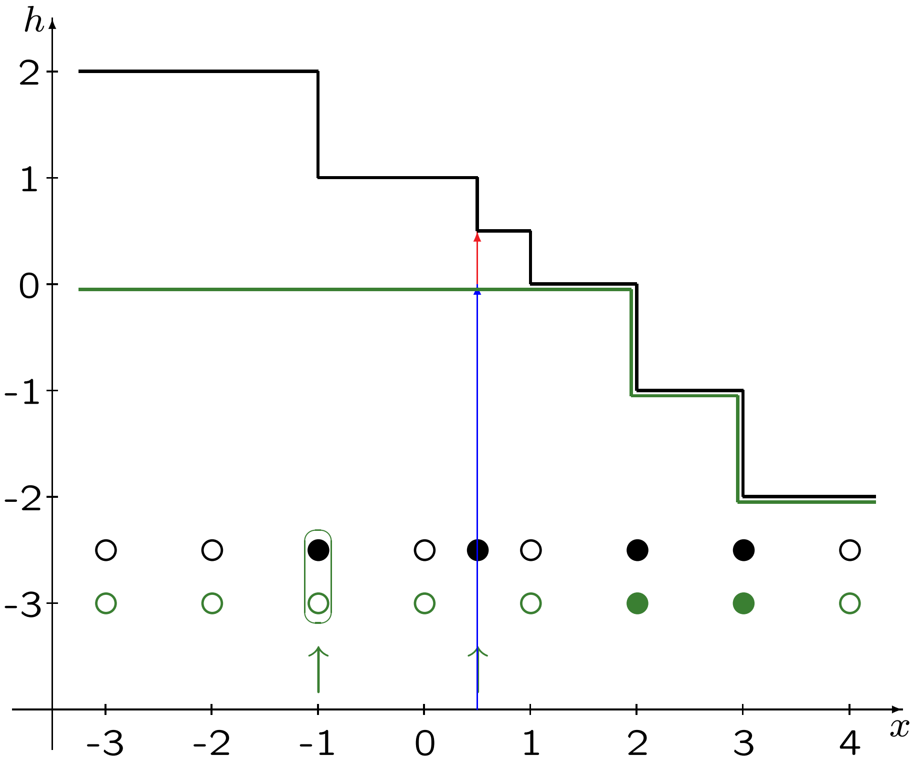
$h_{Vt}(t) - h_{Vt}(t)$ = the net number of $\uparrow$'s passed through the moving window at Vt $(V \in \mathbb{R})$.

According to the product of μ's:



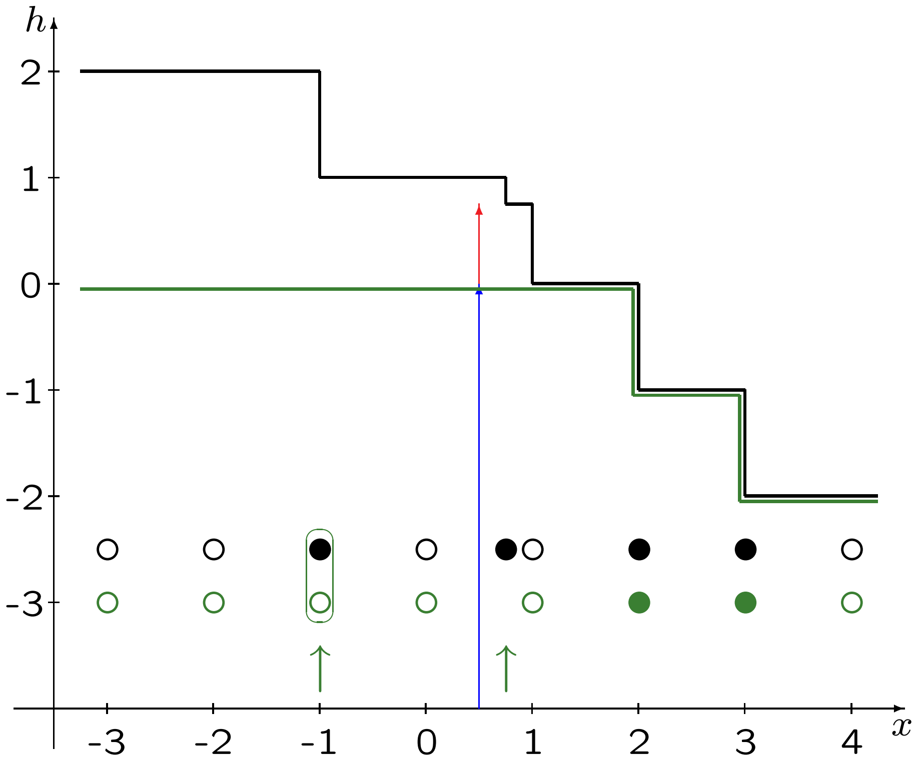
$h_{Vt}(t) - h_{Vt}(t) =$ the net number of $\uparrow$'s passed through the moving window at $Vt \quad (V \in \mathbb{R})$.

According to the product of μ's:



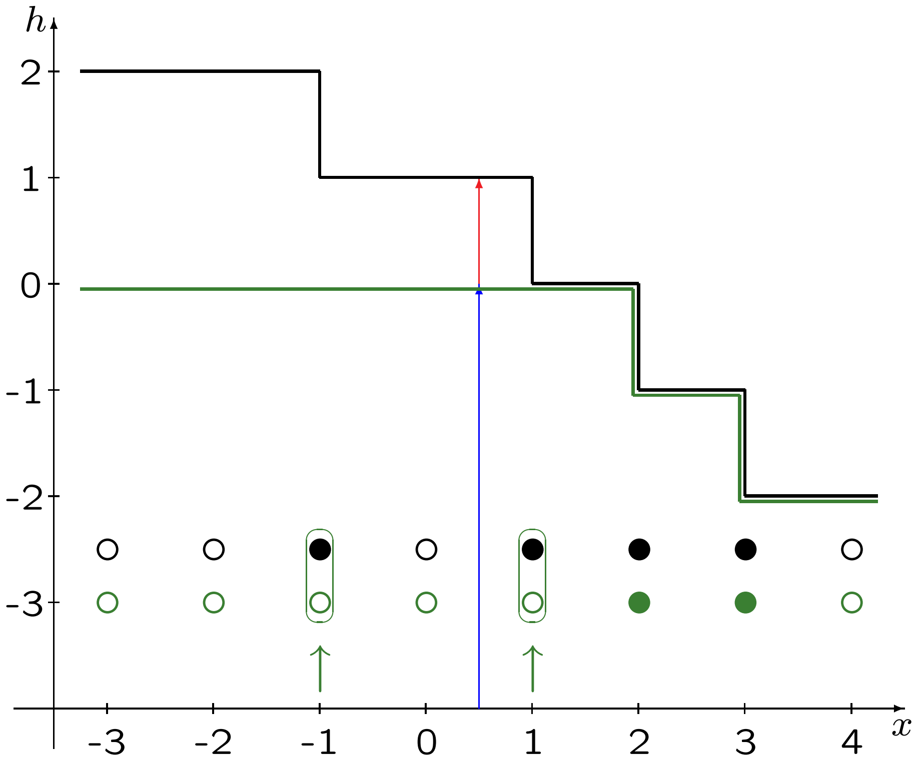
$h_{Vt}(t) - h_{Vt}(t) =$ the net number of $\uparrow$'s passed through the moving window at Vt $(V \in \mathbb{R})$.

According to the product of μ's:



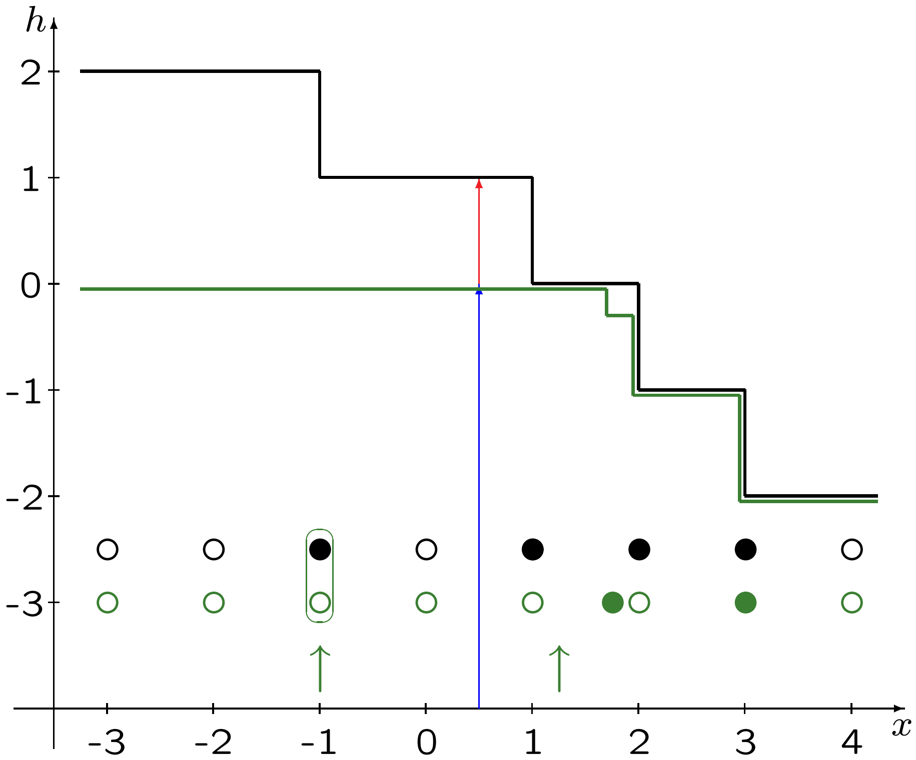
$h_{Vt}(t) - h_{Vt}(t) =$ the net number of $\uparrow$'s passed through the moving window at $Vt \quad (V \in \mathbb{R})$.

According to the product of μ's:



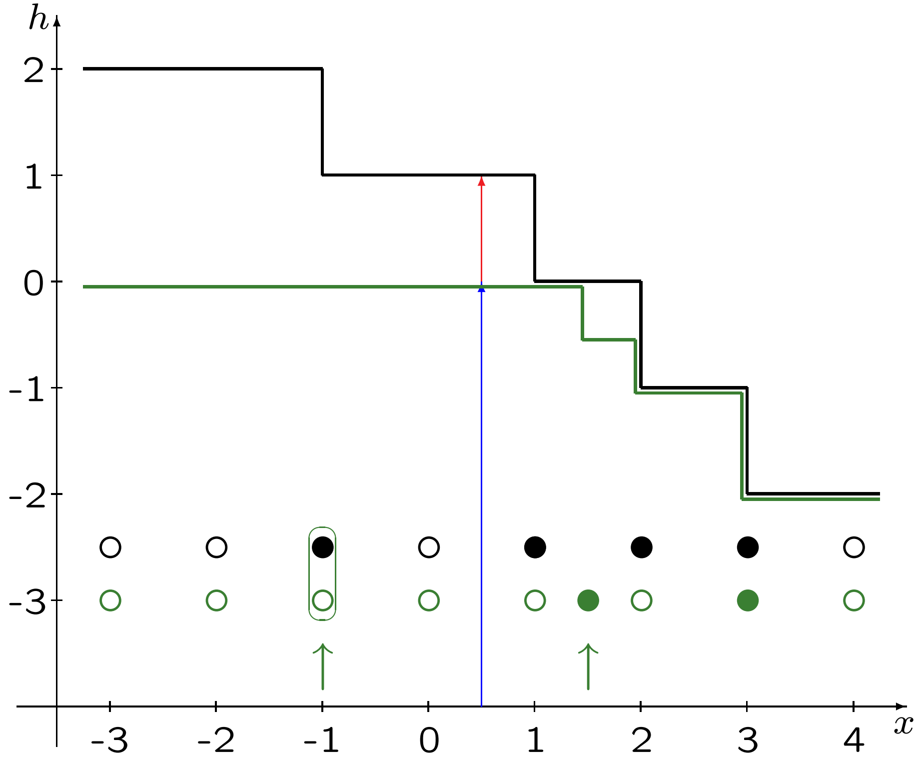
$h_{Vt}(t) - h_{Vt}(t) =$ the net number of $\uparrow$'s passed through the moving window at Vt $(V \in \mathbb{R})$.

According to the product of μ's:



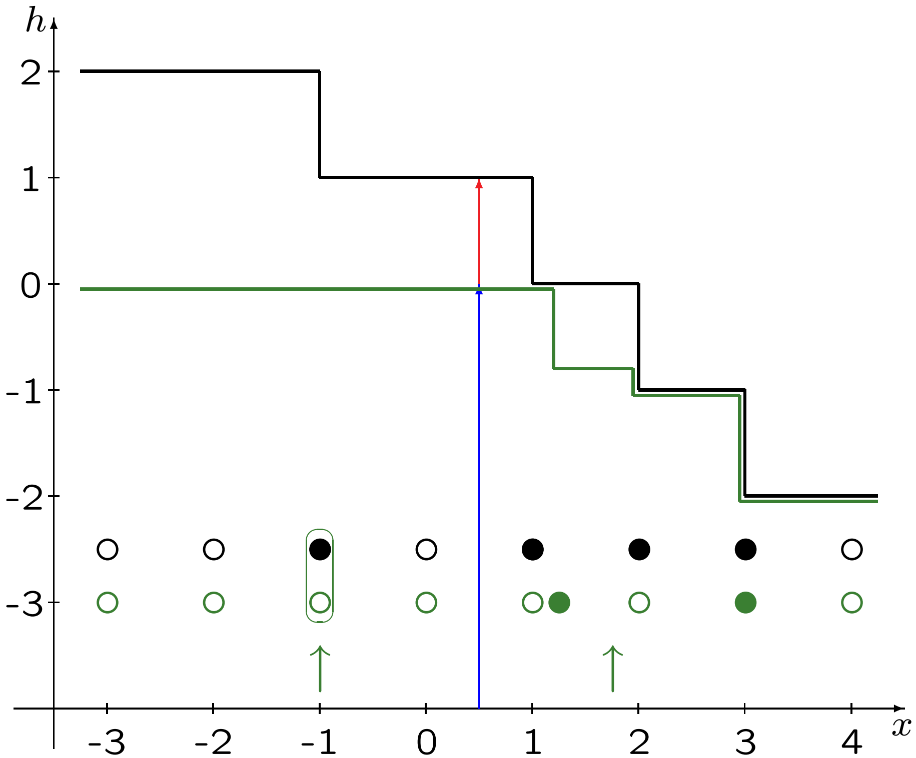
$h_{Vt}(t) - h_{Vt}(t) =$ the net number of ↑'s passed through the moving window at $Vt \quad (V \in \mathbb{R})$.

According to the product of μ's:



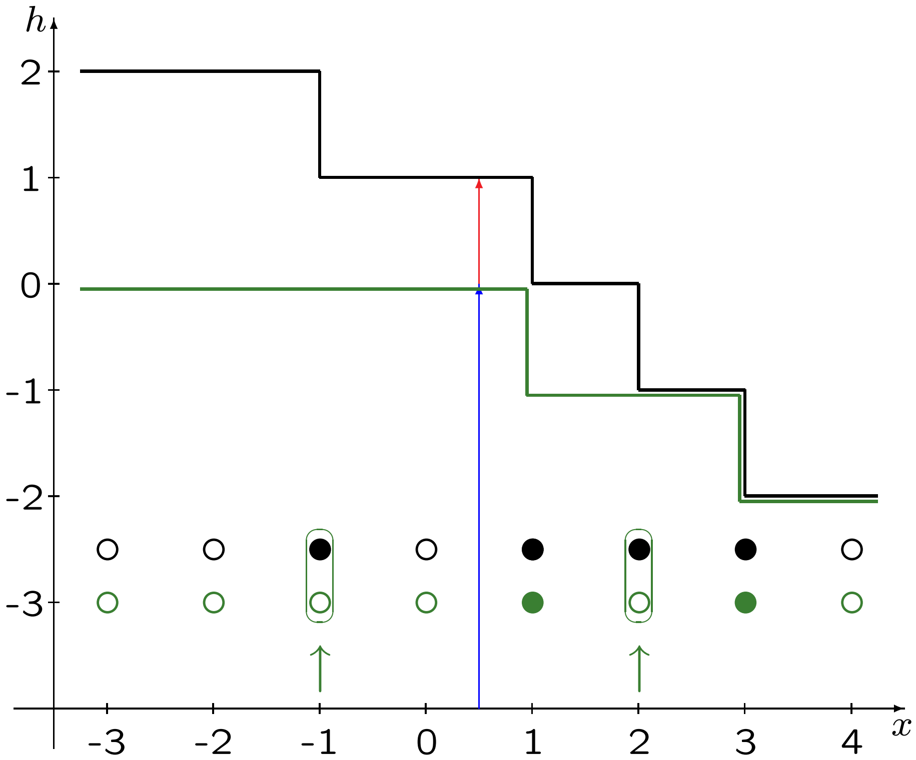
$h_{Vt}(t) - h_{Vt}(t)$ = the net number of $\uparrow$'s passed through the moving window at Vt $(V \in \mathbb{R})$.

According to the product of μ's:



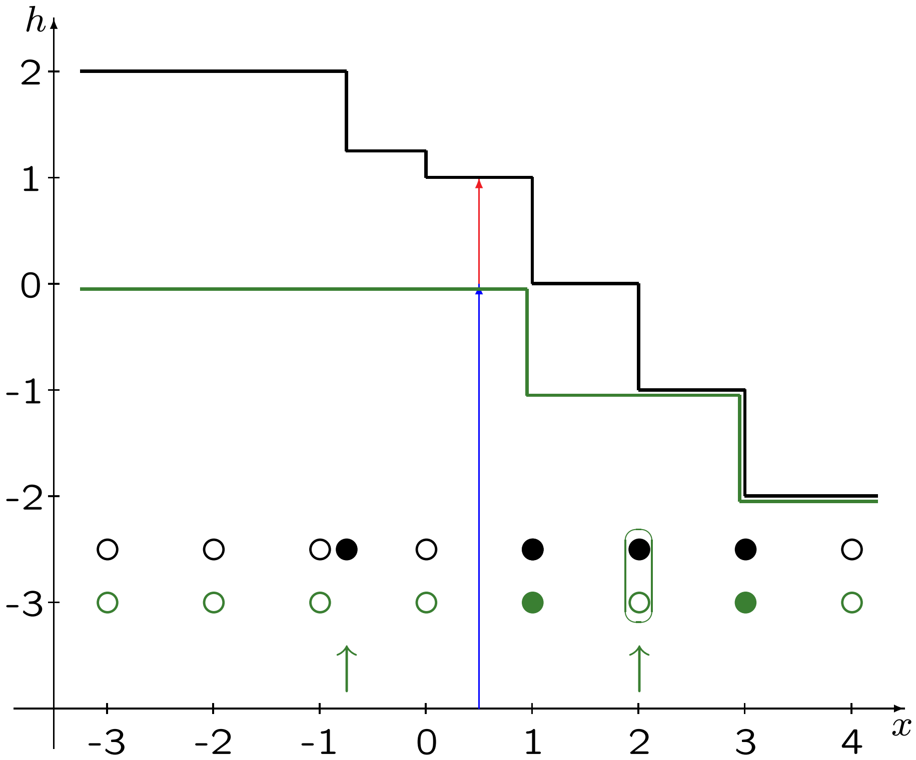
$h_{Vt}(t) - h_{Vt}(t) =$ the net number of $\uparrow$'s passed through the moving window at $Vt \quad (V \in \mathbb{R})$.

According to the product of μ's:



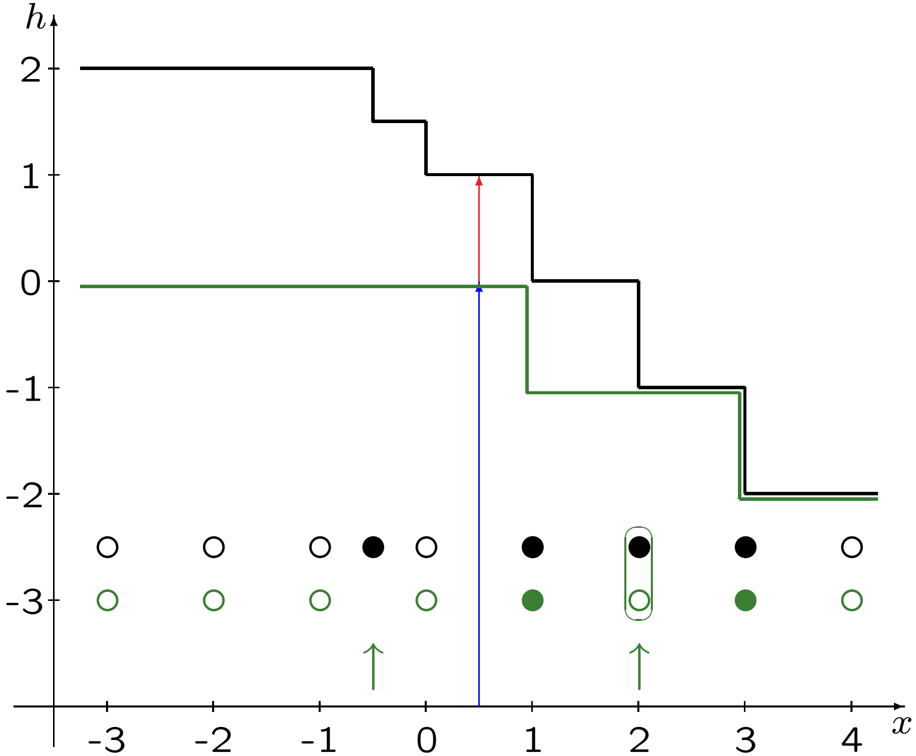
$h_{Vt}(t) - h_{Vt}(t) =$ the net number of $\uparrow$'s passed through the moving window at $Vt \quad (V \in \mathbb{R})$.

According to the product of μ's:



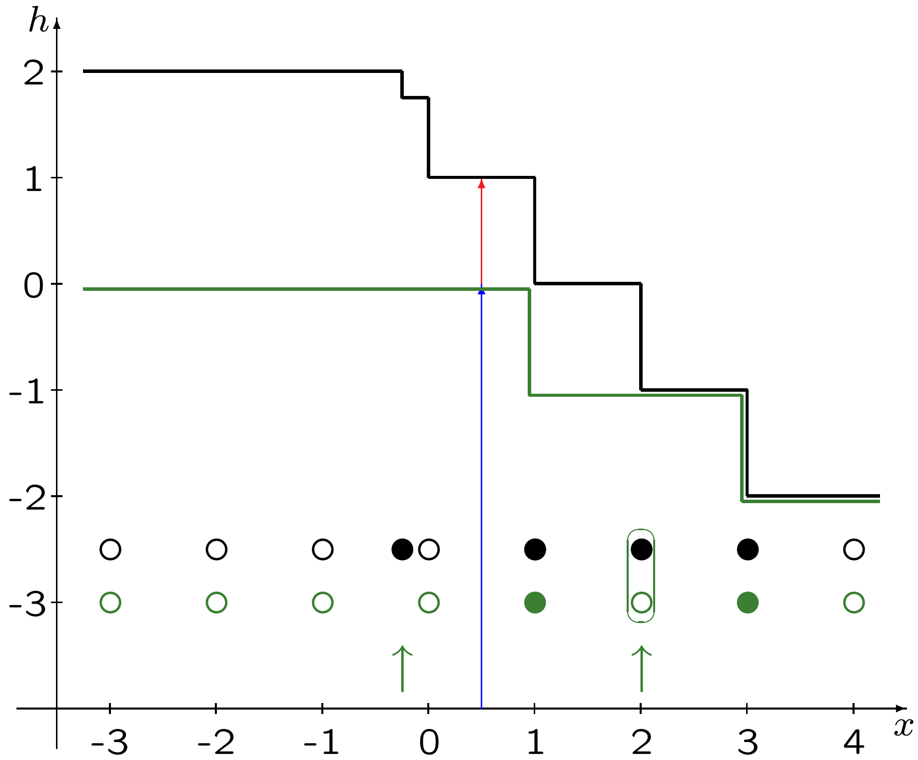
$h_{Vt}(t) - h_{Vt}(t) =$ the net number of $\uparrow$'s passed through the moving window at Vt $(V \in \mathbb{R})$.

According to the product of μ's:



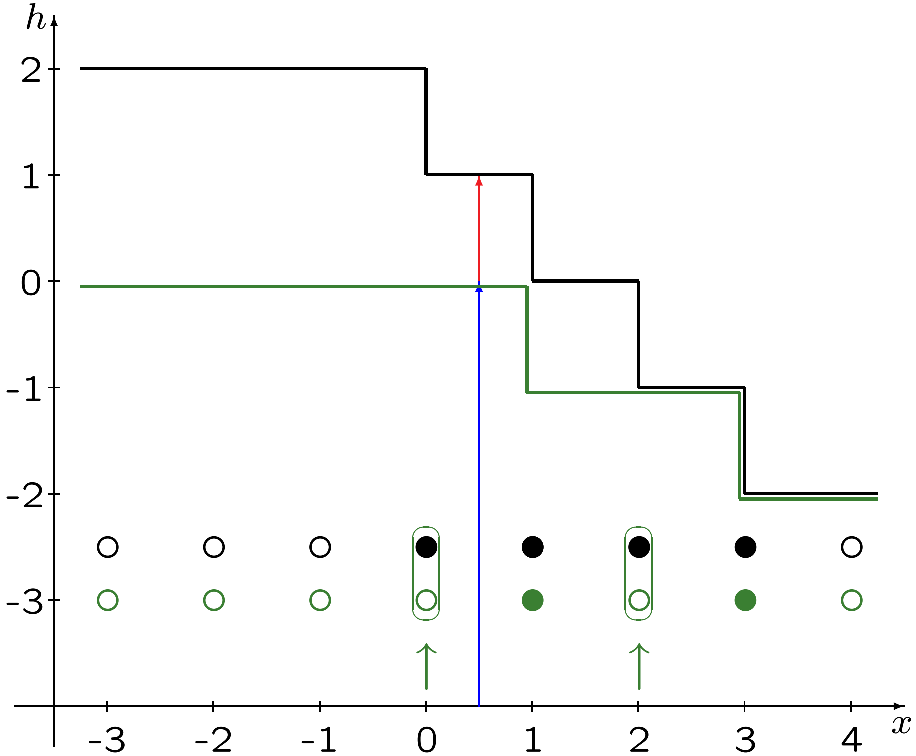
$h_{Vt}(t) - h_{Vt}(t)$ = the net number of $\uparrow$'s passed through the moving window at Vt $(V \in \mathbb{R})$.

According to the product of μ's:



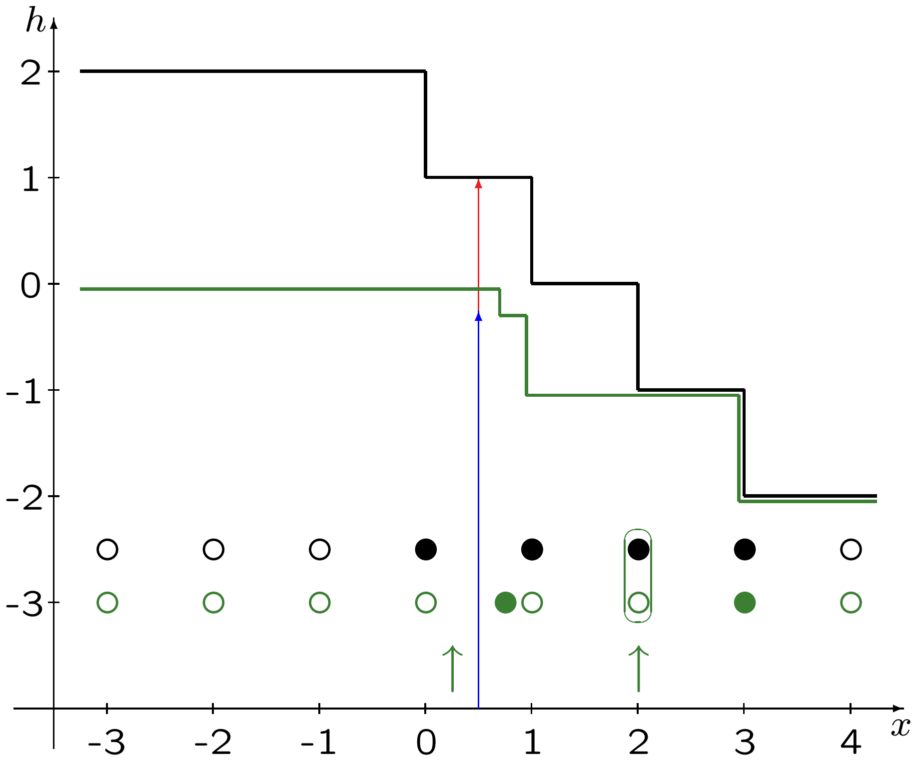
$h_{Vt}(t) - h_{Vt}(t) =$ the net number of $\uparrow$'s passed through the moving window at Vt $(V \in \mathbb{R})$.

According to the product of μ's:



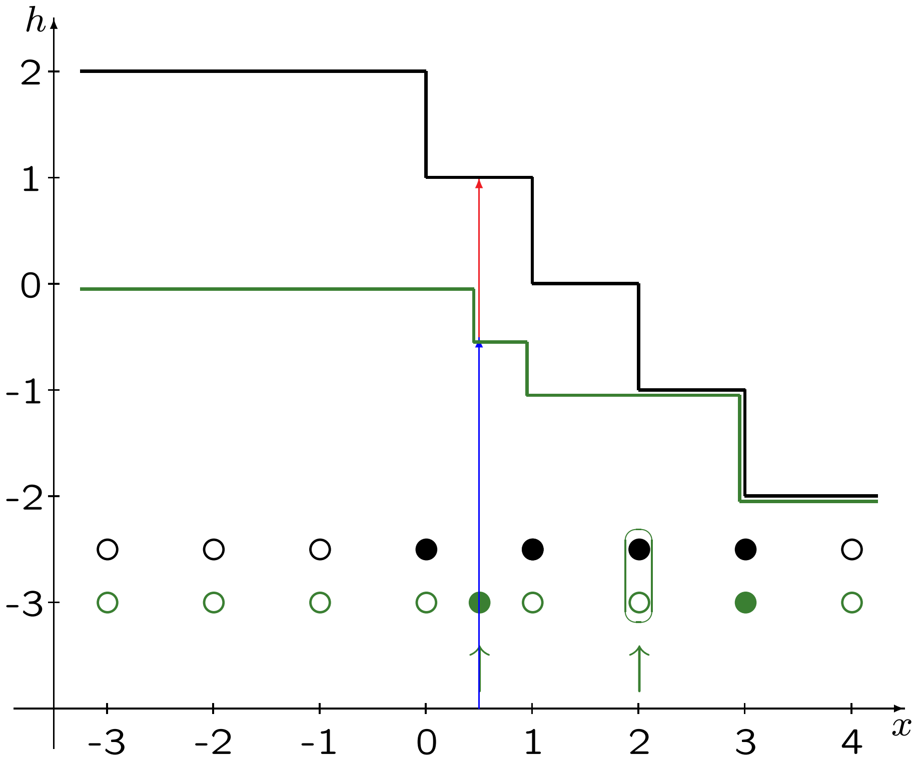
$h_{Vt}(t) - h_{Vt}(t) =$ the net number of $\uparrow$'s passed through the moving window at $Vt \quad (V \in \mathbb{R})$.

According to the product of μ's:



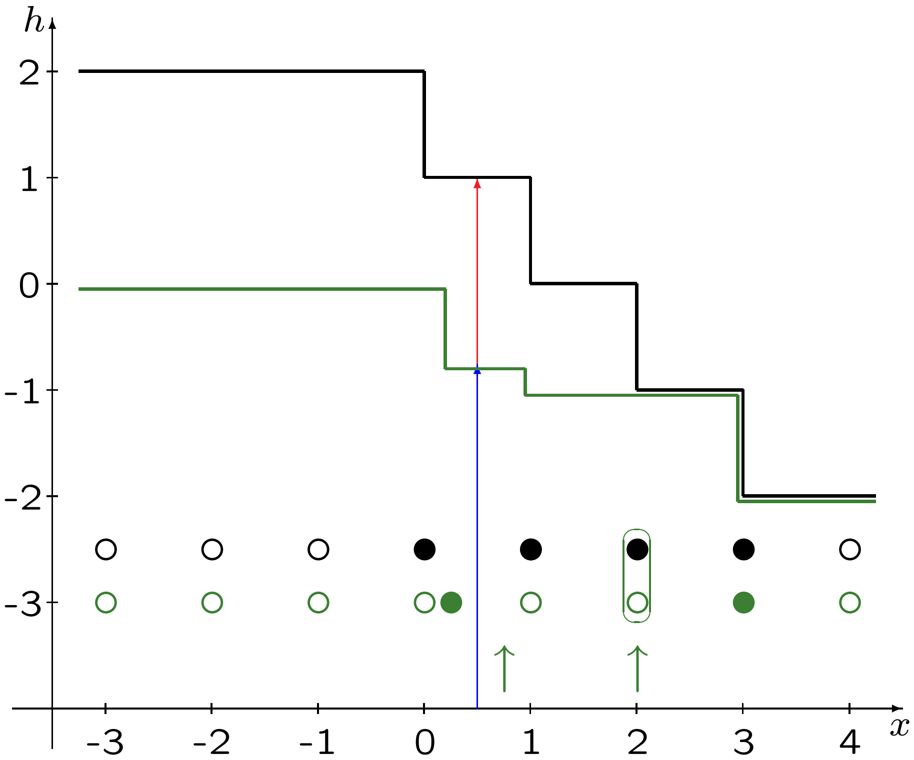
$h_{Vt}(t) - h_{Vt}(t)$ = the net number of $\uparrow$'s passed through the moving window at Vt $(V \in \mathbb{R})$.

According to the product of μ's:



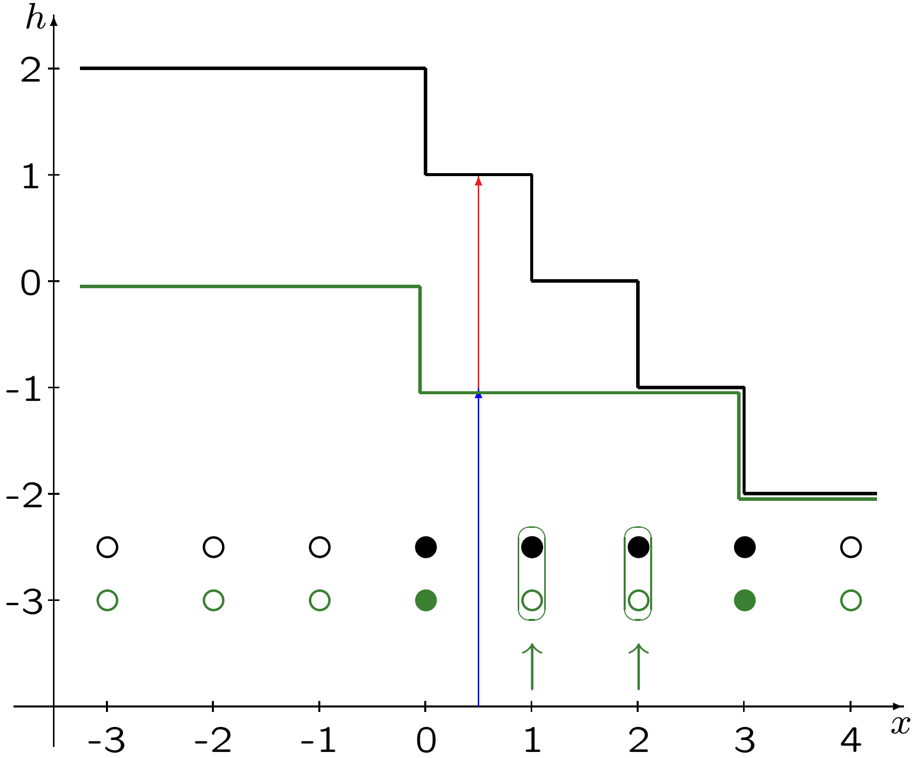
$h_{Vt}(t) - h_{Vt}(t)$ = the net number of $\uparrow$'s passed through the moving window at Vt $(V \in \mathbb{R})$.

According to the product of μ's:



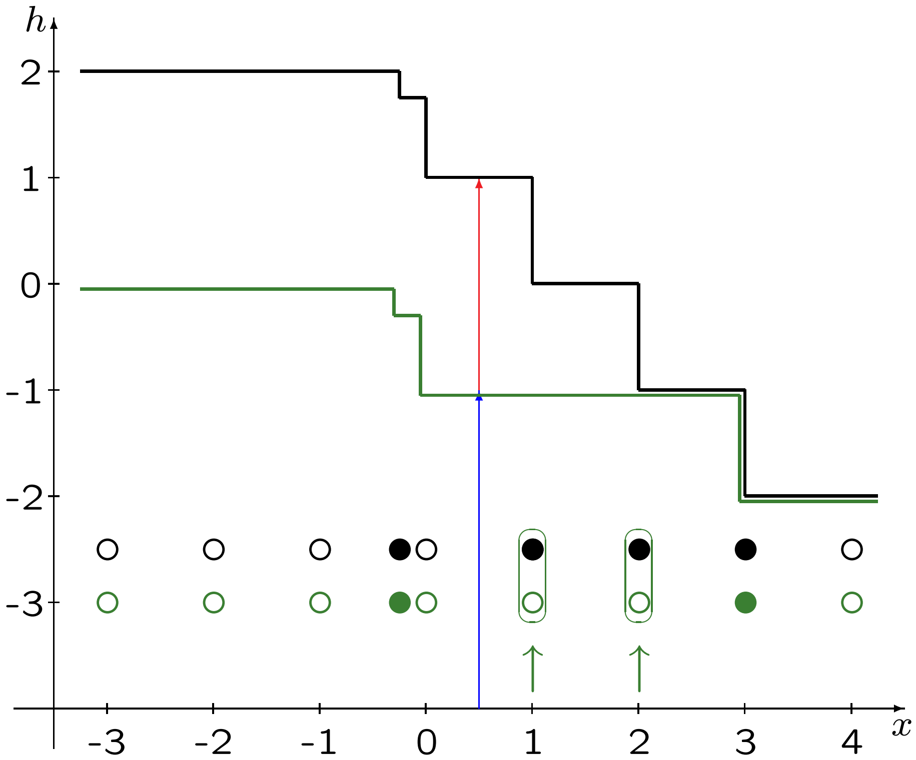
$h_{Vt}(t) - h_{Vt}(t)$ = the net number of $\uparrow$'s passed through the moving window at Vt $(V \in \mathbb{R})$.

According to the product of μ's:



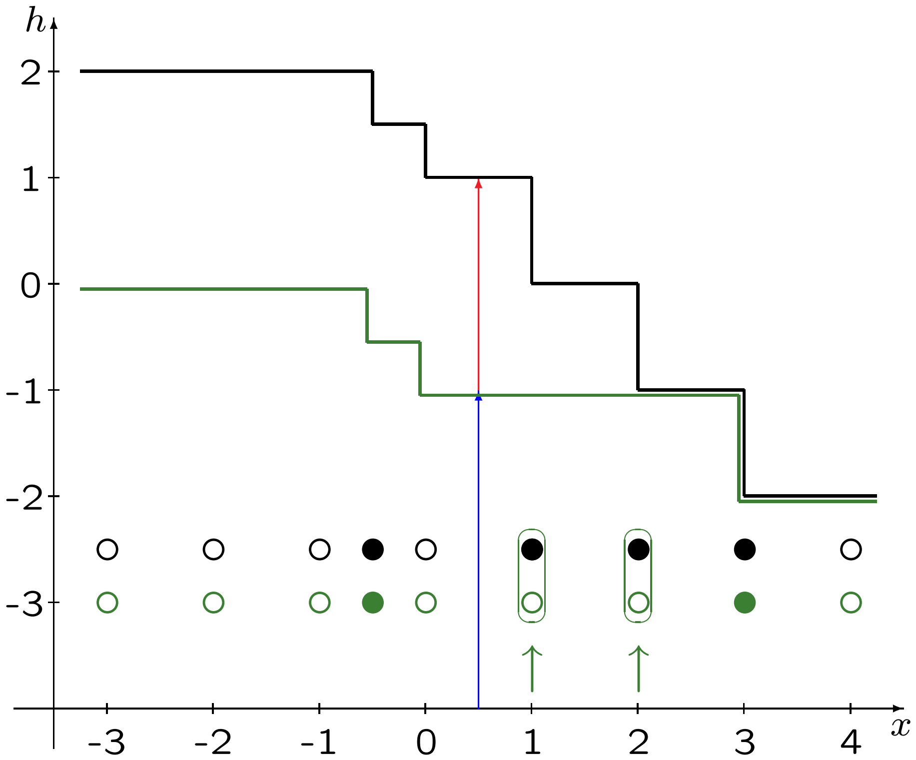
$h_{Vt}(t) - h_{Vt}(t) =$ the net number of $\uparrow$'s passed through the moving window at Vt $(V \in \mathbb{R})$.

According to the product of μ's:



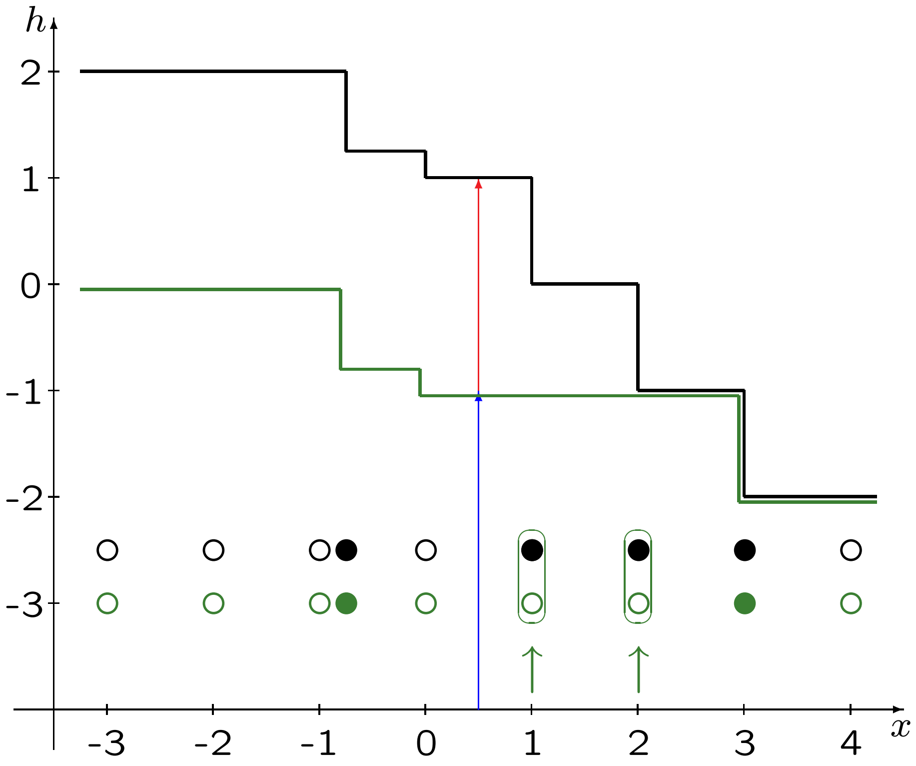
$h_{Vt}(t) - h_{Vt}(t) =$ the net number of $\uparrow$'s passed through the moving window at Vt $(V \in \mathbb{R})$.

According to the product of μ's:



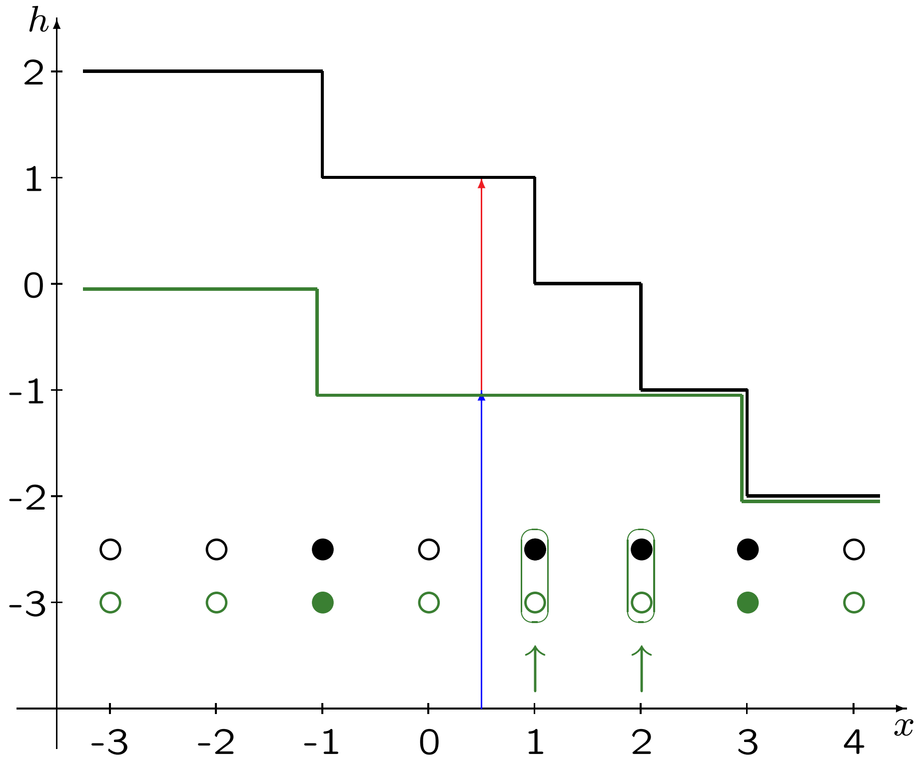
$h_{Vt}(t) - h_{Vt}(t) =$ the net number of $\uparrow$'s passed through the moving window at Vt $(V \in \mathbb{R})$.

According to the product of μ's:



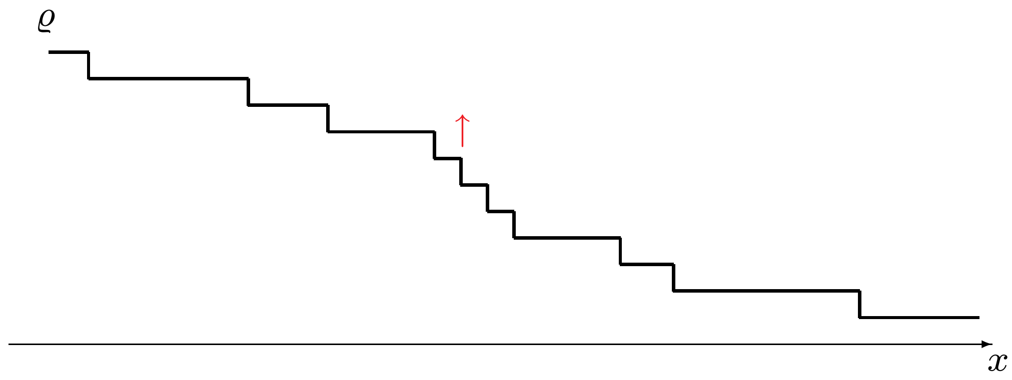
$h_{Vt}(t) - h_{Vt}(t)$ = the net number of $\uparrow$'s passed through the moving window at Vt $(V \in \mathbb{R})$.

According to the product of μ's:

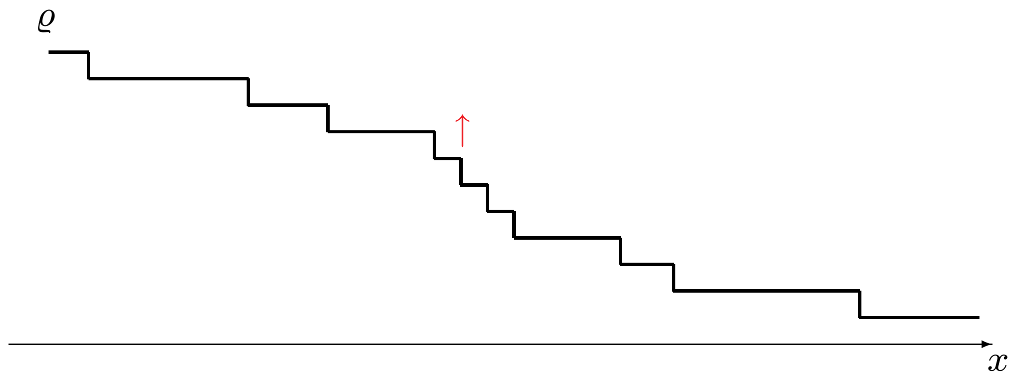


$h_{Vt}(t) - h_{Vt}(t) =$ the net number of $\uparrow$'s passed through the moving window at Vt $(V \in \mathbb{R})$.

5. The upper bound

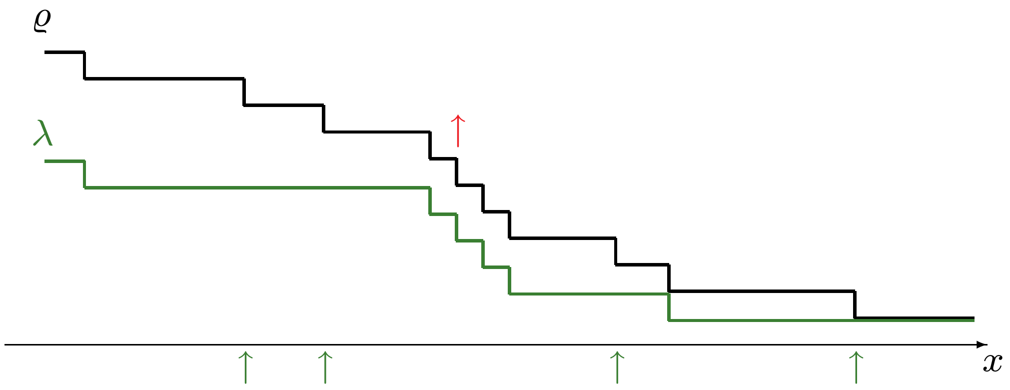


5. The upper bound



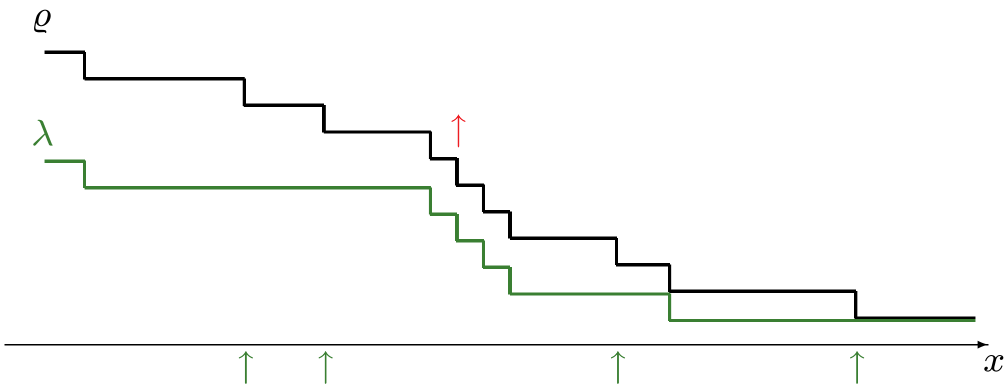
Connect $Q(t)$

5. The upper bound



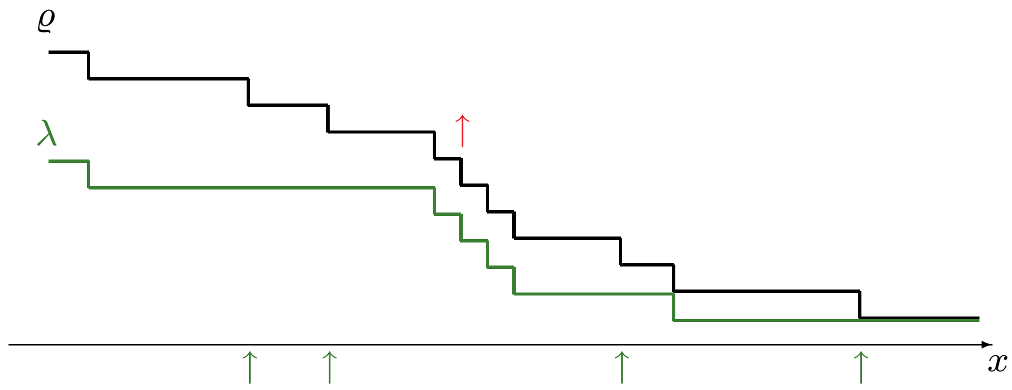
Connect $Q(t)$ with the $\uparrow$'s

5. The upper bound



Connect $Q(t)$ with the $\uparrow$'s (this needs nontrivial couplings):

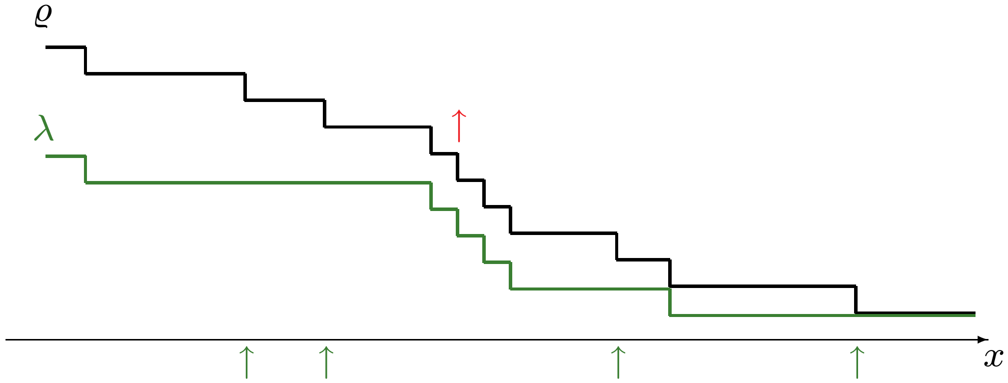
5. The upper bound



Connect $Q(t)$ with the $\uparrow$'s (this needs nontrivial couplings):

$$\mathbf{P}\{Q(t) \text{ is too large}\}$$

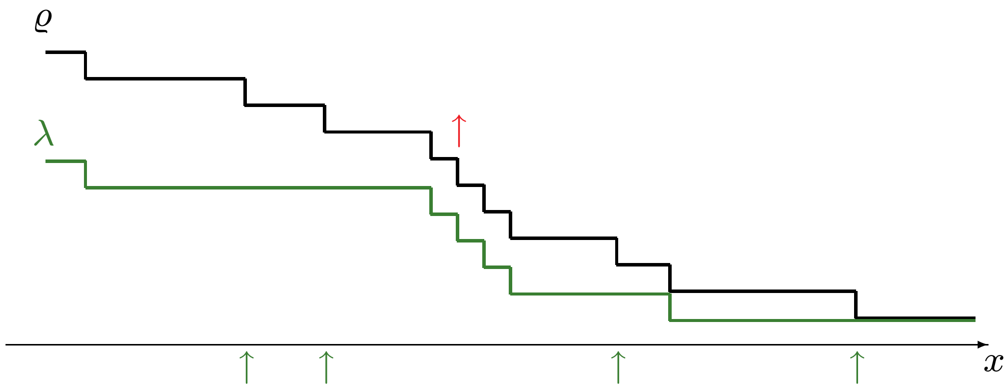
5. The upper bound



Connect $Q(t)$ with the $\uparrow$'s (this needs nontrivial couplings):

$$\mathbf{P}\{Q(t) \text{ is too large}\} \leq \mathbf{P}\{\text{too many } \uparrow\text{'s have crossed } C(\varrho)t\}$$

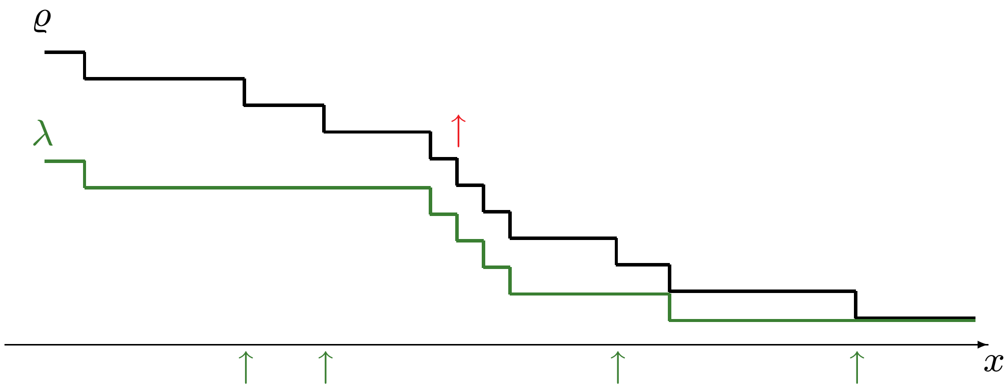
5. The upper bound



Connect $Q(t)$ with the $\uparrow$'s (this needs nontrivial couplings):

$$\begin{aligned}\mathbf{P}\{Q(t) \text{ is too large}\} &\leq \mathbf{P}\{\text{too many } \uparrow\text{'s have crossed } C(\varrho)t\} \\ &\leq \mathbf{P}\{h_{C(\varrho)t}(t) - h_{C(\varrho)t}(t) \text{ is too large}\}.\end{aligned}$$

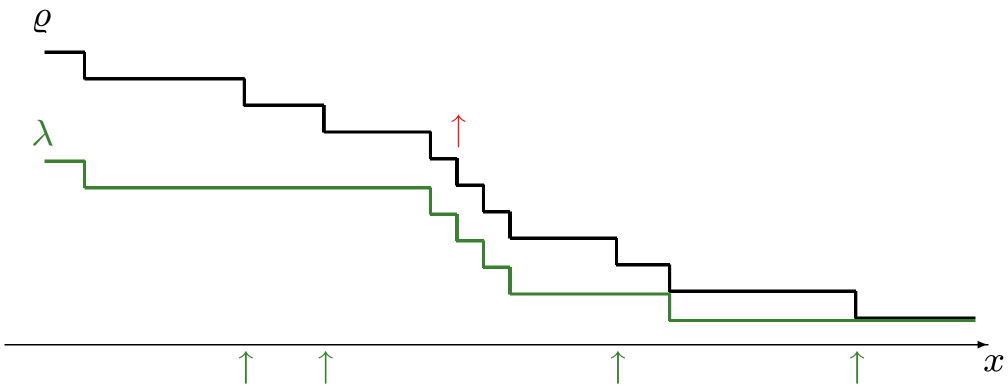
5. The upper bound



Connect $Q(t)$ with the $\uparrow$'s (this needs nontrivial couplings):

$$
\begin{aligned}
\mathbf{P}\{Q(t) \text{ is too large}\} &\le \mathbf{P}\{\text{too many } \uparrow\text{'s have crossed } C(\varrho)t\} \\
&\le \mathbf{P}\{h_{C(\varrho)t}(t) - h_{C(\varrho)t}(t) \text{ is too large}(\lambda)\}.
\end{aligned}
$$

5. The upper bound

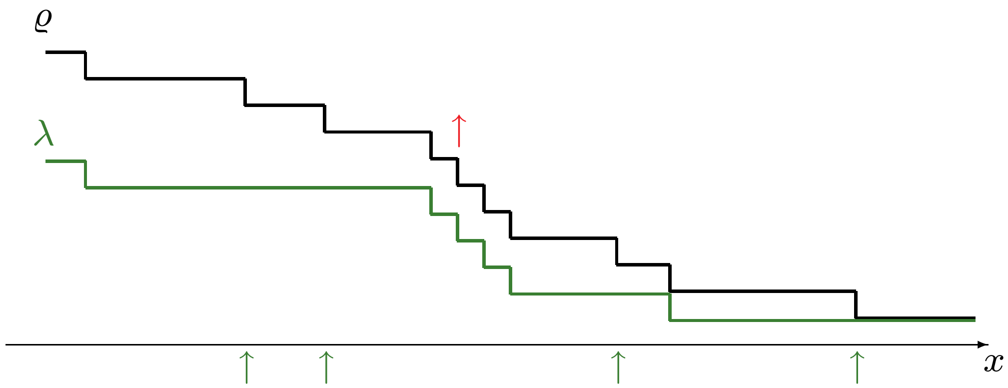


Connect $Q(t)$ with the $\uparrow$'s (this needs nontrivial couplings):

$$\begin{aligned}\mathbf{P}\{Q(t) \text{ is too large}\} &\le \mathbf{P}\{\text{too many } \uparrow\text{'s have crossed } C(\varrho)t\}\\ &\le \mathbf{P}\{h_{C(\varrho)t}(t) - h_{C(\varrho)t}(t) \text{ is too large}(\lambda)\}.\end{aligned}$$

Optimize "too large(λ)" in λ,

5. The upper bound



Connect $Q(t)$ with the $\uparrow$'s (this needs nontrivial couplings):

$$\begin{aligned}\mathbf{P}\{Q(t) \text{ is too large}\} &\le \mathbf{P}\{\text{too many } \uparrow\text{'s have crossed } C(\varrho)t\}\\ &\le \mathbf{P}\{h_{C(\varrho)t}(t) - h_{C(\varrho)t}(t) \text{ is too large}(\lambda)\}.\end{aligned}$$

Optimize "too large(λ)" in λ, use Chebyshev's inequality and relate **Var**$(h_{C(\varrho)t}(t))$ to **Var**$(h_{C(\varrho)t}(t))$.

The computations result in

$$\mathbf{P}\{Q(t) - C(\varrho)t \geq u\} \quad \leq \quad c \cdot \frac{t^2}{u^4} \cdot \mathbf{Var}(h_{C(\varrho)t}(t))$$

The computations result in (remember $\mathbf{E}(Q(t)) = C(\varrho)t$)

$$\mathbf{P}\{Q(t) - C(\varrho)t \geq u\} \quad \leq \quad c \cdot \frac{t^2}{u^4} \cdot \mathbf{Var}(h_{C(\varrho)t}(t))$$

The computations result in (remember $\mathbf{E}(Q(t)) = C(\varrho)t$)

$$\begin{aligned} \mathbf{P}\{Q(t) - C(\varrho)t \geq u\} &\leq c \cdot \frac{t^2}{u^4} \cdot \mathbf{Var}(h_{C(\varrho)t}(t)) \\ &\overset{\text{Thm}}{=} c \cdot \frac{t^2}{u^4} \cdot \mathbf{E}|Q(t) - C(\varrho)t|. \end{aligned}$$

The computations result in (remember $\mathbf{E}(Q(t)) = C(\varrho)t$)

$$\mathbf{P}\{Q(t) - C(\varrho)t \geq u\} \leq c \cdot \frac{t^2}{u^4} \cdot \mathbf{Var}(h_{C(\varrho)t}(t))$$

$$\overset{\text{Thm}}{=} c \cdot \frac{t^2}{u^4} \cdot \mathbf{E}|Q(t) - C(\varrho)t|.$$

With

$$\widetilde{Q}(t) := Q(t) - C(\varrho)t \qquad \text{and} \qquad E := \mathbf{E}|\widetilde{Q}(t)|,$$

we have (with a similar lower deviation bound)

$$\mathbf{P}\{|\widetilde{Q}(t)| > u\} \leq c \cdot \frac{t^2}{u^4} \cdot E.$$

The computations result in (remember $\mathbf{E}(Q(t)) = C(\varrho)t$)

$$\begin{aligned} \mathbf{P}\{Q(t) - C(\varrho)t \geq u\} &\leq c \cdot \frac{t^2}{u^4} \cdot \mathbf{Var}(h_{C(\varrho)t}(t)) \\ &\overset{\text{Thm}}{=} c \cdot \frac{t^2}{u^4} \cdot \mathbf{E}|Q(t) - C(\varrho)t|. \end{aligned}$$

With

$$\widetilde{Q}(t) := Q(t) - C(\varrho)t \qquad \text{and} \qquad E := \mathbf{E}|\widetilde{Q}(t)|,$$

we have (with a similar lower deviation bound)

$$\mathbf{P}\{|\widetilde{Q}(t)| > u\} \leq c \cdot \frac{t^2}{u^4} \cdot E.$$

Claim: this already implies the $t^{2/3}$ upper bound:

We had $\mathbf{P}\{|\widetilde{Q}(t)| > u\} \leq c \cdot \frac{t^2}{u^4} \cdot E.$

We had $\mathbf{P}\{|\widetilde{Q}(t)| > u\} \leq c \cdot \frac{t^2}{u^4} \cdot E$.

$$E = \mathbf{E}|\widetilde{Q}(t)| = \int_0^\infty \mathbf{P}\{|\widetilde{Q}(t)| > u\} \, \mathrm{d}u$$

We had $\mathbf{P}\{|\widetilde{Q}(t)| > u\} \le c \cdot \frac{t^2}{u^4} \cdot E.$

$$E = \mathbf{E}|\widetilde{Q}(t)| = \int_0^\infty \mathbf{P}\{|\widetilde{Q}(t)| > u\}\, \mathsf{d}u$$
$$= E \int_0^\infty \mathbf{P}\{|\widetilde{Q}(t)| > vE\}\, \mathsf{d}v$$

We had $\mathbf{P}\{|\widetilde{Q}(t)| > u\} \leq c \cdot \frac{t^2}{u^4} \cdot E.$

$$
\begin{aligned}
E = \mathbf{E}|\widetilde{Q}(t)| &= \int_0^\infty \mathbf{P}\{|\widetilde{Q}(t)| > u\} \, \mathrm{d}u \\
&= E \int_0^\infty \mathbf{P}\{|\widetilde{Q}(t)| > vE\} \, \mathrm{d}v \\
&\leq E \int_{1/2}^\infty \mathbf{P}\{|\widetilde{Q}(t)| > vE\} \, \mathrm{d}v + \frac{1}{2}E
\end{aligned}
$$

We had $\mathbf{P}\{|\widetilde{Q}(t)| > u\} \leq c \cdot \frac{t^2}{u^4} \cdot E$.

$$
\begin{aligned}
E = \mathbf{E}|\widetilde{Q}(t)| &= \int_0^\infty \mathbf{P}\{|\widetilde{Q}(t)| > u\} \, \mathrm{d}u \\
&= E \int_0^\infty \mathbf{P}\{|\widetilde{Q}(t)| > vE\} \, \mathrm{d}v \\
&\leq E \int_{1/2}^\infty \mathbf{P}\{|\widetilde{Q}(t)| > vE\} \, \mathrm{d}v + \frac{1}{2}E \\
&\leq c \cdot \frac{t^2}{E^2} + \frac{1}{2}E,
\end{aligned}
$$

that is, $E^3 \leq c \cdot t^2$.

We had $\mathbf{P}\{|\widetilde{Q}(t)| > u\} \le c \cdot \frac{t^2}{u^4} \cdot E$.

$$\begin{aligned}
E = \mathbf{E}|\widetilde{Q}(t)| &= \int_0^\infty \mathbf{P}\{|\widetilde{Q}(t)| > u\} \,\mathrm{d}u \\
&= E \int_0^\infty \mathbf{P}\{|\widetilde{Q}(t)| > vE\} \,\mathrm{d}v \\
&\le E \int_{1/2}^\infty \mathbf{P}\{|\widetilde{Q}(t)| > vE\} \,\mathrm{d}v + \frac{1}{2}E \\
&\le c \cdot \frac{t^2}{E^2} + \frac{1}{2}E,
\end{aligned}$$

that is, $E^3 \le c \cdot t^2$.

$$\mathbf{Var}(h_{C(\varrho)t}(t)) \overset{\text{Thm}}{=} \text{const.} \cdot \mathbf{E}|Q(t) - C(\varrho)t|$$

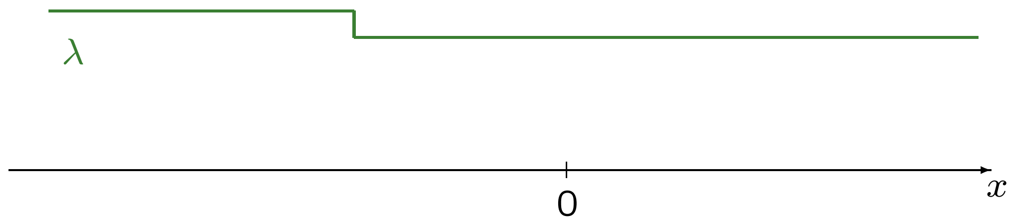
We had $\mathbf{P}\{|\tilde{Q}(t)| > u\} \leq c \cdot \frac{t^2}{u^4} \cdot E.$

$$
\begin{aligned}
E = \mathbf{E}|\tilde{Q}(t)| &= \int_0^\infty \mathbf{P}\{|\tilde{Q}(t)| > u\}\ \mathrm{d}u \\
&= E\int_0^\infty \mathbf{P}\{|\tilde{Q}(t)| > vE\}\ \mathrm{d}v \\
&\leq E\int_{1/2}^\infty \mathbf{P}\{|\tilde{Q}(t)| > vE\}\ \mathrm{d}v + \frac{1}{2}E \\
&\leq c \cdot \frac{t^2}{E^2} + \frac{1}{2}E,
\end{aligned}
$$

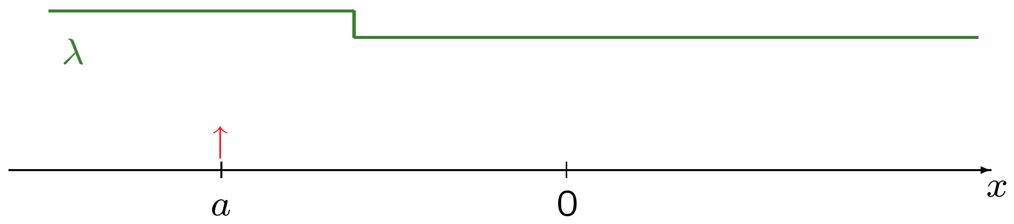
that is, $E^3 \leq c \cdot t^2$.

$$
\begin{aligned}
\mathbf{Var}(h_{C(\varrho)t}(t)) &\overset{\text{Thm}}{=} \text{const.} \cdot \mathbf{E}|Q(t) - C(\varrho)t| \\
&= \text{const.} \cdot E \leq c \cdot t^{2/3}. \qquad \blacksquare
\end{aligned}
$$

6. The lower bound

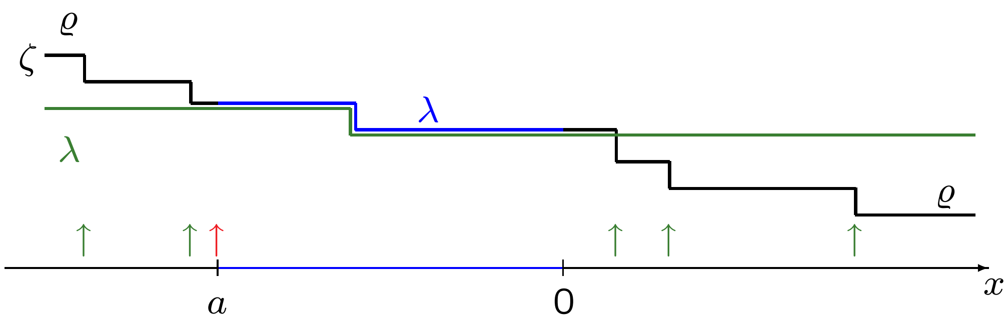


6. The lower bound



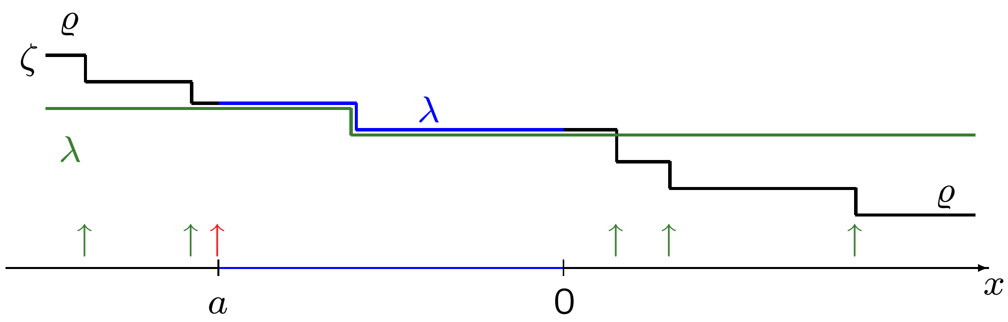
Let $Q^a(0) = a < 0$.

6. The lower bound



Let $Q^a(0) = a < 0$.

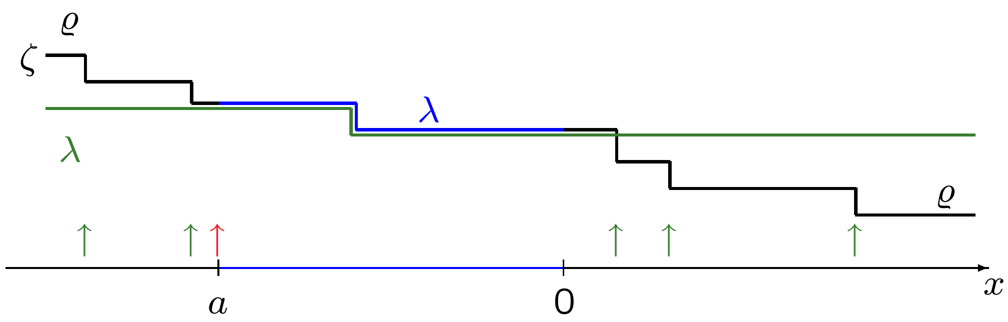
6. The lower bound



Let $Q^a(0) = a < 0$. If $Q^a(t) \le C(\varrho)t$, then the $\uparrow$'s have not crossed the path $C(\varrho)t$ from left to right:

$$\mathbf{P}\{Q^a(t) \le C(\varrho)t\} \le \mathbf{P}\{h_{C(\varrho)t}(t) < h_{C(\varrho)t}(t)\}.$$

6. The lower bound

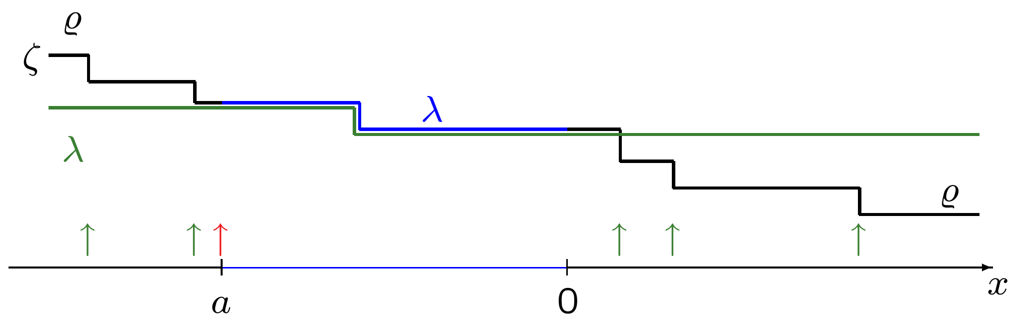


Let $Q^a(0) = a < 0$. If $Q^a(t) \le C(\varrho)t$, then the $\uparrow$'s have not crossed the path $C(\varrho)t$ from left to right:

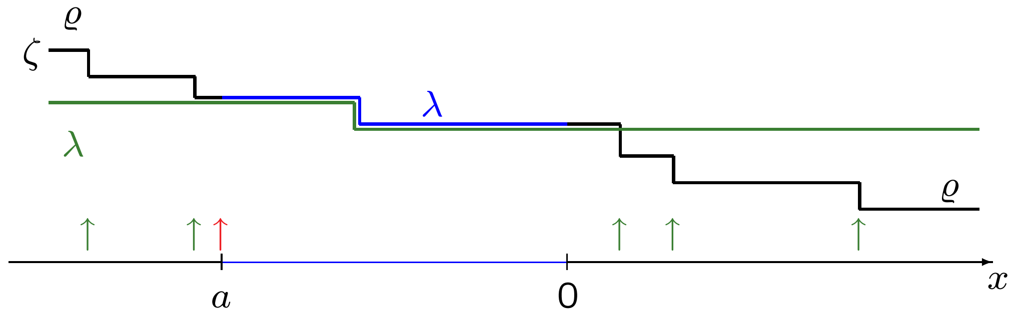
$$\mathbf{P}\{Q^a(t) \le C(\varrho)t\} \le \mathbf{P}\{h_{C(\varrho)t}(t) < h_{C(\varrho)t}(t)\}.$$

Therefore:

$$1 \le \mathbf{P}\{Q^a(t) > C(\varrho)t\} + \mathbf{P}\{h_{C(\varrho)t}(t) < h_{C(\varrho)t}(t)\}.$$

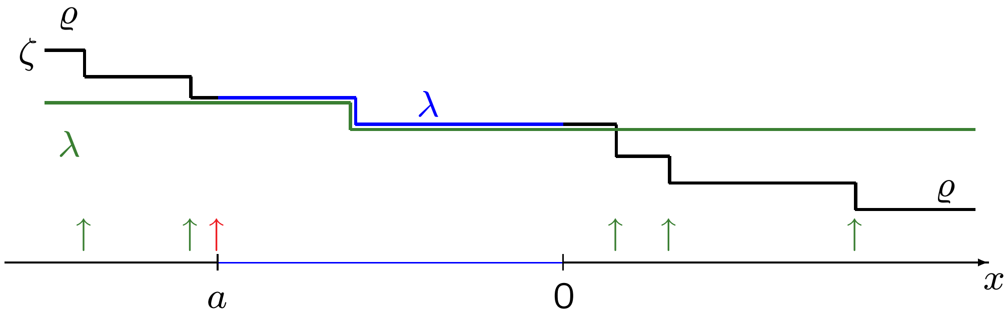


$$1 \leq \mathbf{P}\{Q^a(t) > C(\varrho)t\} + \mathbf{P}\{h_{C(\varrho)t}(t) < h_{C(\varrho)t}(t)\}$$



$$1 \le \mathbf{P}\{Q^a(t) > C(\varrho)t\} + \mathbf{P}\{h_{C(\varrho)t}(t) < h_{C(\varrho)t}(t)\}$$

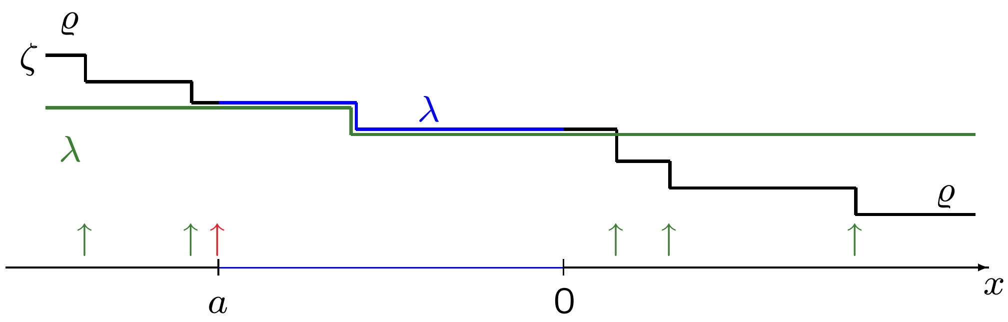
⇝ Set a so that $\mathbf{E}(Q^a(t)) < C(\varrho)t$,



$$1 \le \mathbf{P}\{Q^a(t) > C(\varrho)t\} + \mathbf{P}\{h_{C(\varrho)t}(t) < h_{C(\varrho)t}(t)\}$$

$\rightsquigarrow$ Set a so that $\mathbf{E}(Q^a(t)) < C(\varrho)t$,

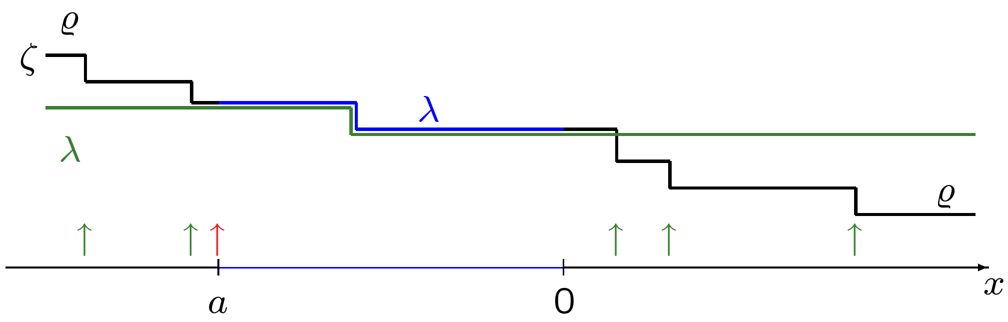
$\rightsquigarrow$ $\mathbf{E}(h_{C(\varrho)t}(t)) - \mathbf{E}(h_{C(\varrho)t}(t)) \sim t(\varrho - \lambda)^2 > 0$ would be the case, if ζ was Bernoulli(ϱ) distributed.



$$1 \le \mathbf{P}\{Q^a(t) > C(\varrho)t\} + \mathbf{P}\{h_{C(\varrho)t}(t) < h_{C(\varrho)t}(t)\}$$

$\rightsquigarrow$ Set a so that $\mathbf{E}(Q^a(t)) < C(\varrho)t$,

$\rightsquigarrow$ $\mathbf{E}(h_{C(\varrho)t}(t)) - \mathbf{E}(h_{C(\varrho)t}(t)) \sim t(\varrho - \lambda)^2 > 0$ would be the case, if ζ was Bernoulli(ϱ) distributed. Instead, $\mathbf{E}(h_{C(\varrho)t}(t))$ will have a harmless Radon-Nikodym factor.



$$1 \le \mathbf{P}\{Q^a(t) > C(\varrho)t\} + \mathbf{P}\{h_{C(\varrho)t}(t) < h_{C(\varrho)t}(t)\}$$

$\rightsquigarrow$ Set a so that $\mathbf{E}(Q^a(t)) < C(\varrho)t$,

$\rightsquigarrow$ $\mathbf{E}(h_{C(\varrho)t}(t)) - \mathbf{E}(h_{C(\varrho)t}(t)) \sim t(\varrho - \lambda)^2 > 0$ would be the case, if ζ was Bernoulli(ϱ) distributed. Instead, $\mathbf{E}(h_{C(\varrho)t}(t))$ will have a harmless Radon-Nikodym factor.

$\Rightarrow$ Both probabilities are deviation probabilities.

$$1 \leq \mathbf{P}\{Q^a(t) > C(\varrho)t\} + \mathbf{P}\{h_{C(\varrho)t}(t) < h_{C(\varrho)t}(t)\}.$$

$$1 \le \mathbf{P}\{Q^a(t) > C(\varrho)t\} + \mathbf{P}\{h_{C(\varrho)t}(t) < h_{C(\varrho)t}(t)\}.$$

Apply Markov's inequality on the first, Chebyshev's on the second probability (use again the connection between **Var**$(h_{C(\varrho)t}(t))$ and **Var**$(h_{C(\varrho)t}(t))$).

$$1 \leq \mathbf{P}\{Q^a(t) > C(\varrho)t\} + \mathbf{P}\{h_{C(\varrho)t}(t) < h_{C(\varrho)t}(t)\}.$$

Apply Markov's inequality on the first, Chebyshev's on the second probability (use again the connection between **Var**$(h_{C(\varrho)t}(t))$ and **Var**$(h_{C(\varrho)t}(t))$).

The correct scaling of the parameters is: $\varrho - \lambda \sim t^{-1/3}, \quad a \sim -t^{2/3}$.

$$1 \leq \mathbf{P}\{Q^a(t) > C(\varrho)t\} + \mathbf{P}\{h_{C(\varrho)t}(t) < h_{C(\varrho)t}(t)\}.$$

Apply Markov's inequality on the first, Chebyshev's on the second probability (use again the connection between **Var**$(h_{C(\varrho)t}(t))$ and **Var**$(h_{C(\varrho)t}(t))$).

The correct scaling of the parameters is: $\varrho - \lambda \sim t^{-1/3}, \quad a \sim -t^{2/3}$.
In this case

$$1 \;\; \leq \;\; c_1 \cdot \frac{\mathbf{E}(|\widetilde{Q^a}(t)|)}{t^{2/3}} + c_2 \cdot \frac{\mathbf{Var}(h_{C(\varrho)t}(t))}{t^{2/3}}$$

$$1 \le \mathbf{P}\{Q^a(t) > C(\varrho)t\} + \mathbf{P}\{h_{C(\varrho)t}(t) < h_{C(\varrho)t}(t)\}.$$

Apply Markov's inequality on the first, Chebyshev's on the second probability (use again the connection between **Var**$(h_{C(\varrho)t}(t))$ and **Var**$(h_{C(\varrho)t}(t))$).

The correct scaling of the parameters is: $\varrho - \lambda \sim t^{-1/3}, \quad a \sim -t^{2/3}$.
In this case

$$\begin{aligned} 1 \ &\le \ c_1 \cdot \frac{\mathbf{E}(|\widetilde{Q^a}(t)|)}{t^{2/3}} + c_2 \cdot \frac{\mathbf{Var}(h_{C(\varrho)t}(t))}{t^{2/3}} \\ &\overset{\text{Thm}}{=} \ c \cdot \frac{\mathbf{Var}(h_{C(\varrho)t}(t))}{t^{2/3}}. \end{aligned}$$

■

7. Open questions

$$\boxed{\mathbf{E}|\widetilde{Q}(t)|^1} \longleftrightarrow \boxed{\mathbf{E}|\widetilde{h}_{C(\varrho)t}(t)|^2}$$

7. Open questions

$$
\begin{tikzcd}
\boxed{\mathbf{E}|\widetilde{Q}(t)|^2} \arrow[d] & \\
\boxed{\mathbf{E}|\widetilde{Q}(t)|^1} \arrow[r, leftrightarrow] & \boxed{\mathbf{E}|\widetilde{h}_{C(\varrho)t}(t)|^2}
\end{tikzcd}
$$

7. Open questions

$$
\begin{tikzcd}
\boxed{\mathbf{E}|\widetilde{Q}(t)|^{3-\varepsilon}} \arrow[d] & \\
\boxed{\mathbf{E}|\widetilde{Q}(t)|^{2}} \arrow[d] & \\
\boxed{\mathbf{E}|\widetilde{Q}(t)|^{1}} \arrow[r, leftrightarrow] & \boxed{\mathbf{E}|\widetilde{h}_{C(\varrho)t}(t)|^{2}}
\end{tikzcd}
$$

7. Open questions

$$
\begin{tikzcd}
\mathbf{E}|\widetilde{Q}(t)|^{4}? \arrow[d, dotted] & \\
\boxed{\mathbf{E}|\widetilde{Q}(t)|^{3-\varepsilon}} \arrow[d] & \\
\boxed{\mathbf{E}|\widetilde{Q}(t)|^{2}} \arrow[d] & \\
\boxed{\mathbf{E}|\widetilde{Q}(t)|^{1}} \arrow[r, leftrightarrow] & \boxed{\mathbf{E}|\widetilde{h}_{C(\varrho)t}(t)|^{2}}
\end{tikzcd}
$$

7. Open questions

$$
\begin{tikzcd}
\mathbf{E}|\widetilde{Q}(t)|^4? \arrow[r, dotted] \arrow[d, dotted] & \mathbf{E}|\widetilde{h}_{C(\varrho)t}(t)|^5? \arrow[d, dotted] \\
\boxed{\mathbf{E}|\widetilde{Q}(t)|^{3-\varepsilon}} \arrow[r, dotted] \arrow[d] & \mathbf{E}|\widetilde{h}_{C(\varrho)t}(t)|^4? \arrow[d, dotted] \\
\boxed{\mathbf{E}|\widetilde{Q}(t)|^2} \arrow[r, dotted] \arrow[d] & \mathbf{E}|\widetilde{h}_{C(\varrho)t}(t)|^3? \arrow[d, dotted] \\
\boxed{\mathbf{E}|\widetilde{Q}(t)|^1} \arrow[r, leftrightarrow] & \boxed{\mathbf{E}|\widetilde{h}_{C(\varrho)t}(t)|^2}
\end{tikzcd}
$$

7. Open questions

$\longrightarrow$ What is the limit $\lim_{t\to\infty} \frac{\mathbf{Var}(h_{C(\varrho)t}(t))}{t^{2/3}} = ?$ What does it have to do with Gaussian random matrices? (Difficult.)

7. Open questions

$\longrightarrow$ What is the limit $\lim_{t\to\infty} \frac{\mathbf{Var}(h_{C(\varrho)t}(t))}{t^{2/3}} = ?$ What does it have to do with Gaussian random matrices? (Difficult.)

$\longrightarrow$ Other processes (zero range, *Bricklayers'*, ...)?

7. Open questions

$\longrightarrow$ What is the limit $\lim_{t\to\infty} \frac{\mathbf{Var}(h_{C(\varrho)t}(t))}{t^{2/3}} =$? What does it have to do with Gaussian random matrices? (Difficult.)

$\longrightarrow$ Other processes (zero range, *Bricklayers'*, ...)?

$\longrightarrow$ Some processes (e.g. symmetric simple exclusion, linear rate zero range) show $t^{1/4}$ scaling (with Gaussian limits), rather than $t^{1/3}$. Where is the borderline? Are there other scalings as well?

Thank you.