A covariant Stinespring theorem

(Quantum Groups Seminar 27-09-21)

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This talk is based on the paper of the same name [14].

Introduction

Quantum information theory: basic setup

- Systems: f.d. C^*-algebras $A, B, \dots$ with chosen trace.
- Channels $f, g, \dots : A \to B$: completely positive trace-preserving (CPTP) linear maps $A \to B$.

Definition

A linear map $f : A \to B$ is *positive* if $x \geq 0 \Rightarrow f(x) \geq 0$.
It is *completely positive* if $\mathrm{id} \otimes f : B(\mathbb{C}^d) \otimes A \to B(\mathbb{C}^d) \otimes B$ is positive for all $d \geq 1$.

Stinespring's theorem

Theorem

Let H, K be f.d. Hilbert spaces. Then for any CP map $f : B(H) \to B(K)$ there exists:

- *a Hilbert space E (the environment)*
- *a linear map $\tau : H \to K \otimes E$ (the dilation)*

such that $f(x) = \mathrm{Tr}_E(\tau x \tau^\dagger)$ (here $\mathrm{Tr}_E : B(K \otimes E) \to B(E)$ is the partial trace over the environment).
The CP map f is trace preserving iff the dilation τ is an isometry.
Different dilations $\tau_1 : H \to K \otimes E_1$, $\tau_2 : H \to K \otimes E_2$ are related by a partial isometry $\alpha : E_1 \to E_2$.

This reduces the study of CPTP maps between matrix algebras to the study of isometries (even unitaries) between Hilbert spaces.

Choi's theorem

- There is an isomorphism of vector spaces $L(B(H), B(K)) = L(H \otimes H^*, K \otimes K^*) \cong B(K^* \otimes H)$.

Theorem

A linear map $f : B(H) \to B(K)$ is completely positive iff the corresponding element $\tilde{f} \in B(K^* \otimes H)$ is positive.

- Allows one to apply e.g. spectral decomposition to CP maps, move to the quantum relation underlying a CP map, consider CP supermaps [3], etc.

Question: covariant Stinespring and Choi's theorems

- Can we extend these theorems to covariant CP maps/channels between finite-dimensional G-C^*-algebras (a.k.a C^*-dynamical systems) for a compact quantum group G?

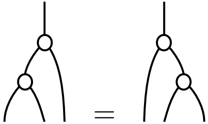
Systems and channels

Overview

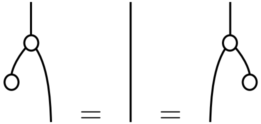
- Our starting point is a formulation of systems and channels in a rigid C^*-tensor category.
- These results are due to a number of authors, e.g. [4, 8, 12, 9, 10, 15].

Frobenius algebras: I

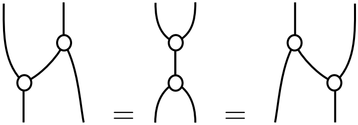
- We define a *Frobenius algebra* in a rigid C^*-tensor category $\mathcal{T}$ to be an object A with *multiplication* and *unit* morphisms $m : A \otimes A \to A$ and $u : \mathbb{1} \to A$ satisfying the following equations (where we draw $m, m^\dagger, u, u^\dagger$ with white vertices):



associativity



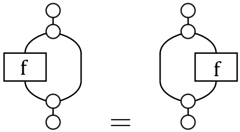
unitality



Frobenius

Systems

- We say that a Frobenius algebra is *special* if it satisfies the following equation:
$$m \circ m^\dagger = \mathrm{id}_A$$
- We say that a Frobenius algebra in a rigid C^*-tensor category $\mathcal{T}$ is *standard* if, for any morphism $f \in \mathrm{End}(A)$, the following scalars are equal:



(This definition generalises straightforwardly to C^*-multitensor categories.)

Definition

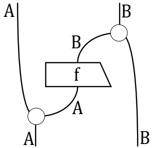
We define a *system* in a rigid C^*-tensor category $\mathcal{T}$ to be a special standard Frobenius algebra in $\mathcal{T}$.

Channels

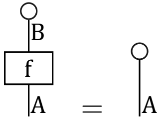
Definition

Let A, B be systems in $\mathcal{T}$.

- We say that a morphism $f : A \to B$ is a *CP morphism* if the element (1) of $\mathrm{End}(A \otimes B)$ is positive.
- We say that it is additionally a *channel* if it preserves the counit (equation (2)).



(1)



(2)

- Obtain categories $\mathrm{Chan}(\mathcal{T}) \subset \mathrm{CP}(\mathcal{T})$.

Systems generalise f.d. G-C^*-algebras

- Let $F : \mathcal{T} \to \mathrm{Hilb}$ be a fibre functor.
- By Tannaka-Krein-Woronowicz duality, F is associated with a compact quantum group G.
- System A in $\mathcal{T} \Rightarrow$ f.d. G-C^*-algebra $F(A)$, equipped with canonical G-invariant functional.[1]
- CP morphism $f : A \to B \Rightarrow$ covariant CP linear map $F(A) \to F(B)$, preserving canonical functional iff f is a channel.
- Induces equivalences

$$\mathrm{CP}(\mathcal{T}) \cong \mathrm{CP}(G) \qquad \mathrm{Chan}(\mathcal{T}) \cong \mathrm{Chan}(G)$$

[1] If the G-C^*-algebra $F(A)$ admits an invariant trace (i.e. if $\dim(F(A)) = d(A)$, guaranteed if G is of Kac type), this functional is tracial.

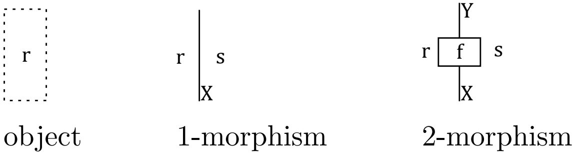
Summary

- In a rigid C^*-tensor category $\mathcal{T}$:
  - *Systems* are special standard Frobenius algebras.
  - *Channels* are counit preserving CP morphisms.
- This generalises G-C^*-algebras and covariant channels for a CQG G (the case where $\mathcal{T} \cong \mathrm{Rep}(G)$).
- We will prove Stinespring's theorem for systems and channels in a general rigid C^*-tensor category.

Background: rigid C^*-2-categories

Diagrammatic calculus for 2-categories

- We use the standard diagrammatic calculus for 2-categories.
  - Objects $r, s, \ldots$ are represented by labelled regions.
  - 1-morphisms $X, Y, \cdots : r \to s$ are represented by labelled wires separating an r-region (on the left) from an s-region (on the right).
  - 2-morphisms $f, g, \cdots : X \to Y$ are represented by boxes with labelled input and output wires.

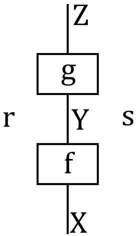


- Identity 1-morphism wires and identity 2-morphisms boxes are invisible.

Diagrammatic calculus: vertical composition

- Vertical composition of 2-morphisms is represented by vertical juxtaposition in the diagram. For example:
  - Let $X, Y, Z : r \to s$ be 1-morphisms.
  - Let $f : X \to Y$ and $g : Y \to Z$ be 2-morphisms.

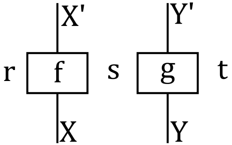
  Then $g \circ f : X \to Z$ is represented as follows:



Diagrammatic calculus: horizontal composition

- Horizontal composition is represented by horizontal juxtaposition in the diagram. For example:
  - Let $X, X' : r \to s$ and $Y, Y' : s \to t$ be 1-morphisms.
  - Let $f : X \to X'$ and $g : Y \to Y'$ be 2-morphisms.

  Then the horizontal composite $f \otimes g : X \otimes Y \to X' \otimes Y'$ is represented as follows:



C^*-2-categories

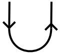
- For any 1-morphisms $X, Y : r \to s$ the set $\mathrm{Hom}(X, Y)$ is a Banach space. Vertical and horizontal composition induce linear maps on 2-morphism spaces.
- Every 2-morphism $f : X \to Y$ has a *dagger* 2-morphism $f^\dagger : Y \to X$. Taking the dagger induces an antilinear map on Hom-spaces. The dagger satisfies the following properties:

  $$(f^\dagger)^\dagger = f \qquad (f \otimes g)^\dagger = f^\dagger \otimes g^\dagger \qquad ||f^\dagger \circ f|| = ||f||^2$$

  The last property implies that, for any 1-morphism X, the algebra $\mathrm{End}(X)$ is a C^*-algebra with involution given by the dagger.
- For any 2-morphism $f : X \to Y$, the 2-morphism $f^\dagger \circ f$ is a positive element of the C^*-algebra $\mathrm{End}(X)$.

Rigid C^*-2-categories

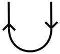
- In a *rigid* C^*-2-category every 1-morphism $X : r \to s$ has a *dual* 1-morphism $X^* : s \to r$.
- In order to represent duality we orient the 1-morphism wires: X^* is represented by a wire with the opposite orientation to X.
- Duality of X and X^* is characterised by the following 2-morphisms, called *cups* and *caps*:



$\eta_X : \mathrm{id}_s \to X^* \otimes X$

$\epsilon_X : X \otimes X^* \to \mathrm{id}_r$

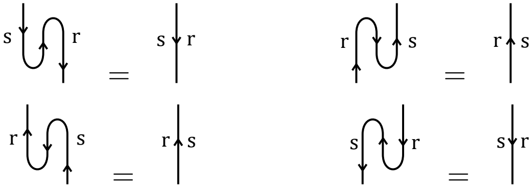
$\eta_X^\dagger : X^* \otimes X \to \mathrm{id}_s$



$\epsilon_X^\dagger : \mathrm{id}_r \to X \otimes X^*$

Rigid C^*-2-categories

These cups and caps obey the *snake equations*:



Local semisimplicity

- All our rigid C^*-2-categories are *locally semisimple* (i.e. the Hom-categories are semisimple). In particular:
  - For any pair of 1-morphisms $X, Y : r \to s$ there is a 1-morphism $X_1 \oplus X_2 : r \to s$ (called the *direct sum*), with isometries $i_1 : X_1 \to X_1 \oplus X_2$, $i_2 : X_2 \to X_1 \oplus X_2$ such that $i_1 \circ i_1^\dagger + i_2 \circ i_2^\dagger = \mathrm{id}_{X_1 \oplus X_2}$.
  - There is a *zero 1-morphism* $\mathbf{0}_{r,s} : r \to s$ such that $\mathrm{End}(\mathbf{0})$ is zero-dimensional.
  - For any 1-morphism $X : r \to s$, every projection $f \in \mathrm{End}(X)$ has a *splitting*, i.e. a 1-morphism $V : r \to s$ together with an isometry $\iota_f : V \to X$ such that $f = \iota_f \circ \iota_f^\dagger$.
  - The C^*-algebra $\mathrm{End}(X)$ is finite-dimensional for every 1-morphism X.

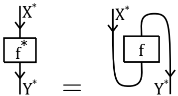
Presemisimplicity

- Our rigid C^*-2-categories are additionally *presemisimple*, in the sense that they have additive structure on objects [5]. In particular:
  - For any pair of objects r_1, r_2 there is an object $r_1 \boxplus r_2$ (called the *direct sum*), with injection and projection 1-morphisms $\iota_i : r_i \to r_1 \boxplus r_2$ and $\rho_i : r_1 \boxplus r_2 \to r_i$ obeying conditions similar to those for 1-morphisms.
  - There is a *zero object* whose endomorphism category is the terminal category.
  - Every object splits as a finite direct sum of *simple* objects, i.e. objects r_i such that $\mathrm{End}(\mathrm{id}_{r_i}) \cong \mathbb{C}$.
- In a presemisimple rigid C^*-2-category[2] we can choose *standard duals* for all 1-morphisms (unique up to unitary isomorphism). From now on we assume that such duals are chosen.

[2] In fact, such duals may be chosen more generally [6].

Dagger, transpose and conjugate

- Let $\mathcal{C}$ be a presemisimple rigid C^*-2-category. For any 2-morphism $f : X \to Y$, we define:
  - Its *transpose* (a.k.a. *mate*) $f^* : Y^* \to X^*$:



  - Its *conjugate* $f_* : X^* \to Y^*$:

$$f_* := (f^*)^\dagger = (f^\dagger)^*$$

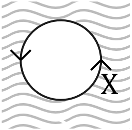
- To represent these in the diagrammatic calculus we draw 2-morphism boxes with an offset edge:

$f : X \to Y$ $\quad f^\dagger : Y \to X$ $\quad f^* : Y^* \to X^*$ $\quad f_* : X^* \to Y^*$

Semisimplicity

Pair of pants algebras

- We identified systems with special standard Frobenius algebras in $\mathcal{T}$.
- In a presemisimple rigid C^*-2-category $\mathcal{C}$, systems in endomorphism categories arise as *pair of pants algebras*.
- Let r, s be objects of $\mathcal{C}$. Let $X : r \to s$ be a 1-morphism such that the following positive element $\dim_L(X) \in \mathrm{End}(\mathrm{id}_s)$ is invertible:

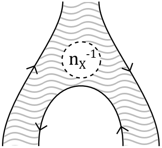


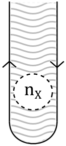
If this condition is obeyed we call X *special* and write $n_X := \sqrt{\dim_L(X)}$.

Pair of pants algebras

Proposition

The object $X \otimes X^*$ of $\mathrm{End}(r)$ is a system, with multiplication and unit defined as follows:

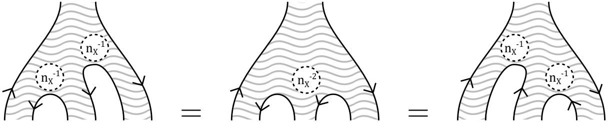




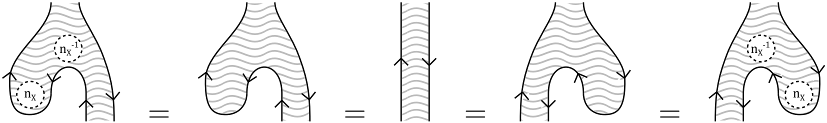
Pair of pants algebras

Proof.

- Associativity:



- Unitality:

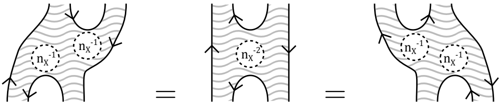


□

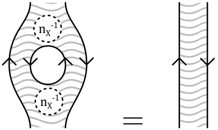
Pair of pants algebras

Proof (cont.)

- Frobenius:



- Special:



- Standardness follows by standardness of the duals in $\mathcal{C}$.

□

Splitting algebras and semisimplicity

Definition

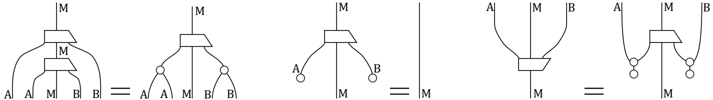
Let $\mathcal{C}$ be a presemisimple rigid C^*-2-category.
We say that a system A in an endomorphism category $\mathrm{End}(r)$ *splits* if it is isomorphic to a pair of pants algebra.
We say that $\mathcal{C}$ is *semisimple* if all systems in all endomorphism categories split.
(Refs:[5, 2].)

- Stinespring's theorem characterises channels between pair of pants algebras.
- We therefore want to find a semisimple rigid C^*-2-category in which $\mathcal{T}$ embeds as an endomorphism category, i.e. $\mathcal{T} \cong \mathrm{End}(r)$ for some object r.

Candidate 1: $\mathrm{Bimod}(\mathcal{T})$

Definition

Let A, B be two special standard Frobenius algebras in $\mathcal{T}$.
We define an $A - B$ *dagger bimodule* ${}_AX_B$ to be an object X together with an *action* morphism $A \otimes X \otimes B \to X$ (drawn as a white rectangle) satisfying the following equations:



Let ${}_AX_B, {}_AY_B$ be $A - B$ dagger bimodules. We define a *bimodule homomorphism* to be a morphism $f : X \to Y$ intertwining the actions.
The A-B dagger bimodules and bimodule homomorphisms form a category A-Mod-B.

Candidate 1: $\mathrm{Bimod}(\mathcal{T})$

Definition (The 2-category $\mathrm{Bimod}(\mathcal{T})$)

- *Objects*: Special standard Frobenius algebras $A, B, \dots$ in $\mathcal{T}$.
- Hom-*categories* $\mathrm{Hom}(A, B)$: $A\text{-}\mathrm{Mod}\text{-}B$.

Horizontal composition is defined using the usual relative tensor product of bimodules.

The category $\mathcal{T}$ embeds (isomorphically) in $\mathrm{Bimod}(\mathcal{T})$ as $\mathrm{End}(\mathbb{1}) = \mathbb{1}\text{-}\mathrm{Mod}\text{-}\mathbb{1}$.

- Pros: very concrete.
- Cons: we made several choices in the definition. Not a strict 2-category.

Candidate 2: $\mathrm{Mod}(\mathcal{T})$

Definition

A *semisimple left $\mathcal{T}$-module category* is a semisimple C^*-category $\mathcal{M}$ together with:

- A unitary linear bifunctor $\tilde{\otimes} : \mathcal{T} \times \mathcal{M} \to \mathcal{M}$.
- Unitary natural isomorphisms $l_X : \mathbb{1}\tilde{\otimes}X \cong X$ and $m_{U,V,X} : (U \otimes V)\tilde{\otimes}X \cong U\tilde{\otimes}(V\tilde{\otimes}X)$ satisfying analogues of the pentagon and triangle equations.

We say that the module category $\mathcal{M}$ is:

- *Cofinite* (a.k.a. *proper*) if for any $X, Y \in \mathcal{M}$ we have $\mathrm{Hom}_{\mathcal{M}}(X, U_i\tilde{\otimes}Y) = 0$ for all but finitely many i, where $\{U_i\}$ are representatives of the isomorphism classes of simple objects in $\mathcal{T}$.
- *Indecomposable* if it does not split as a nontrivial direct sum.
- *Finitely decomposable* if it splits as a finite direct sum of indecomposables.

Candidate 2: $\mathrm{Mod}(\mathcal{T})$

Definition (The 2-category $\mathrm{Mod}(\mathcal{T})$)

- *Objects*: Cofinite semisimple finitely decomposable left $\mathcal{T}$-module categories.
- *1-morphisms*: Unitary $\mathcal{T}$-module functors.
- *2-morphisms*: Natural transformations of $\mathcal{T}$-module functors.

$\mathcal{T}$ embeds in $\mathrm{Mod}(\mathcal{T})$ as $\mathrm{End}_{\mathcal{T}}(\mathcal{T})$.

- Pros: very natural definition, strict 2-category.
- Cons: we rely on $\mathrm{Bimod}(\mathcal{T})$ for our definition of the rigid structure (see next slide).

A theorem about unpacking

Theorem

$\mathrm{Bimod}(\mathcal{T})$ and $\mathrm{Mod}(\mathcal{T})$ are equivalent semisimple rigid C^*-2-categories.

Proof.

Main steps:

1. We show that $\mathrm{Bimod}(\mathcal{T}) \cong \mathrm{Mod}(\mathcal{T})$ as a $\mathbb{C}$-linear dagger 2-category.
2. We show that $\mathrm{Bimod}(\mathcal{T})$ is a semisimple rigid C^*-2-category. (Thus so is $\mathrm{Mod}(\mathcal{T})$.)

- *Step 1*: The equivalence $\Psi : \mathrm{Bimod}(\mathcal{T}) \xrightarrow{\sim} \mathrm{Mod}(\mathcal{T})$ is defined in the usual way:
    - *Objects*: $\Psi(A) := \mathbb{1}\text{-}\mathrm{Mod}\text{-}A$, where the left action of $\mathcal{T}$ is by tensor product.
    - *1-morphisms*: $\Psi({}_A M_B) := - \otimes_A {}_A M_B : \mathbb{1}\text{-}\mathrm{Mod}\text{-}A \to \mathbb{1}\text{-}\mathrm{Mod}\text{-}B$.
    - *2-morphisms*: $\Psi(f) := \mathrm{id} \otimes_A f : \Psi({}_A M_B) \to \Psi({}_A N_B)$.

A theorem about unpacking (cont.)

Proof.

- *Step 1 (cont.)*: Most of the hard work in proving that Ψ is an equivalence was done in [12]. It was shown there that Ψ is essentially surjective on objects and that $\Psi_{\mathbb{1},\mathbb{1}} : \mathrm{End}(\mathbb{1}) = \mathcal{T} \to \mathrm{End}_{\mathcal{T}}(\mathcal{T})$ is an equivalence. The rest is quite straightforward [14, Thm. 3.41].
- *Step 2*: The C^*-structure on $\mathrm{Bimod}(\mathcal{T})$ is inherited directly from $\mathcal{T}$. Dual bimodules from [16].[3] Direct sum on objects clear. Semisimplicity proven in two steps:
  - We show constructively that any system in $\mathcal{T} = \mathrm{End}(\mathbb{1})$ splits [14, Lem. 3.30]). (Actually we show a little more than this, allowing us to classify systems in $\mathcal{T}$ [14, Thm. 4.4].)
  - We then show (again constructively) that this implies splitting in the other endomorphism categories too [14, Prop. 3.31]).

□

[3] We rely on standardness of the Frobenius algebras to show that the dual bimodule is dagger, although see [7].

This is the only way to unpack

Definition

A semisimple rigid C^*-2-category is *connected* if the Hom-category between any pair of nonzero objects is nonzero.

Theorem ([14, Prop. 3.33])

Every connected semisimple rigid C^*-2-category $\mathcal{C}$ is equivalent to $\mathrm{Mod}(\mathrm{End}(r))$ *for any simple object r in $\mathcal{C}$.*

Proof.

- An equivalence $\Delta : \mathcal{C} \xrightarrow{\sim} \mathrm{Bimod}(\mathrm{End}(r))$ is defined as follows:
  - *Objects*: For every object $s \in \mathcal{C}$, pick a special 1-morphism $P_s : r \to s$. Then $\Delta(s) := P_s \otimes P_s^*$.
  - *1-morphisms*: For every 1-morphism $X : s \to t$, define $\Delta(X) := P_s \otimes X \otimes P_t^*$, with the obvious bimodule structure.
  - *2-morphisms*: For every 2-morphism $f : X \to Y$, define $\Delta(f) := \mathrm{id}_{P_s} \otimes f \otimes \mathrm{id}_{P_t^*}$.

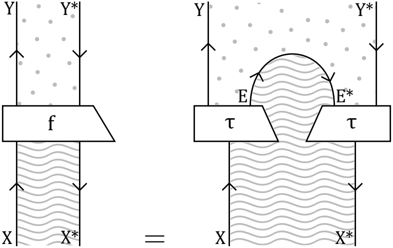
Back to physics

- We have found a (strict) semisimple rigid C^*-2-category $\mathrm{Mod}(\mathcal{T})$ into which $\mathcal{T}$ embeds as $\mathcal{T} \cong \mathrm{End}_{\mathcal{T}}(\mathcal{T})$.
- Now every system in $\mathcal{T}$ is of the form $X \otimes X^*$.

Covariant Stinespring's theorem

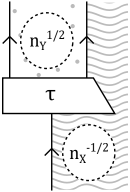
Theorem (Covariant Stinespring's theorem)

Let $X : \mathcal{T} \to \mathcal{M}_1$ and $Y : \mathcal{T} \to \mathcal{M}_2$ be special 1-morphisms in $\mathrm{Mod}(\mathcal{T})$, let $X \otimes X^*$ and $Y \otimes Y^*$ be the corresponding systems in $\mathcal{T}$, and let $f : X \otimes X^* \to Y \otimes Y^*$ be a CP morphism. Then there exists a 1-morphism $E : \mathcal{M}_2 \to \mathcal{M}_1$ (the environment) and a 2-morphism $\tau : X \to Y \otimes E$ (the dilation), such that:



Theorem (Covariant Stinespring's theorem (cont.))

The morphism



is an isometry if and only if f is a channel.
In the other direction, for any 1-morphism $E : t \to s$ and 2-morphism $\tau : X \to Y \otimes E$, the corresponding morphism $f : X \otimes X^* \to Y \otimes Y^*$ is a CP morphism with dilation τ.

Theorem (Covariant Stinespring's theorem (cont.))

Different dilations for a CP morphism $f : X \otimes X^* \to Y \otimes Y^*$ are related by a partial isometry on the environment. Specifically, let $\tau_1 : X \to Y \otimes E_1$, $\tau_2 : X \to Y \otimes E_2$ be two dilations of f. Then there exists a partial isometry $\alpha : E_1 \to E_2$ such that

$$(\mathrm{id}_Y \otimes \alpha) \circ \tau_1 = \tau_2 \qquad (\mathrm{id}_Y \otimes \alpha^\dagger) \circ \tau_2 = \tau_1$$

In particular, the minimal dilation minimising the quantum dimension of the environment $d(E)$ is unique up to unitary α.

Example: matrix G-C^*-algebras

- Let G be a compact quantum group and let $\mathcal{T} = \mathrm{Rep}(G)$.
- Now 1-morphisms $X, Y : \mathcal{T} \to \mathcal{T}$ in $\mathrm{Mod}(\mathcal{T})$ are objects in $\mathrm{End}(\mathcal{T}) \cong \mathcal{T}$: in other words, they are f.d. continuous unitary G-representations.
- The corresponding $X \otimes X^*$, $Y \otimes Y^*$ are the induced matrix G-C^*-algebras $B(X), B(Y)$.
- Now the covariant Stinespring theorem may be stated as follows:
    - Let X, Y be f.d. continuous unitary G-representations and let $B(X), B(Y)$ be the corresponding matrix G-C^*-algebras.
    - For any completely positive map $f : B(X) \to B(Y)$ there exists an f.d. continuous unitary G-representation E and an intertwiner $V : X \to Y \otimes E$ such that $f(x) = \mathrm{Tr}_E(VxV^\dagger)$.
    - The CP map f preserves the canonical invariant functional iff V is an isometry.

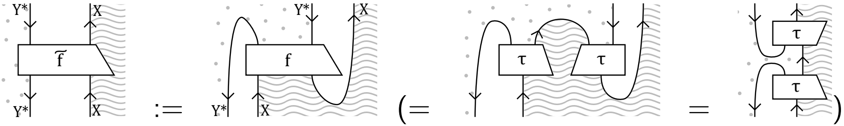
Remarks on the example

- We could have proven this for CP maps between matrix G-C^*-algebras without ever leaving $\mathrm{Rep}(G)$.
- However, in general there are indecomposable f.d. G-C^*-algebras which are not matrix G-C^*-algebras; that is, they come from 1-morphisms $\mathcal{T} \to \mathcal{M}$ in $\mathrm{Mod}(\mathcal{T})$, where $\mathcal{M}$ is a simple object inequivalent to $\mathcal{T}$. (This is precisely to say that G is not *torsion-free* in general [1].)
- Even in the torsion-free case, the covariant Stinespring theorem also applies to maps between decomposable G-C^*-algebras (corresponding to 1-morphisms $\mathcal{T} \to \boxplus_i \mathcal{M}_i$ in $\mathrm{Mod}(\mathcal{T})$).
- We have to embed $\mathcal{T}$ in the semisimple 2-category $\mathrm{Mod}(\mathcal{T})$ to dilate CP morphisms between all pairs of G-C^*-algebras.

Choi's theorem

Theorem

Let $f : X \otimes X^* \to Y \otimes Y^*$ be a CP morphism in $\mathcal{T}$. Then the following element $\tilde{f} \in \mathrm{End}(Y^* \otimes X)$ is positive:



This gives a bijective correspondence (in fact, an isomorphism of convex cones) between positive elements of $\mathrm{End}(Y^* \otimes X)$ and CP morphisms $X \otimes X^* \to Y \otimes Y^*$.

- Remark: $Y^* \otimes X$ is an object in the C^*-category $\mathrm{Hom}_{\mathcal{T}}(\mathcal{M}_2, \mathcal{M}_1)$.

Applications

Application 1: Characterising covariant channels

- In classical/quantum information theory, we often want to find a channel $A \to B$ optimising some quantity.
- Often we can assume some kind of symmetry which allows us to simplify the problem.

Example ([11, § 4.2.2])

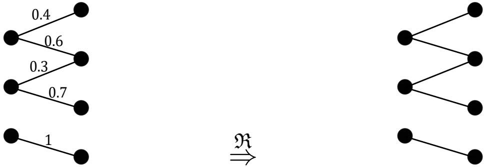
- Want to find an optimal measurement on n copies of a quantum state $\rho \in B(\mathbb{C}^d)$ that tests whether they have some *unitarily invariant* property.
- Translation: optimal covariant channel $B(\mathbb{C}^d)^{\otimes n} \to \mathbb{C} \oplus \mathbb{C}$, where $B(\mathbb{C}^d)^{\otimes n}$ has the action of $U(d) \times S_n$ and $\mathbb{C} \oplus \mathbb{C}$ has the trivial action.
- Can show using covariant Choi theorem that such a channel is defined by a pair of positive operators $M_{\mathrm{yes}}, M_{\mathrm{no}}$ in $\mathrm{End}((\mathbb{C}^d)^{\otimes n})$ which are intertwiners for the permutation action of S_n and the tensor product action of $U(d)$.
- Such operators are strongly constrained by Schur-Weyl duality.

Application 1: Characterising covariant channels

- To use such methods in general we need to understand the category $\mathrm{Mod}(\mathcal{T})$ well.
- Even for a finite group G, while we know the objects of $\mathrm{Mod}(\mathrm{Rep}(G))$ [13], we do not have a good description of many of the Hom-categories (as far as I know).

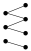
Application 2: Quantum relations, constructing covariant channels

- We saw that, by the covariant Choi's theorem, CP morphisms $f : X \otimes X^* \to Y \otimes Y^*$ correspond to positive operators $\tilde{f} \in \mathrm{End}(Y^* \otimes X)$.
- If $\tilde{f}$ is a projection, we say that f is a *quantum relation*.
- Every CP morphism f has an *underlying relation* $\mathfrak{R}(f)$ whose projection $\widetilde{\mathfrak{R}(f)}$ is $s(\tilde{f})$ (least projection p such that $p\tilde{f} = \tilde{f}$).
- The relation $\mathfrak{R}(f)$ encodes the *zero-error communication theory* of the channel f.



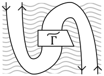
Application 2: Quantum relations, constructing covariant channels

- We can move from a quantum relation $X \otimes X^* \to Y \otimes Y^*$ to its *quantum confusability graph* on $X \otimes X^*$:



$$\Rightarrow$$

- In general, we can define a quantum confusability graph Γ on $X \otimes X^*$ to be a projector $\tilde{\Gamma} \in \mathrm{End}(X^* \otimes X)$ obeying the following equations:



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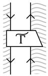

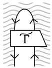

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Lemma

Let Γ be a quantum confusability graph on $X \otimes X^*$. Then there exists a system $Y \otimes Y^*$ in $\mathcal{T}$ and a channel $f : X \otimes X^* \to Y \otimes Y^*$ such that $\Gamma = \mathfrak{R}(f^\dagger \circ f)$.

Thanks for listening!

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