SYMMETRY PROPERTIES OF MINIMIZERS OF A PERTURBED DIRICHLET ENERGY WITH A BOUNDARY PENALIZATION

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Abstract. We consider $\mathbb{S}^2$-valued maps on a domain $\Omega \subset \mathbb{R}^N$ minimizing a perturbation of the Dirichlet energy with vertical penalization in Ω and horizontal penalization on $\partial\Omega$. We first show the global minimality of universal constant configurations in a specific range of the physical parameters using a Poincaré-type inequality. Then, we prove that any energy minimizer takes its values into a fixed great circle $\mathbb{S}^1 \subset \mathbb{S}^2$, and deduce uniqueness under Dirichlet boundary conditions. Finally, we show radial symmetry and monotonicity of minimizers in a ball. Our results can be applied to the Oseen–Frank energy for nematic liquid crystals and micromagnetic energy in a thin-film regime.

1. Introduction

The field of thin structures is a branch of material science that is experiencing rapid growth. The interest relies on their applications in miniaturization and integration of electronic devices, but even more on their capability to support the emergence of new physics [12,18]. Indeed, atomically thin materials can be employed to achieve physical properties that are hardly visible in bulk materials. Moreover, combining several atomically thin layers to create new heterostructures allows for the design of novel materials with prescribed properties [28].

In the last twenty years, the role of thin-structures in micromagnetics and nematic liquid crystals has been an area of active research in both applied mathematics and condensed matter physics (see, e.g., [2, 3, 6, 7, 14–16, 19, 22, 27, 30]). Recent advances in manufacturing thin films and curved layers provide a possibility to design new materials composed of several magnetic monolayers of atomic thickness [12, 31]. These new materials exhibit some unconventional properties, including perpendicular magnetocrystalline anisotropy [4] and Dzyaloshinskii–Moriya interaction (DMI) (or antisymmetric exchange) [10, 25] and require a new set of reduced theoretical models to predict the magnetization behavior in ferromagnetic samples. This new physics is often dominated by surface and edge effects, and leads to a surprising behavior near the material boundaries, giving rise to novel magnetization structures [18, 24, 32].

In this paper, we are interested in studying the ground states of a simplified two-dimensional model (cf. eq. (2)), concentrating on their reduced symmetry properties. The model we investigate is closely related to a reduced model for ferromagnetic thin films with strong perpendicular anisotropy in the regime when magnetocrystalline and shape anisotropies are of the same order of magnitude, leading to the preference for in-plane magnetization inside the sample and out-of-plane magnetization behavior on the boundary [8, 22]. Since the energy functionals governing micromagnetic interactions and defects in nematic liquid crystals are mathematically related, our analysis also applies to the analysis of ground states in the thin-film limit Oseen–Frank theory of nematic liquid crystals under weak anchoring conditions.

1.1. Our model. Let $\Omega \subset \mathbb{R}^N$, $N \in \mathbb{N}^*$, be a bounded domain and let $\mathbb{S}^2 \subset \mathbb{R}^3$ be the two-dimensional unit sphere. We consider the energy of a configuration $\boldsymbol{m} \in H^1(\Omega, \mathbb{S}^2)$, defined by

$$\mathcal{E}_\kappa(\boldsymbol{m}) = \int_\Omega |\nabla \boldsymbol{m}|^2 + \kappa^2 \int_\Omega (\boldsymbol{m} \cdot \boldsymbol{e}_3)^2, \tag{1}$$

where $|\nabla \boldsymbol{m}|^2 = \sum_{i=1}^{N} |\partial_i \boldsymbol{m}|^2$, $\boldsymbol{e}_3$=(0,0,1), and $\kappa \in [0, +\infty)$ is some fixed (material-dependent) parameter which takes into account in-plane anisotropic effects. Under natural boundary conditions (which is the typical case in micromagnetics), the only minimizers of $\mathcal{E}_\kappa$ are the constant in-plane configurations. In this note, we are interested in the problem of minimizing the energy $\mathcal{E}_\kappa$ under an additional penalization term on the boundary of Ω that makes the problem non-trivial: for every $\gamma > 0$ we consider the energy functional defined for every $\boldsymbol{m} \in H^1(\Omega, \mathbb{S}^2)$ by

$$\mathcal{E}_{\kappa,\gamma}(\boldsymbol{m}) = \int_\Omega |\nabla \boldsymbol{m}|^2 + \kappa^2 \int_\Omega (\boldsymbol{m} \cdot \boldsymbol{e}_3)^2 + \frac{1}{\gamma^2} \int_{\partial\Omega} |\boldsymbol{m} \times \boldsymbol{e}_3|^2, \tag{2}$$

where $\gamma \in (0, +\infty)$ fixes the intensity of the perpendicular anisotropy on $\partial\Omega$. The energy in this form naturally appears in the Oseen-Frank model of liquid crystals [11] and as a thin film limit of micromagnetic energy for ferromagnetic materials with strong perpendicular anisotropy [8].

A straightforward application of the Direct method of the Calculus of Variations assures that for every $\kappa \in [0, +\infty)$ and $\gamma \in (0, +\infty)$, there exists a global minimizer of the energy $\mathcal{E}_{\kappa,\gamma}$. Global minimizers satisfy the following Euler-Lagrange equation in the weak sense, i.e., for every $\boldsymbol{\varphi} \in H^1(\Omega, \mathbb{R}^3)$ such that $\boldsymbol{m}(x) + \boldsymbol{\varphi}(x) \in \mathbb{S}^2$ for a.e} $x \in \Omega$,

$$\begin{aligned}\int_\Omega \nabla \boldsymbol{m} : \nabla \boldsymbol{\varphi} + \kappa^2 (\boldsymbol{m} \cdot \boldsymbol{e}_3)(\boldsymbol{\varphi} \cdot \boldsymbol{e}_3) = \int_\Omega &(|\nabla \boldsymbol{m}|^2 + \kappa^2 (\boldsymbol{m} \cdot \boldsymbol{e}_3)^2) \boldsymbol{m} \cdot \boldsymbol{\varphi} \\ &+ \frac{1}{\gamma^2} \int_{\partial\Omega} \left[(\boldsymbol{m} \cdot \boldsymbol{e}_3) \boldsymbol{e}_3 - (\boldsymbol{m} \cdot \boldsymbol{e}_3)^2 \boldsymbol{m}\right] \cdot \boldsymbol{\varphi}.\end{aligned} \tag{3}$$

If a global minimizer $\boldsymbol{m}$ is $\mathcal{C}^2(\Omega, \mathbb{S}^2)$, this means that $\boldsymbol{m}$ solves

$$-\Delta \boldsymbol{m} + \kappa^2 (\boldsymbol{m} \cdot \boldsymbol{e}_3) \boldsymbol{e}_3 = (|\nabla \boldsymbol{m}|^2 + \kappa^2 (\boldsymbol{m} \cdot \boldsymbol{e}_3)^2) \boldsymbol{m} \quad \text{in } \Omega \tag{4}$$

together with the nonlinear Robin boundary condition:

$$\partial_{\boldsymbol{n}} \boldsymbol{m} = \frac{1}{\gamma^2} \left[(\boldsymbol{m} \cdot \boldsymbol{e}_3) \boldsymbol{e}_3 - (\boldsymbol{m} \cdot \boldsymbol{e}_3)^2 \boldsymbol{m}\right] \quad \text{on } \partial\Omega. \tag{5}$$

In the limiting case $\gamma \to 0^+$, we have a non-trivial Dirichlet boundary value problem since $\mathcal{E}_{\kappa,\gamma}$ tends to the energy $\mathcal{E}_{\kappa,0}$ defined for every $\boldsymbol{m} \in H^1(\Omega, \mathbb{S}^2)$ as

$$\mathcal{E}_{\kappa,0}(\boldsymbol{m}) := \begin{cases} \mathcal{E}_\kappa(\boldsymbol{m}) \text{ if } \boldsymbol{m} \pm \boldsymbol{e}_3 \in H_0^1(\Omega, \mathbb{R}^3), \\ +\infty \quad \text{otherwise}. \end{cases} \tag{6}$$

Note that, we shall write $\mathcal{E}_{\kappa,\gamma}$ for both the boundary penalization problem, corresponding to (2) when $\gamma > 0$, and the boundary value problem (with boundary value $\pm e_3$), corresponding to (6) when $\gamma = 0$. This is more convenient since many of our results apply to both problems. As for $\gamma > 0$, the existence of global minimizers of $\mathcal{E}_{\kappa,0}$ follows from the Direct method in the Calculus of Variations.

1.2. Contributions of the present work. The aim of the paper is to show the symmetry properties of minimizers of $\mathcal{E}_{\kappa,\gamma}$. In particular, we prove that any minimizer of $\mathcal{E}_{\kappa,\gamma}$ has values in some meridian of the sphere. As a consequence, restricting domain Ω to a ball we also show that any minimizer is radially symmetric and monotone.

In what follows, we describe the results in more detail. Our first result concerns the minimality of universal configurations, i.e., vector fields $\boldsymbol{m} \in H^1(\Omega, \mathbb{S}^2)$ which solve the Euler-Lagrange equations (3) regardless of the value of the boundary penalization constant $\gamma > 0$. Given the dependence of the boundary term in (3) on γ, such configurations must satisfy

$$(\boldsymbol{m} \cdot \boldsymbol{e}_3)(\boldsymbol{e}_3 - (\boldsymbol{m} \cdot \boldsymbol{e}_3)\boldsymbol{m}) = 0 \quad \text{a.e on } \partial\Omega. \tag{7}$$

It is easy to check that the *constant* vector fields $\pm \boldsymbol{e}_3$, as well as any *constant* in-plane vector field $\boldsymbol{e}_\perp \in \mathbb{S}^2$, $\boldsymbol{e}_\perp \cdot \boldsymbol{e}_3 = 0$, are universal configurations. Concerning these configurations, we prove the following result, which clarifies how to tune the parameters κ and γ so that these configurations emerge as ground states.

Theorem 1. *Let $\Omega \subseteq \mathbb{R}^N$ be a bounded Lipschitz domain. The following assertions hold:*

i) For any $\gamma \in [0, +\infty)$, there exists $\kappa_\gamma > 0$, depending only on γ and Ω, such that for any $\kappa \in [0, \kappa_\gamma)$ the constant out-of-plane vector field $\pm \boldsymbol{e}_3$ are the unique global minimizers of $\mathcal{E}_{\kappa,\gamma}$. In particular, $\pm \boldsymbol{e}_3$ are the unique solutions of the Dirichlet boundary value problem $\min \mathcal{E}_{\kappa,0}$ if $\kappa \in [0, \kappa_0)$.

ii) For any $\kappa \in (0, +\infty)$, there exists $\gamma_\kappa > 0$, depending only on κ and Ω, such that for any $\gamma \in (\gamma_\kappa, +\infty)$ the constant in-plane vector fields $\boldsymbol{e}_\perp \in \mathbb{S}^2$, $\boldsymbol{e}_\perp \cdot \boldsymbol{e}_3 = 0$, are the unique global minimizers of $\mathcal{E}_{\kappa,\gamma}$.

The statements in Theorem 1 characterize the energy landscape under restrictions on the control parameters κ and γ. Our second result retrieves information on the properties of minimizers under no additional assumptions on the system parameters κ and γ. Exploiting the symmetries of the system, we prove that minimizers of $\mathcal{E}_{\kappa,\gamma}$ have values in a quadrant of $\mathbb{S}^1$.

In what follows, we denote by

$$O(3, \boldsymbol{e}_3) := \{\sigma \in O(3) : \sigma(\boldsymbol{e}_3) = \boldsymbol{e}_3 \text{ or } \sigma(\boldsymbol{e}_3) = -\boldsymbol{e}_3\}$$

the group of isometries preserving the $\boldsymbol{e}_3$-axis.

Theorem 2. *Let $N \in \mathbb{N}^*$ and $\Omega \subset \mathbb{R}^N$ be a smooth domain and let $\kappa, \gamma \in [0, +\infty)$. If $\boldsymbol{m} \in H^1(\Omega, \mathbb{S}^2)$ is a global minimizer of $\mathcal{E}_{\kappa,\gamma}$, then $\boldsymbol{m} \in C^\infty(\Omega)$, there exists $\sigma \in O(3, \boldsymbol{e}_3)$ and a lifting map $\varphi \in H^1(\Omega)$ such that $0 \leqslant \varphi \leqslant \frac{\pi}{2}$ a.e., and*

$$\boldsymbol{m} = \sigma \circ (\sin\varphi, 0, \cos\varphi) \quad \text{a.e. in } \Omega.$$

Moreover, either $\varphi \equiv 0$ in Ω so that $\boldsymbol{m}$ is constant out-of-plane (i.e., $\boldsymbol{m} \equiv \pm\boldsymbol{e}_3$), or $\varphi \equiv \frac{\pi}{2}$ in Ω so that $\boldsymbol{m}$ is constant in-plane (i.e., $\boldsymbol{m} \cdot \boldsymbol{e}_3 \equiv 0$), or $0 < \varphi < \frac{\pi}{2}$ a.e. in Ω. In any case,

$$\varphi \in \underset{\psi \in H^1(\Omega)}{\operatorname{argmin}} \left\{ \int_\Omega |\nabla\psi|^2 + \kappa^2 \int_\Omega \cos^2\psi + \frac{1}{\gamma^2} \int_{\partial\Omega} \sin^2\psi \right\}, \quad \text{if } \gamma > 0, \tag{8}$$

$$\varphi \in \underset{\psi \in H^1_0(\Omega)}{\operatorname{argmin}} \left\{ \int_\Omega |\nabla\psi|^2 + \kappa^2 \int_\Omega \cos^2\psi \right\}, \quad \text{if } \gamma = 0. \tag{9}$$

In particular, the Dirichlet problem (9) has a unique solution such that $\operatorname{Im}(\varphi) \subset (0, \frac{\pi}{2}]$ thanks to classical results about sublinear elliptic equations using that $y \mapsto \frac{\sin y}{y}$ is decreasing on $(0, \pi]$ (see Appendix II in [1]). However, this argument does not exclude a priori the possibility of having coexistence of a solution φ such that $\operatorname{Im}(\varphi) \subset (0, \frac{\pi}{2}]$ and the constant solution $\varphi \equiv 0$, corresponding to $\boldsymbol{m} \equiv \pm\boldsymbol{e}_3$. This is not possible by our next result:

Theorem 3. *Let $\Omega \subset \mathbb{R}^N$, $N \in \mathbb{N}^*$ be a smooth bounded domain and let $\kappa \in [0, +\infty)$. If $\boldsymbol{m}$ and $\bar{\boldsymbol{m}}$ are two minimizers of the problem $\min \mathcal{E}_{\kappa,0}$, then there exists $\sigma \in O(3, \boldsymbol{e}_3)$ such that $\bar{\boldsymbol{m}} = \sigma \circ \boldsymbol{m}$.*

The statements in Theorems 1 to 3 hold without any assumption on the geometry of the domain Ω. Our last result focuses on spherical domains and proves that if Ω is a ball, then any global minimizer of $\mathcal{E}_{\kappa,\gamma}$ is radially symmetric, i.e. $\boldsymbol{m} = \boldsymbol{m}(|x|)$ and by Theorem 2 has values in a quadrant of $\mathbb{S}^1$ (a particular case is illustrated in Figure 1).

Theorem 4. *Let $\kappa, \gamma \in [0, +\infty)$. Let $\Omega = B_R$ be a ball of radius $R > 0$ centered at the origin in $\mathbb{R}^N$, then any global minimizer $\boldsymbol{m}$ of $\mathcal{E}_{\kappa,\gamma}$ is radially symmetric. More precisely, there exists $\sigma \in O(3, \boldsymbol{e}_3)$ such that*

$$\sigma \circ \boldsymbol{m}(x) = \Big(\sin\Big(\frac{u(|x|)}{2}\Big), 0, \cos\Big(\frac{u(|x|)}{2}\Big)\Big) \quad \text{in } \Omega$$

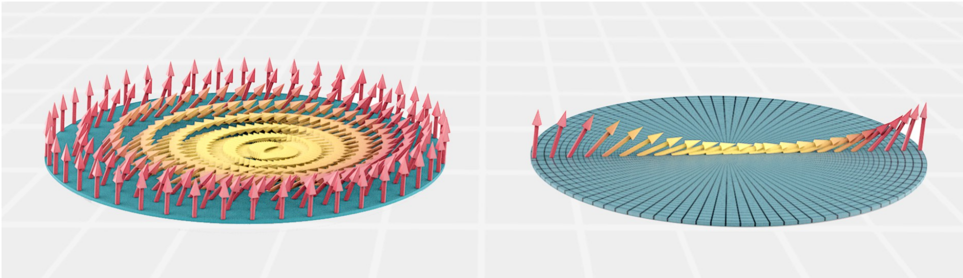

FIGURE 1. On the left, a minimizer of $\mathcal{E}_{\kappa,\gamma}$, with $\kappa^2 = 5$, $\gamma = 0.1$, in the unit disk of $\mathbb{R}^2$. On the right, we isolated a ray in order to visualize the profile of the minimizer better.

for some non-increasing function $u : [0, R] \to [0, \pi]$ which solves

$$u''(r) + \frac{N-1}{r} u' + \kappa^2 \sin u = 0 \quad \textit{in } (0, R), \tag{10}$$

with $u'(0) = 0$ and either a Dirichlet condition or a nonlinear Robin condition at $r = R$, namely

$$\begin{cases} u(R) = 0 & \textit{if } \gamma = 0, \\ u'(R) + \frac{1}{\gamma^2} \sin u(R) = 0 & \textit{if } \gamma > 0. \end{cases} \tag{11}$$

By Theorem 3,the global minimizer u when $\gamma = 0$ is unique. It is either the steady state $u \equiv 0$ or an increasing function into $(0, \pi)$.

1.3. Outline. The paper is organized as follows. In Section 2, we prove the minimality of universal configurations (Theorem 1). For that, we need a Poincaré-type inequality with a remainder, which is proved in Proposition 1. Section 3 is devoted to the analysis of symmetries of the minimizers and their range. There we prove Theorem 2. In Section 4, we show the uniqueness of minimizers under Dirichlet boundary conditions (Theorem 3). Finally, in Section 5, we focus on the case when domain is a ball, and we prove radial symmetry of energy minimizers (Theorem 4).

2. Minimality of universal configurations: Proof of Theorem 1

To investigate the minimality of the constant out-of-plane configurations $\pm \boldsymbol{e}_3$ we need the following Poincaré-type inequality, which can be of some interest on its own.

Proposition 1 (Poincaré-type inequality)**.** *Let $\Omega \subseteq \mathbb{R}^N$ be a bounded smooth domain. Then, there exists $c_\Omega > 0$ such that for every $u \in H^1(\Omega)$ and every $\delta > 0$, we have*

$$\delta\,(c_\Omega - \delta) \int_\Omega u^2(x)\mathrm{d}x \leqslant \int_\Omega |\nabla u(x)|^2\,\mathrm{d}x + \delta \int_{\partial\Omega} u^2(x)\mathrm{d}\mathcal{H}^{N-1}(x). \tag{12}$$

Moreover, in the previous relation, the constant c_Ω can be taken $c_\Omega = \frac{N}{\operatorname{diam}(\Omega)}$.

Proof. We argue along the lines in [5]. Without loss of generality, we can assume that $0 \in \Omega$. Also, by density, it is sufficient to prove (12) for every $u \in C^\infty(\overline{\Omega})$. By the divergence theorem we have

$$\int_\Omega \left(2u(x)\nabla u(x)\cdot x + N u^2(x)\right)\mathrm{d}x = \int_\Omega \operatorname{div}[u^2(x)x]\mathrm{d}x = \int_{\partial\Omega} u^2(x)\boldsymbol{n}(x)\cdot x\,\mathrm{d}\mathcal{H}^{N-1}(x),$$

where, for a.e. $x \in \partial\Omega$, we denoted by $\boldsymbol{n}(x)$ the unitary normal vector field at $\xi \in \partial\Omega$. By Young's inequality, it follows that for every $\delta > 0$, one has

$$N \int_\Omega u^2(x)\mathrm{d}x \leqslant \sup_{x\in\partial\Omega} |x| \int_{\partial\Omega} u^2(x)\mathrm{d}\mathcal{H}^{N-1}(x) + \sup_{x\in\Omega} |x| \int_\Omega \left[\frac{1}{\delta}|\nabla u(x)|^2 + \delta u^2(x)\right]\mathrm{d}x.$$

Since $\sup_{x\in\partial\Omega}|x| \leqslant \operatorname{diam}(\Omega)$ and $\sup_{x\in\Omega}|x| \leqslant \operatorname{diam}(\Omega)$, we have

$$(N-\delta\operatorname{diam}(\Omega))\int_\Omega u^2(x)\mathrm{d}x \leqslant \frac{\operatorname{diam}(\Omega)}{\delta}\int_\Omega |\nabla u(x)|^2\,\mathrm{d}x + \operatorname{diam}(\Omega)\int_{\partial\Omega} u^2(x)\mathrm{d}\mathcal{H}^{N-1}(x).$$

From the previous estimate, we get that for every $\delta>0$ there holds

$$\delta\,(c_\Omega-\delta)\int_\Omega u^2(x)\mathrm{d}x \;\leqslant \int_\Omega |\nabla u(x)|^2\,\mathrm{d}x + \delta\int_{\partial\Omega} u^2(x)\mathrm{d}\mathcal{H}^{N-1}(x),$$

with $c_\Omega := \frac{N}{\operatorname{diam}(\Omega)}$. This concludes the proof. □

Proof of Theorem 1, item i. We first consider the case where $\gamma>0$. Without loss of generality, we can focus on the configuration $\boldsymbol{m}=+\boldsymbol{e}_3$. We observe that for any $\boldsymbol{v}\in H^1(\Omega,\mathbb{R}^3)$ such that $|\boldsymbol{v}+\boldsymbol{e}_3|=1$ or, equivalently, such that $|\boldsymbol{v}|^2=-2\,(\boldsymbol{v}\cdot\boldsymbol{e}_3)$, we have

$$\begin{aligned}\mathcal{E}_{\kappa,\gamma}(\boldsymbol{e}_3+\boldsymbol{v})-\mathcal{E}_{\kappa,\gamma}(\boldsymbol{e}_3) &= \int_\Omega |\nabla\boldsymbol{v}|^2+\kappa^2\int_\Omega(\boldsymbol{v}\cdot\boldsymbol{e}_3)^2+2\,(\boldsymbol{v}\cdot\boldsymbol{e}_3)+\frac{1}{\gamma^2}\int_{\partial\Omega}|\boldsymbol{v}\times\boldsymbol{e}_3|^2\\ &= \int_\Omega|\nabla\boldsymbol{v}|^2-\kappa^2\int_\Omega|\boldsymbol{v}_\perp|^2\;+\;\frac{1}{\gamma^2}\int_{\partial\Omega}|\boldsymbol{v}_\perp|^2,\end{aligned} \tag{13}$$

with $\boldsymbol{v}_\perp=\boldsymbol{v}-(\boldsymbol{v}\cdot\boldsymbol{e}_3)\,\boldsymbol{e}_3$. Estimating the energy increment $\mathcal{E}_{\kappa,\gamma}(\boldsymbol{e}_3+\boldsymbol{v})-\mathcal{E}_{\kappa,\gamma}(\boldsymbol{e}_3)$ through the Poincaré inequality (12) we get for every $\delta>0$,

$$\begin{aligned}\mathcal{E}_{\kappa,\gamma}(\boldsymbol{e}_3+\boldsymbol{v})-\mathcal{E}_{\kappa,\gamma}(\boldsymbol{e}_3) &\geqslant \delta\,(c_\Omega-\delta)\int_\Omega|\boldsymbol{v}_\perp|^2-\delta\int_{\partial\Omega}|\boldsymbol{v}_\perp|^2-\kappa^2\int_\Omega|\boldsymbol{v}_\perp|^2\;+\;\frac{1}{\gamma^2}\int_{\partial\Omega}|\boldsymbol{v}_\perp|^2\\ &\geqslant \left(\delta\;(c_\Omega-\delta)-\kappa^2\right)\int_\Omega|\boldsymbol{v}_\perp|^2+\left(\frac{1}{\gamma^2}-\delta\right)\int_{\partial\Omega}|\boldsymbol{v}_\perp|^2.\end{aligned} \tag{14}$$

If we set $\delta_\gamma:=\min\{\frac{c_\Omega}{2},\frac{1}{\gamma^2}\}$ and $\kappa_\gamma:=(\delta_\gamma(c_\Omega-\delta_\gamma))^{1/2}>0$, then for every $\kappa\in[0,\kappa_\gamma)$ there exists $\delta\in(0,\delta_\gamma)$ such that $\delta\;(c_\Omega-\delta)>\kappa^2$ and $\frac{1}{\gamma^2}>\delta$. Hence, by (14), $\boldsymbol{e}_3$ (and so $-\boldsymbol{e}_3$) is a minimum point of $\mathcal{E}_{\kappa,\gamma}$, and any other minimum point $\boldsymbol{m}$ can only be obtained by perturbations in the $\boldsymbol{e}_3$ direction. This means that the constant out-of-plane vector fields $\pm\boldsymbol{e}_3$ are the only minimizers of $\mathcal{E}_{\kappa,\gamma}$.

A simpler argument gives a similar result for $\mathcal{E}_{\kappa,0}$. Indeed, in this case, $\boldsymbol{v}\in H_0^1(\Omega,\mathbb{R}^3)$ and (13) reads as

$$\mathcal{E}_{\kappa,0}(\boldsymbol{e}_3+\boldsymbol{v})-\mathcal{E}_{\kappa,0}(\boldsymbol{e}_3)=\int_\Omega|\nabla\boldsymbol{v}|^2-\kappa^2\int_\Omega|\boldsymbol{v}_\perp|^2.$$

But then the result follows from classical Poincaré inequality in $H_0^1(\Omega,\mathbb{R}^3)$, by taking $\kappa_0:=c_\Omega$ where c_Ω is the Poincaré constant. □

Proof of Theorem 1, item ii. The range of parameters under which the minimality of the constant in-plane configurations holds depends essentially on γ, and can be easily investigated through the classical trace inequality:

$$c_{\partial\Omega}\|u\|_{L^2(\partial\Omega)}\leqslant\|u\|_{H^1(\Omega)}, \tag{15}$$

for some $c_{\partial\Omega}>0$ and every $u\in H^1(\Omega)$. Indeed, let $\boldsymbol{e}_\perp\in\mathbb{S}^2$ such that $\boldsymbol{e}_\perp\cdot\boldsymbol{e}_3=0$ and let $\boldsymbol{v}\in H^1(\Omega,\mathbb{R}^3)$ such that $|\boldsymbol{v}+\boldsymbol{e}_\perp|=1$. In particular,

$$\begin{aligned}|(\boldsymbol{v}+\boldsymbol{e}_\perp)\times\boldsymbol{e}_3|^2-|\boldsymbol{e}_\perp\times\boldsymbol{e}_3|^2 &= |\boldsymbol{v}\times\boldsymbol{e}_3|^2+2(\boldsymbol{v}\times\boldsymbol{e}_3)(\boldsymbol{e}_\perp\times\boldsymbol{e}_3)\\ &= |\boldsymbol{v}\times\boldsymbol{e}_3|^2+2\boldsymbol{v}\cdot\boldsymbol{e}_\perp\\ &= |\boldsymbol{v}\times\boldsymbol{e}_3|^2-|\boldsymbol{v}|^2=-(\boldsymbol{v}\cdot\boldsymbol{e}_3)^2.\end{aligned}$$

Hence, we have

$$\begin{aligned}\mathcal{E}_{\kappa,\gamma}(\boldsymbol{e}_\perp+\boldsymbol{v})-\mathcal{E}_{\kappa,\gamma}(\boldsymbol{e}_\perp) &= \int_\Omega |\nabla \boldsymbol{v}|^2+\kappa^2\int_\Omega (\boldsymbol{v}\cdot\boldsymbol{e}_3)^2-\frac{1}{\gamma^2}\int_{\partial\Omega}(\boldsymbol{v}\cdot\boldsymbol{e}_3)^2\\ &\geqslant \int_\Omega |\nabla \boldsymbol{v}_\perp|^2+\left(c_{\partial\Omega}^2\cdot\min\{1,\kappa^2\}-\frac{1}{\gamma^2}\right)\int_{\partial\Omega}(\boldsymbol{v}\cdot\boldsymbol{e}_3)^2,\end{aligned}$$

where $\boldsymbol{v}_\perp=\boldsymbol{v}-(\boldsymbol{v}\cdot\boldsymbol{e}_3)\,\boldsymbol{e}_3$. Therefore, as soon as

$$\gamma\geqslant\gamma_\kappa:=\frac{1}{c_{\partial\Omega}\cdot\min\{1,\kappa\}},$$

we obtain that $\boldsymbol{e}_\perp$ is a global minimizer of $\mathcal{E}_{\kappa,\gamma}$. Moreover, if $\gamma>\gamma_\kappa$, we have that the constant in-plane vector fields $\boldsymbol{e}_\perp\in\mathbb{S}^2$, with $\boldsymbol{e}_\perp\cdot\boldsymbol{e}_3=0$, are the only minimizers of $\mathcal{E}_{\kappa,\gamma}$. Indeed, if $\mathcal{E}_{\kappa,\gamma}(\boldsymbol{e}_\perp+\boldsymbol{v})-\mathcal{E}_{\kappa,\gamma}(\boldsymbol{e}_\perp)=0$ then $\boldsymbol{v}_\perp$ is constant a.e. in Ω and, therefore, so is $(\boldsymbol{v}\cdot\boldsymbol{e}_3)$ due to constraint $|\boldsymbol{e}_\perp+\boldsymbol{v}|=1$ imposed on $\boldsymbol{v}$. Since $\boldsymbol{v}\cdot\boldsymbol{e}_3=0$ a.e. on $\partial\Omega$, we conclude that $\boldsymbol{v}$ is constant and in-plane. This concludes the proof. □

3. Symmetries in the target space and range of minimizers

Thanks to the symmetries, we shall see that the range of any minimizer is contained in a meridian of $\mathbb{S}^2$.

3.1. Symmetries of the energy functional in the target space. First, it is clear that the energy is invariant under the group of isometries that preserve the vertical coordinate axis $\mathbb{R}\boldsymbol{e}_3$, i.e.,

$$O(3,\boldsymbol{e}_3):=\{\sigma\in O(3):\sigma(\boldsymbol{e}_3)=\boldsymbol{e}_3 \text{ or } \sigma(\boldsymbol{e}_3)=-\boldsymbol{e}_3\};$$

this group is generated by the isotropy group $\{\sigma\in O(3):\sigma(\boldsymbol{e}_3)=\boldsymbol{e}_3\}$ and the reflection $\sigma_{\boldsymbol{e}_3}$ through the plane orthogonal to $\boldsymbol{e}_3$.

Proposition 2. *For every $\kappa,\gamma\in[0,+\infty)$, $\sigma\in O(3,\boldsymbol{e}_3)$ and $\boldsymbol{m}\in H^1(\Omega,\mathbb{S}^2)$, $\mathcal{E}_{\kappa,\gamma}(\boldsymbol{m})=\mathcal{E}_{\kappa,\gamma}(\sigma\circ\boldsymbol{m})$.*

Proposition 2 applies in particular to the reflection $\sigma=\sigma_{\boldsymbol{v}}$, defined by $\sigma_{\boldsymbol{v}}(\boldsymbol{w})=\boldsymbol{w}-2(\boldsymbol{v}\cdot\boldsymbol{w})\boldsymbol{v}$, through the plane orthogonal to a vector $\boldsymbol{v}\in\mathbb{S}^2$ which is either equal to $\boldsymbol{e}_3$ or orthogonal to $\boldsymbol{e}_3$. Using the fact that the H^1 seminorm is preserved by taking the positive or negative parts, we also have the following result.

Proposition 3. *Let $\kappa\in[0,+\infty)$, $\boldsymbol{v}\in\mathbb{S}^2$ and $\boldsymbol{m}\in H^1(\Omega,\mathbb{S}^2)$. If either $\boldsymbol{v}=\boldsymbol{e}_3$ or $\boldsymbol{v}\cdot\boldsymbol{e}_3=0$, then $\mathcal{E}_{\kappa,\gamma}(\boldsymbol{m})=\mathcal{E}_{\kappa,\gamma}(\sigma_{\boldsymbol{v}}^+\circ\boldsymbol{m})$, where*

$$\sigma_{\boldsymbol{v}}^+(\boldsymbol{w}):=\begin{cases}\boldsymbol{w} & \text{if } \boldsymbol{w}\cdot\boldsymbol{v}\geqslant 0,\\ \boldsymbol{w}-2(\boldsymbol{v}\cdot\boldsymbol{w})\boldsymbol{v} & \text{if } \boldsymbol{w}\cdot\boldsymbol{v}<0.\end{cases}\tag{16}$$

This applies for instance to $\left(\sigma_{\boldsymbol{e}_1}^+\circ\boldsymbol{m}\right)=(|m_1|,m_2,m_3)$, $\left(\sigma_{\boldsymbol{e}_2}^+\circ\boldsymbol{m}\right)=(m_1,|m_2|,m_3)$ and $\left(\sigma_{\boldsymbol{e}_3}^+\circ\boldsymbol{m}\right)=(m_1,m_2,|m_3|)$.

3.2. Regularity of minimizers. For Ω a two-dimensional domain, the regularity of minimizers follows from the classical regularity theory of Schoen-Uhlenbeck [29]. However the regularity in dimension $N\geqslant 3$ is not trivially guaranteed in our problem, as there may exist singular homogeneous harmonic maps into $\mathbb{S}^2$ such as $x\mapsto\frac{x}{|x|}$ in $\mathbb{R}^3$. Here, we can prove regularity by using the symmetries. We start with an easy lemma.

Lemma 1. *Let $u\in W^{1,p}(\Omega)$ be a Sobolev function defined on an open set $\Omega\subset\mathbb{R}^N$, $p\geqslant 1$. If $|u|$ is continuous, then u is continuous.*

Proof. If $u(x) = 0$, then u is continuous at x. If $u(x) \neq 0$, then, as $|u|$ is continuous, there exists a non empty ball $B_r(x) \subset \Omega$ where $|u| \geqslant \alpha > 0$. Let $v \in W^{1,p}(B_r(x))$ be defined by $v(x) := \max\{\min\{\frac{1}{\alpha} u(x), 1\}, -1\}$. We have that $v(x) \in \{-1, 1\}$ everywhere in $B_r(x)$, which for a Sobolev function means that v is equal to a constant a.e. in $B_r(x)$. This means that the sign of u does not change on $B_r(x)$, i.e. that $u = |u|$ a.e. in $B_r(x)$ or $u = -|u|$ a.e. in $B_r(x)$. Thus, u is continuous. □

Proposition 4. *Let $\kappa, \gamma \in [0, +\infty)$ and let $\boldsymbol{m} \in H^1(\Omega, \mathbb{S}^2)$ be a global minimizer of $\mathcal{E}_{\kappa,\gamma}$. Then $\boldsymbol{m} \in \mathcal{C}^\infty(\Omega, \mathbb{S}^2)$.*

Proof. By Proposition 3, $|\boldsymbol{m}| := (|m_1|, |m_2|, |m_3|)$ is still a global minimizer of $\mathcal{E}_{\kappa,\gamma}$. In particular, $|\boldsymbol{m}|$ is a global minimizer of $\mathcal{E}_\kappa$ under its own boundary condition. Since $|\boldsymbol{m}|$ is valued into a strictly convex subset of the sphere $\mathbb{S}^2$ and since $\mathcal{E}_\kappa$ is nothing but a perturbation of the Dirichlet energy by a lower order term (namely, the zero-order term of energy density $\kappa^2(\boldsymbol{m}\cdot\boldsymbol{e}_3)^2$), we deduce from [29, Theorem IV and its corollary] that $|\boldsymbol{m}|$ is continuous in Ω[1]. Hence $\boldsymbol{m}$ is continuous by Lemma 1. But it is then standard to prove that $\boldsymbol{m}$ is smooth (see [29] for instance). □

3.3. Range of minimizers.

We start with the following consequence of the maximum principle.

Lemma 2. *Let $\kappa, \gamma \in [0, +\infty)$ and $\boldsymbol{v} \in \mathbb{S}^2$ such that either $\boldsymbol{v} \cdot \boldsymbol{e}_3 = 0$ or $\boldsymbol{v} \in \{-\boldsymbol{e}_3, \boldsymbol{e}_3\}$. If $\boldsymbol{m}$ is a global minimizer of $\mathcal{E}_{\kappa,\gamma}$, then either $\boldsymbol{m} \cdot \boldsymbol{v} \equiv 0$ in Ω or $\boldsymbol{m} \cdot \boldsymbol{v}$ never vanishes in Ω.*

Proof. By Proposition 3, $\sigma_{\boldsymbol{v}}^+ \circ \boldsymbol{m}$ is still a minimizer of $\mathcal{E}_{\kappa,\gamma}$. By Proposition 4, $\sigma_{\boldsymbol{v}}^+ \circ \boldsymbol{m}$ is smooth. In particular, $\sigma_{\boldsymbol{v}}^+ \circ \boldsymbol{m}$ (and not only $\boldsymbol{m}$) solves the Euler-Lagrange equation (4); projecting this equation on $\boldsymbol{v}$, we obtain that $(\sigma_{\boldsymbol{v}}^+ \circ \boldsymbol{m}) \cdot \boldsymbol{v} = |\boldsymbol{m} \cdot \boldsymbol{v}|$ solves the elliptic equation

$$\Delta |\boldsymbol{m} \cdot \boldsymbol{v}| + c(x) |\boldsymbol{m} \cdot \boldsymbol{v}| = 0 \quad \text{in } \Omega,$$

with

$$c(x) = \begin{cases} |\nabla (\sigma_{\boldsymbol{v}}^+ \circ \boldsymbol{m})|^2 + \kappa^2 m_3^2 & \text{if } \boldsymbol{v} \cdot \boldsymbol{e}_3 = 0, \\ |\nabla (\sigma_{\boldsymbol{v}}^+ \circ \boldsymbol{m})|^2 + \kappa^2 (m_3^2 - 1) & \text{if } \boldsymbol{v} = \boldsymbol{e}_3. \end{cases}$$

We then apply the maximum principle [17, Theorem 2.10] to find that either $\boldsymbol{m} \cdot \boldsymbol{v} \equiv 0$ or $\boldsymbol{m} \cdot \boldsymbol{v}$ does not vanish in Ω. □

Proof of Theorem 2. By Proposition 4, $\boldsymbol{m}$ is smooth. For the rest of the proof, we proceed in three steps,

Step 1. $\boldsymbol{m}$ is valued into a meridian. For $\boldsymbol{v} \in \mathbb{S}^1 \times \{0\}$, we denote by $\mathbb{S}^2_+(\boldsymbol{v})$ the closed hemisphere directed by $\boldsymbol{v}$, i.e., the closed subset of $\mathbb{S}^2$ obtained intersecting $\mathbb{S}^2$ with the closed half-space $\{z \in \mathbb{R}^3 \,:\, z \cdot \boldsymbol{v} \geq 0\}$. If $\boldsymbol{m} \equiv \pm \boldsymbol{e}_3$ in Ω there is nothing to prove. If not, there exists $x_0 \in \Omega$ such that the projection $\boldsymbol{m}_\perp(x_0)$ of $\boldsymbol{m}(x_0)$ onto the plane orthogonal to $\boldsymbol{e}_3$ is different from zero. We set $\boldsymbol{v}_0 := \boldsymbol{m}_\perp(x_0)/|\boldsymbol{m}_\perp(x_0)|$ and we claim that the target space of $\boldsymbol{m}$ is contained in the meridian passing through $\boldsymbol{v}_0$. By construction, $\boldsymbol{m}(x_0) \cdot \boldsymbol{v} > 0$ for every $\boldsymbol{v} \in \mathbb{S}^1 \times \{0\}$ such that $\boldsymbol{v} \cdot \boldsymbol{v}_0 > 0$. Therefore, by Lemma 2 and the continuity of $\boldsymbol{m}$, we get that for every $x \in \Omega$ there holds

$$\boldsymbol{m}(x) \in \bigcap_{\{\boldsymbol{v} \in \mathbb{S}^1 \times \{0\} : \boldsymbol{v} \cdot \boldsymbol{v}_0 > 0\}} \mathbb{S}^2_+(\boldsymbol{v}).$$

As the intersection on the right-hand side is the meridian passing through $\boldsymbol{v}_0$ we conclude.

[1]Note that the Shoen-Uhlenbeck regularity theory gives smoothness of $\boldsymbol{m}$ with no restriction on the image of $\boldsymbol{m}$ in dimension $N = 2$; in dimension $N \geqslant 3$, the presence of singularities is ruled out thanks to the condition that $|\boldsymbol{m}|$ is valued into a strictly convex subset of $\mathbb{S}^2$.

Step 2. The image of $\boldsymbol{m}$ is contained in a quarter of meridian. Indeed, let $\boldsymbol{v}_0^\perp \in \mathbb{S}^2$ satisfy $\boldsymbol{v}_0^\perp \cdot \boldsymbol{v}_0 = \boldsymbol{v}_0^\perp \cdot \boldsymbol{e}_3 = 0$. Applying again Lemma 2 to $\boldsymbol{v} = \boldsymbol{e}_3$ and $\boldsymbol{v} = \boldsymbol{v}_0^\perp$, we obtain that

$$\boldsymbol{m} \cdot \boldsymbol{v}_0 = 0, \quad \pm \boldsymbol{m} \cdot \boldsymbol{v}_0^\perp \geqslant 0, \quad \pm \boldsymbol{m} \cdot \boldsymbol{e}_3 \geqslant 0 \quad \text{a.e. in } \Omega.$$

Step 3. Conclusion. Since $O(3, \boldsymbol{e}_3)$ acts transitively on the quadrant of meridians, we can express $\boldsymbol{m}$ in terms of a particular solution valued into the quadrant of meridian $\{m_2 = 0\} \cap \{m_1, m_3 \geqslant 0\}$. Namely, there exists $\sigma \in O(3, \boldsymbol{e}_3)$ such that $\boldsymbol{m} = \sigma \circ \boldsymbol{u}$, where $\boldsymbol{u} \in H^1(\Omega, \mathbb{S}^1)$ is of the form $\boldsymbol{u} = (u_1, 0, u_2)$ with $u_1, u_2 \in H^1(\Omega, \mathbb{R})$ such that $u_1^2 + u_2^2 = 1$ and $u_1, u_2 \geqslant 0$ a.e. in Ω. We then lift the map $\boldsymbol{u}$ to $\mathbb{R}$ by writing $\boldsymbol{u} = (\sin\varphi, 0, \cos\varphi)$ with $\varphi \in H^1(\Omega)$ and $0 \leqslant \varphi \leqslant \frac{\pi}{2}$ a.e. in Ω. We conclude by noticing that Lemma 2 also tells us that either $\varphi \equiv 0$, or $\varphi \equiv \frac{\pi}{2}$, or $0 < \varphi < \frac{\pi}{2}$ a.e. in Ω. □

4. Uniqueness of minimizers under Dirichlet boundary conditions

Theorem 3 is a direct consequence of the following estimate.

Lemma 3. *Let $\boldsymbol{m} \in H^1(\Omega, \mathbb{S}^2)$ and $\boldsymbol{v} \in H_0^1(\Omega, \mathbb{R}^3)$ satisfy $\boldsymbol{m} + \boldsymbol{v} \in \mathbb{S}^2$ a.e. in Ω. If $\boldsymbol{m}$ satisfies the Euler-Lagrange equations (3) and if $m_1 > 0$ a.e. in Ω, then m_1 is bounded below by positive constants on compact subsets of Ω and*

$$\mathcal{E}_\kappa(\boldsymbol{m} + \boldsymbol{v}) - \mathcal{E}_\kappa(\boldsymbol{m}) \geqslant \int_\Omega m_1^2 \left| \nabla \left(\frac{\boldsymbol{v}}{m_1} \right) \right|^2 + \kappa^2 \int_\Omega (\boldsymbol{v} \cdot \boldsymbol{e}_3)^2. \tag{17}$$

Proof of Lemma 3. We follow the ideas of [9, Theorem 4.3] and [20, Theorem 5.1]. We have

$$\mathcal{E}_\kappa(\boldsymbol{m} + \boldsymbol{v}) - \mathcal{E}_\kappa(\boldsymbol{m}) = \int_\Omega |\nabla \boldsymbol{v}|^2 + \kappa^2 \int_\Omega (\boldsymbol{v} \cdot \boldsymbol{e}_3)^2 + 2 \int_\Omega \nabla \boldsymbol{m} : \nabla \boldsymbol{v} + 2\kappa^2 \int_\Omega (\boldsymbol{m} \cdot \boldsymbol{e}_3)(\boldsymbol{v} \cdot \boldsymbol{e}_3).$$

Note that since $|\boldsymbol{m}| = |\boldsymbol{m} + \boldsymbol{v}| = 1$ a.e., we also have $|\boldsymbol{v}| \leqslant 2$ a.e. in Ω. In particular, $\boldsymbol{v} \in H_0^1(\Omega, \mathbb{R}^3) \cap L^\infty(\Omega, \mathbb{R}^3)$. Since $\boldsymbol{m}$ satisfies the Euler-Lagrange equations (3), we get

$$\mathcal{E}_\kappa(\boldsymbol{m} + \boldsymbol{v}) - \mathcal{E}_\kappa(\boldsymbol{m}) = \int_\Omega |\nabla \boldsymbol{v}|^2 + \kappa^2 \int_\Omega (\boldsymbol{v} \cdot \boldsymbol{e}_3)^2 + 2 \int_\Omega (|\nabla \boldsymbol{m}|^2 + \kappa^2 (\boldsymbol{m} \cdot \boldsymbol{e}_3)^2) \boldsymbol{m} \cdot \boldsymbol{v}.$$

On the other hand, since $|\boldsymbol{m} + \boldsymbol{v}| = 1$, we have $2\boldsymbol{m} \cdot \boldsymbol{v} = -|\boldsymbol{v}|^2$ and, therefore,

$$\mathcal{E}_\kappa(\boldsymbol{m} + \boldsymbol{v}) - \mathcal{E}_\kappa(\boldsymbol{m}) = \int_\Omega |\nabla \boldsymbol{v}|^2 + \kappa^2 \int_\Omega (\boldsymbol{v} \cdot \boldsymbol{e}_3)^2 - \int_\Omega (|\nabla \boldsymbol{m}|^2 + \kappa^2 (\boldsymbol{m} \cdot \boldsymbol{e}_3)^2) |\boldsymbol{v}|^2. \tag{18}$$

Hence, (17) will follow once we prove that for all $\boldsymbol{v} \in H_0^1 \cap L^\infty(\Omega, \mathbb{R}^3)$,

$$\int_\Omega |\nabla \boldsymbol{v}|^2 \geqslant \int_\Omega (|\nabla \boldsymbol{m}|^2 + \kappa^2 (\boldsymbol{m} \cdot \boldsymbol{e}_3)^2) |\boldsymbol{v}|^2 + \int_\Omega m_1^2 \left| \nabla \left(\frac{\boldsymbol{v}}{m_1} \right) \right|^2. \tag{19}$$

We first assume that $\boldsymbol{v} \in C_c^2(\Omega, \mathbb{R}^3)$, the general case will follow by density.

Now, by the Euler-Lagrange equation of m_1 in (3) and since m_1 is assumed to be positive in Ω, we have in particular that m_1 is a positive weak superharmonic function, i.e. $\Delta m_1 \leqslant 0$ weakly in Ω; we deduce from the weak Harnack-Moser inequality (see [21, Theorem 14.1.2.]) that m_1 is bounded from below by a positive constant on the support of $\boldsymbol{v}$. Hence, we can write $\boldsymbol{v}$ in the form

$$\boldsymbol{v} = m_1 \boldsymbol{u}, \tag{20}$$

where $\boldsymbol{u} = \frac{\boldsymbol{v}}{m_1} \in H_0^1(\Omega, \mathbb{R}^3) \cap L^\infty(\Omega, \mathbb{R}^3)$. We then compute

$$\int_\Omega |\nabla \boldsymbol{v}|^2 = \sum_{j=1}^N \int_\Omega |\boldsymbol{u}\partial_j m_1 + m_1 \partial_j \boldsymbol{u}|^2 \tag{21}$$

$$= \int_\Omega |\boldsymbol{u}|^2 |\nabla m_1|^2 + m_1^2 |\nabla \boldsymbol{u}|^2 + m_1 \nabla m_1 \cdot \nabla |\boldsymbol{u}|^2 \tag{22}$$

$$= \int_\Omega m_1^2 |\nabla \boldsymbol{u}|^2 + \nabla m_1 \cdot \nabla (m_1 |\boldsymbol{u}|^2). \tag{23}$$

Now, testing the Euler-Lagrange equations (3) against $\boldsymbol{\varphi} := m_1 |\boldsymbol{u}|^2 \boldsymbol{e}_1 \in H_0^1(\Omega, \mathbb{R}^3) \cap L^\infty(\Omega, \mathbb{R}^3)$, we obtain

$$\int_\Omega \nabla m_1 \cdot \nabla (m_1 |\boldsymbol{u}|^2) = \int_\Omega (|\nabla \boldsymbol{m}|^2 + \kappa^2 (\boldsymbol{m} \cdot \boldsymbol{e}_3)^2) m_1^2 |\boldsymbol{u}|^2. \tag{24}$$

Combining the previous two relations, and recalling that $\boldsymbol{v} = m_1 \boldsymbol{u}$, we obtain the following identity:

$$\int_\Omega |\nabla \boldsymbol{v}|^2 = \int_\Omega m_1^2 |\nabla \boldsymbol{u}|^2 + (|\nabla \boldsymbol{m}|^2 + \kappa^2 (\boldsymbol{m} \cdot \boldsymbol{e}_3)^2) |\boldsymbol{v}|^2. \tag{25}$$

This proves (19) in the case where $\boldsymbol{v} \in C_c^\infty(\Omega, \mathbb{R}^3)$. In general, we have $\boldsymbol{v} \in H_0^1(\Omega, \mathbb{R}^3) \cap L^\infty(\Omega, \mathbb{R}^3)$ and there thus exists a sequence $(\boldsymbol{v}_n)_{n\in\mathbb{N}}$ in $C_c^\infty(\Omega, \mathbb{R}^3)$ such that

$$\sup_{n\in\mathbb{N}} \|\boldsymbol{v}_n\|_\infty \leqslant \|\boldsymbol{v}\|_\infty + 1$$

and

$$\boldsymbol{v}_n \to \boldsymbol{v} \quad \text{in } H_0^1(\Omega, \mathbb{R}^3). \tag{26}$$

By the previous computations in the smooth case, we have for every compact $K \subset \Omega$ and $n \in \mathbb{N}$,

$$\int_\Omega |\nabla \boldsymbol{v}_n|^2 \geqslant \int_K (|\nabla \boldsymbol{m}|^2 + \kappa^2 (\boldsymbol{m} \cdot \boldsymbol{e}_3)^2) |\boldsymbol{v}_n|^2 + \int_K m_1^2 \left| \nabla \left(\frac{\boldsymbol{v}_n}{m_1} \right) \right|^2.$$

The conclusion follows by passing to the limit $n \to \infty$ using the dominated convergence theorem, and then taking the supremum over compacts $K \subset \Omega$ using the monotone convergence theorem. □

Proof of Theorem 3. If the constant out-of-plane configurations $\pm \boldsymbol{e}_3$ are the only global minimizers of $\mathcal{E}_{\kappa,0}$, we are done. If not, this means by Theorem 2 that $\mathcal{E}_{\kappa,0}$ has a global minimizer of the form $\boldsymbol{m} = (\sin\varphi, 0, \cos\varphi)$ with $\varphi \in H^1(\Omega)$ such that $0 < \varphi \leqslant \frac{\pi}{2}$ a.e. in Ω. If $\bar{\boldsymbol{m}} = \boldsymbol{m} + \boldsymbol{v}$ is another minimizer with $\boldsymbol{v} \in H_0^1(\Omega, \mathbb{R}^3)$, then we have by Lemma 3 that $\boldsymbol{v} = m_1 \boldsymbol{v}_0$ for some $\boldsymbol{v}_0 \in \mathbb{R}^3$. But, in order to satisfy the constraint $\boldsymbol{m} + m_1 \boldsymbol{v}_0 \in \mathbb{S}^2$, we must have $\boldsymbol{v}_0 \cdot (m_1 \boldsymbol{v}_0 + 2\boldsymbol{m}) = 0$. Restricted to the boundary $\partial\Omega$, where we have $\boldsymbol{m} = \boldsymbol{e}_3$, this condition yields $\boldsymbol{v}_0 \cdot \boldsymbol{e}_3 = 0$. Hence, since $m_2 \equiv 0$, we arrive at the equation $0 = \boldsymbol{v}_0 \cdot (m_1 \boldsymbol{v}_0 + 2m_1 e_1)$ which means that $|\boldsymbol{v}_0 + e_1|^2 = 1$. Hence, $\boldsymbol{v}_0 = (\cos\theta - 1, \sin\theta, 0)$ for some $\theta \in \mathbb{R}$, which means that $\bar{\boldsymbol{m}} = (\cos\theta \sin\varphi, \sin\theta \sin\varphi, \cos\varphi)$, i.e. $\boldsymbol{m}$ is a rotation of $\boldsymbol{m}$ of angle θ around the x_3-axis. □

5. Radial symmetry of minimizers in a ball: Proof of Theorem 4

Numerical simulations suggest that when the domain Ω has spherical symmetry, the minimizers of $\mathcal{E}_{\kappa,\gamma}$ are radially symmetric (cf. Figure 1). The aim of this section is to turn this observation into a quantitative statement.

The proof we give below for the radial symmetry of minimizers $\mathcal{E}_{\kappa,\gamma}$ also works for the boundary value problem associated with $\mathcal{E}_{\kappa,0}$. However, radiality of the minimizers of $\mathcal{E}_{\kappa,0}$ immediately follows from a celebrated result of Gidas-Ni-Nirenberg [13] about radial symmetry for semilinear elliptic equations. We give the details below.

Proposition 5. *If Ω is a ball centered at the origin, then any minimizer $\boldsymbol{m}$ of the energy $\mathcal{E}_{\kappa,0}$ is radially symmetric.*

More precisely, $\boldsymbol{m}$ is either constant with $\boldsymbol{m}\cdot\boldsymbol{e}_3 \in \{0,-1,1\}$, or there exist $\sigma \in O(3,\boldsymbol{e}_3)$ and a solution $\varphi : \mathbb{R}^+ \to (0,\frac{\pi}{2})$ in (9) such that

$$\boldsymbol{m}(x) = \sigma \circ (\sin\varphi(|x|), 0, \cos\varphi(|x|)) \quad \textit{a.e. in } \Omega.$$

Proof of Proposition 5. Without loss of generality, one can assume that $\boldsymbol{m}$ is not constant. By Theorem 2, there exists $\sigma \in O(3,\boldsymbol{e}_3)$ and a solution $\varphi \in H_0^1(\Omega)$ of (9) such that $\boldsymbol{m} = \sigma(\sin\varphi, 0, \cos\varphi)$ and $0 < \varphi < \frac{\pi}{2}$ a.e. in Ω. In particular, φ solves the Euler-Lagrange equation $\Delta(2\varphi)+\kappa^2\sin(2\varphi) = 0$ in the weak sense in Ω. By elliptic regularity, φ is smooth and the equation holds in the strong sense. The radial symmetry of φ then follows from Gidas-Ni-Nirenberg [13]. □

In the case of the penalization of the boundary datum, we use a reflection method introduced in [23] and the unique continuation principle for elliptic equations (see, for instance, [26]). Note that this method also works for the boundary value problem associated with $\mathcal{E}_{\kappa,0}$, and the following proof also covers Proposition 5.

Proof of Theorem 4. We concentrate on the case $\gamma > 0$. (The case $\gamma = 0$ is similar, and also covered by Proposition 5.) Without loss of generality, one can assume that $\boldsymbol{m}$ is not constant. By Theorem 2, there exists $\sigma \in O(3,\boldsymbol{e}_3)$ and a solution $\varphi \in H^1(B_R)$ of (8) such that $\boldsymbol{m} = \sigma(\sin\varphi, 0, \cos\varphi)$ and $0 < \varphi < \frac{\pi}{2}$ a.e. in B_R. As before, we get that φ is a solution of

$$\Delta(2\varphi) + \kappa^2 \sin(2\varphi) = 0 \quad \text{in } B_R. \tag{27}$$

Now, let H be a hyperplane passing through the origin and dividing $\mathbb{R}^N$ into two half-spaces H^+ and H^-. Up to interchange H^+ and H^-, one can assume that

$$\int_{H^-\cap B_R} |\nabla\varphi|^2 + \kappa^2\cos^2\varphi + \frac{1}{\gamma^2}\int_{H^-\cap\partial B_R} \sin^2\varphi \leqslant \int_{H^+\cap B_R} |\nabla\varphi|^2 + \kappa^2\cos^2\varphi + \frac{1}{\gamma^2}\int_{H^+\cap\partial B_R} \sin^2\varphi.$$

Let $\varphi_* \in H^1(B_R)$ be defined by $\varphi_* = \varphi$ on $H^-\cap B_R$ and $\varphi_* = \varphi\circ\sigma_H$ on $H^+\cap B_R$ where σ_H stands for the reflection through H. By the previous inequality, we have that $\mathcal{E}_{\kappa,\gamma}(\varphi_*) \leqslant \mathcal{E}_{\kappa,\gamma}(\varphi)$, i.e., φ_* is also a global minimizer. Hence it also solves (27). However, since $\varphi_* = \varphi$ on $H^-\cap B_R$, we deduce by the unique continuation principle (see Theorem III in [26]) that $\varphi_* = \varphi$, i.e., $\varphi = \varphi\circ\sigma_H$ in B_R. Since the hyperplane H is arbitrary, this means that φ is radially symmetric. This means that we can write $\varphi(x) = \frac{u(|x|)}{2}$ for every $x \in B_R$, for some function $u : [0,R] \to [0,\pi]$. Moreover, since φ is smooth, u is smooth.

We now argue that u is nonincreasing. Indeed, define the nonincreasing rearrangement of u by $u^*(r) = \sup_{s\in[r,R]} u(s)$. We have that u^* is Lipschitz with $|(u^*)'| \leqslant |u'|$ on $[0,R]$ since if $0 \leqslant r_1 \leqslant r_2 \leqslant R$, then $u^*(r_2) \leqslant u^*(r_1)$ and

$$u^*(r_1) \leqslant \sup_{s\in[r_2,R]} u(s) + \sup_{s\in[r_1,r_2]} |u(s) - u(r_2)| \leqslant u^*(r_2) + (r_2 - r_1) \sup_{s\in[r_1,r_2]} |u'(s)|.$$

But then, the function $\varphi^* \in W^{1,2}(B_R)$, defined by $\varphi^*(x) = u^*(|x|)$ for every $x \in B_R$, satisfies $\varphi = \varphi^*$ a.e. on ∂B_R, and $\cos\varphi^* \leqslant \cos\varphi$ and $|\nabla\varphi^*| \leqslant |\nabla\varphi|$ a.e. in B_R. Hence, $\cos\varphi^* = \cos\varphi$ a.e., and so $\varphi = \varphi^*$ a.e., since otherwise, φ^* would have strictly less energy than φ in (8).

Last of all, as a solution of (8), $\varphi(x) = \frac{u(|x|)}{2}$ must be a solution of the associated Euler-Lagrange equation, which means that u solves the system (10)-(11). □

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