# GINZBURG–LANDAU RELAXATION FOR HARMONIC MAPS ON PLANAR DOMAINS INTO A GENERAL COMPACT VACUUM MANIFOLD

ANTONIN MONTEIL, RÉMY RODIAC, AND JEAN VAN SCHAFTINGEN

ABSTRACT. We study the asymptotic behaviour, as a small parameter $\varepsilon$ tends to zero, of minimisers of a Ginzburg–Landau type energy with a nonlinear penalisation potential vanishing on a compact submanifold $\mathcal{N}$ and with a given $\mathcal{N}$–valued Dirichlet boundary data. We show that minimisers converge up to a subsequence to a singular $\mathcal{N}$–valued harmonic map, which is smooth outside a finite number of points around which the energy concentrates and whose singularities' location minimises a renormalised energy, generalising known results by Bethuel, Brezis and Hélein for the circle $\mathbb{S}^1$. We also obtain $\Gamma$–convergence results and uniform Marcinkiewicz weak $L^2$ or Lorentz $L^2$ estimates on the derivatives. We prove that solutions to the corresponding Euler–Lagrange equation converge uniformly to the constraint and converge to harmonic maps away from singularities.

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## 1. INTRODUCTION

Given a smooth compact connected manifold $\mathcal{N}$ which can be assumed, thanks to Nash's embedding theorem [44], to be isometrically embedded into $\mathbb{R}^\nu$ for some $\nu \in \mathbb{N}_*$, given a bounded domain $\Omega \subset \mathbb{R}^2$ with Lipschitz boundary and given $g \in W^{1/2,2}(\partial\Omega, \mathcal{N})$, a *minimising harmonic map* $u$ is a map $u : \Omega \to \mathcal{N}$ which minimises the *Dirichlet energy*

$$\int_\Omega \frac{|Du|^2}{2} \tag{1.1}$$

on the nonlinear subspace

$$W^{1,2}_g(\Omega, \mathcal{N}) \coloneqq \{u \in W^{1,2}(\Omega, \mathbb{R}^\nu) \ : \ u \in \mathcal{N} \text{ almost everywhere in } \Omega \text{ and } \operatorname{tr}_{\partial\Omega} u = g\} \tag{1.2}$$

of the *Sobolev space* $W^{1,2}(\Omega, \mathbb{R}^\nu)$ of functions having a square-summable weak derivative. It is known since Morrey's work that, when the domain $\Omega$ is two-dimensional, any minimising harmonic map is smooth [42].

Because of topological obstructions, the set $W^{1,2}_g(\Omega, \mathcal{N})$ can happen to be empty; if $g \in \mathcal{C}(\partial\Omega, \mathcal{N})$, this will be the case if and only if the map $g$ cannot be extended to a continuous map from $\Omega$ to $\mathcal{N}$ (see [50]). This occurs for example when the domain $\Omega$ is simply connected while the manifold $\mathcal{N}$ is not simply-connected and the map $g$ is not homotopic to a constant map.

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The *Ginzburg–Landau relaxation* strategy consists in replacing the constraint that $u \in \mathcal{N}$ almost everywhere in $\Omega$ by an additional penalisation term to the Dirichlet energy (1.1). Fixing a nonnegative function $F \in \mathcal{C}(\mathbb{R}^\nu, [0, +\infty))$ such that $F^{-1}(\{0\}) = \mathcal{N}$, one defines for every $\varepsilon \in (0, +\infty)$, the *Ginzburg–Landau energy* as

$$\mathcal{E}^\varepsilon_F(u) = \int_\Omega \frac{|Du|^2}{2} + \frac{F(u)}{\varepsilon^2}. \tag{1.3}$$

In the present work, we will require $F$ to satisfy the following non-degeneracy condition:

$$\begin{gathered}\text{there exist } \delta_F, m_F, M_F \in (0, +\infty) \text{ such that for every } z \in \mathbb{R}^\nu \text{ with } \operatorname{dist}(z, \mathcal{N}) < \delta_F,\\ \frac{m_F}{2} \operatorname{dist}(z, \mathcal{N})^2 \le F(z) \le \frac{M_F}{2} \operatorname{dist}(z, \mathcal{N})^2.\end{gathered} \tag{1.4}$$

The existence of minimisers of $\mathcal{E}^\varepsilon_F$ under the Dirichlet boundary condition $\operatorname{tr}_{\partial\Omega} u = g$ follows from a classical result in the direct method of calculus of variations (see for example [23, Corollary 3.24]). When $\varepsilon \to 0$, one expects the function $u_\varepsilon$ to eventually take its value into $\mathcal{N}$ except in some small singular regions; the limiting map can then play a role of generalised solution of the Dirichlet problem for harmonic maps into $\mathcal{N}$.

Our first result (Theorem 7.3) describes this asymptotic behaviour of minimisers of the Ginzburg–Landau energy when $\varepsilon \to 0$: if for each $\varepsilon > 0$, $u_\varepsilon$ is a minimiser of the Ginzburg–Landau energy $\mathcal{E}^\varepsilon_F$ under the boundary condition $\operatorname{tr}_{\partial\Omega} u_\varepsilon = g$, then there exists a sequence $(\varepsilon_n)_{n\in\mathbb{N}}$ converging to 0, a finite set of points $\{a_1, \dots, a_k\} \subset \Omega$ and a map $u_* \in W^{1,2}_{\mathrm{loc}}(\bar\Omega \setminus \{a_1, \dots, a_k\})$ such that $u_{\varepsilon_n} \to u_*$ strongly in $W^{1,2}_{\mathrm{loc}}(\bar\Omega \setminus \{a_1, \dots, a_k\})$, $u_*$ is an $\mathcal{N}$-valued harmonic map in $\Omega \setminus \{a_1, \dots, a_k\}$ and the configuration of points $\{a_1, \dots, a_k\}$ minimizes a *renormalised energy*. This renormalised energy is defined as the sum of a renormalised energy for harmonic maps that we have defined in [39] and that we present in §3, and a term defined in §4 depending on the singularities and on the penalisation nonlinearity $F$.

When $\mathcal{N} = \mathbb{S}^1 \subset \mathbb{R}^2$and $F(z) = (1 - |z|^2)^2$, we recover the seminal results of Bethuel, Brezis & Hélein [9], for the original Ginzburg–Landau functional used to model the behaviour of *type II superconductors* for a star-shaped domain $\Omega$; the results were later extended to simply-connected domains in [52]; here we do not assume that $\Omega$ is simply connected in our work and provide thus new results for the original Ginzburg–Landau functional in the multiply connected case. In the case of a general target manifold $\mathcal{N}$, the leading-order asymptotics and the topological charges of singularities in our results (Theorem 7.3 (ii) and (vi) at the $o(\log 1/\varepsilon)$ level) are due to Canevari [15].

Functionals of the form (1.3) appear in various other physical models besides the Ginzburg–Landau model in superconductivity. The *Landau–de Gennes theory* describes the state of a *nematic liquid crystal* via a field of symmetric traceless $3 \times 3$ matrix which minimises an energy of the form (1.3) with $\mathcal{N} \simeq \mathbb{RP}^2$; the study of such minimisers has been the object of many works [4, 5, 15, 30]. Energies of the form (1.3) also appear in physics in *Chern-Simon-Higgs theory* [5] with $\mathcal{N} = \mathbb{S}^1 \times \{0\} \simeq \mathbb{S}^1$ and other phase transitions problems like *biaxial molecules in nematic phase* ($\mathcal{N} \simeq SU(2)/Q$, where $Q$ is the quaternion group), *superfluid $^3$He in dipole-free phase* with $\mathcal{N} \simeq SU(2) \times SU(2)/H$ where $H$ is a subgroup of $SU(2) \times SU(2)$ isomorphic to four copies of $\mathbb{S}^1$ and *superfluid $^3$He in dipole-locked phase* with $\mathcal{N} \simeq \mathbb{RP}^3$ [37].

Minimisation of Ginzburg–Landau type energies has also appeared as a strategy in meshing algorithms for numerical analysis and computer graphics: in order to generate a quadrangular meshing of a surface or a hexahedral meshing of a three-dimensional domain, one constructs first a guiding cross-field or frame-field which is mathematically a map taking its value into $SO(2)/C_4$ and $SO(3)/O$, where $C_4$ is the cyclic group of order 4 of direct symmetries of a square, and $O$ is the octahedral group of direct symmetries of the cube [6, 18, 31, 36, 53]. Mathematically, in the latter case $\pi_1(SO(3)/O) = 2O$ is the *nonabelian* binary octahedral group. Since one would like these cross-fields or frame-fields to minimise a Dirichlet energy and since one can face topological obstructions as described earlier in this introduction, the strategy consists in constructing these fields using a Ginzburg–Landau relaxation. The cross-fields and frame-fields will necessarily have singularities and one expect to place these singularities in an optimal way using this procedure.

The asymptotics that we obtain imply in particular that when the domain $\Omega$ is a disk and the boundary data $g$ is an atomic minimising geodesic in $\mathcal{N}$ (see §3.2), then the asymptotic profile is of the form $u_*(x) = g(x/|x|)$ (Theorem 8.1). This generalises the answer of Bethuel, Brezis & Hélein to Matano's original problem on the Ginzburg–Landau equation [25].

As another consequence of our results, the *stress-energy tensor* of the limit $u_*$ has vanishing flux around the singularities — equivalently, the residue of the *Hopf differential* of $u_*$ vanishes at each singularity.

The results presented above are not confined to minimisers of the Ginzburg–Landau energy, and imply in particular $\Gamma$−convergence results at first and second order similar to the classical case, see [32,35,48] for $\Gamma$−convergence results at first order and [1] for $\Gamma$−convergence results at second order. All the results also come with Marcinkiewicz weak $L^2$ estimates — or equivalently estimates in the endpoint Lorentz space $L^{2,\infty}$ — on the gradient as for the original Ginzburg–Landau functional [51].

We consider next the improvements in the asymptotics that can be obtained when $u_\varepsilon$ is a weak solution to

$$\Delta u_\varepsilon = \frac{\nabla F(u_\varepsilon)}{\varepsilon^2} \qquad \text{in } \Omega, \tag{1.5}$$

We refer to (1.5) as the generalised Ginzburg–Landau equation. Minimisers of the Ginzburg–Landau energy $\mathcal{E}^\varepsilon_F$ satisfy the corresponding Euler–Lagrange equation, i.e., (1.5) is satisfied under reasonable assumptions (see §9.1). We prove in Theorem 9.3 that under a boundedness assumption on $\nabla F(u_\varepsilon)$, the distance to the manifold $\operatorname{dist}(u_{\varepsilon_n}, \mathcal{N})$ *converges uniformly to* 0 up to the boundary and away from singularities for any boundary data $g \in W^{1/2,2}(\partial\Omega, \mathcal{N})$ — which is not continuous in general. We next prove in Theorem 9.6 that weakly converging solutions of (1.5) converge to harmonic maps. Finally, we obtain higher-order convergence up to the boundary under a higher regularity assumption on the boundary data (Theorem 9.10).

Another strategy to study phase-transition problems where one deals with manifold-valued order-parameters has been implemented in [16, 17] by constructing a substitute to the Jacobian determinant used in the classical $\mathbb{S}^1$-valued Ginzburg-Landau theory to obtain first order $\Gamma$−convergence results; this substitute is obtained by using flat chains in the setting of manifolds with abelian fundamental groups. Other types of topological obstructions have been analysed via a Ginzburg–Landau relaxation in the case of two-dimensional Riemannian manifolds [33, 34]; the authors prove the convergence of vector fields minimising some Ginzburg–Landau type energy to a canonical unit-length harmonic tangent field with a finite number of singularities; the singularities arise form a non-vanishing Euler-Poincaré characteristic, their number is determined by the Poincaré–Hopf index theorem and their position is governed by a renormalised energy.

We continue the present work with a preliminary section on the projection onto the manifold and on non-degeneracy conditions on $F$ (§2). We next recall in §3 the definitions and properties of singular energy, geometric renormalised energy, renormalisable singular mappings and synharmony from [39]. In §4, we introduce a quantity measuring the energy of a vortex with a given boundary condition at infinity. We combine then the different tools to obtain an *upper bound* on the energy of minimisers in §5.

In §6, we obtain by Sandier's vortex-ball method [48] a first lower-bound on the energy and then following Jerrard's strategy [35] we obtain localised estimates. We apply then these estimates to energy convergence results, implying convergence of minimisers and $\Gamma$−convergence results (§7). We also explain how our results locate singularities on a disk with an atomic minimising geodesic as boundary data (§8).

In the last section §9 we give sufficient conditions for minimisers to be solutions of the Ginzburg–Landau equation. Then we study solutions to this equation and we prove uniform convergence of these solutions to the constraint manifold $\mathcal{N}$, weak convergence to harmonic maps and higher-order convergence away from singularities.

## 2. Retraction on the manifold and non-degeneracy of the relaxation potential

### 2.1. **Embedding and nearest point retraction.**

The Ginzburg–Landau relaxation procedure requires an isometric embedding of the vacuum manifold $\mathcal{N}$ into $\mathbb{R}^\nu$. The classical Nash embedding theorem [44] provides such an embedding. When $\mathcal{N} = G/H$ where $G$ is a Lie group and $H \subset G$ is a closed subgroup, it can be relevant to use an *equivariant isometric embedding* due to Moore [40] (see also [41]): there exists an isometric embedding $\Psi : G/H \to \mathbb{R}^\nu$ and a representation

$R : G \to \mathrm{Lin}(\mathbb{R}^\nu)$ such that for every $g \in G$ and $y \in G/H$, $\Psi(gy) = R(g)(\Psi(y))$; in contrast with Nash's embedding theorem, the dimension $\nu$ of the target space $\mathbb{R}^\nu$ depends on the metric on $G$ and on the choice of the subgroup $H$, and the compactness of $G/H$ is essential (there is no such embedding if $G/H$ is the hyperbolic plane $\mathbb{H}^2 \simeq SO(1,2)/\times SO(2)$).

We define the function $\mathrm{dist}_{\mathcal{N}} : \mathbb{R}^\nu \to [0, +\infty)$ by setting for each $y \in \mathbb{R}^\nu$,

$$\mathrm{dist}_{\mathcal{N}}(y) \coloneqq \mathrm{dist}(y, \mathcal{N}) \coloneqq \inf \big\{|y - z| \;:\; z \in \mathcal{N}\big\}.$$

We define the set

$$\mathcal{N}_\delta \coloneqq \big\{y \in \mathbb{R}^\nu \;:\; \mathrm{dist}(y, \mathcal{N}) < \delta\big\}.$$

The next lemma describes the nearest point retraction of a neighbourhood of $\mathcal{N}$ on $\mathcal{N}$.

**Lemma 2.1.** *There exists $\delta_{\mathcal{N}} > 0$ such that the nearest point retraction $\Pi_{\mathcal{N}} : \mathcal{N}_{\delta_{\mathcal{N}}} \to \mathcal{N}$ characterized by*

$$|y - \Pi_{\mathcal{N}}(y)| = \mathrm{dist}(y, \mathcal{N})$$

*is well-defined and smooth. Moreover, if the mappings $P^\top_{\mathcal{N}} : \mathcal{N} \to \mathrm{Lin}(\mathbb{R}^\nu, \mathbb{R}^\nu)$ and $P^\perp_{\mathcal{N}} : \mathcal{N} \to \mathrm{Lin}(\mathbb{R}^\nu, \mathbb{R}^\nu)$ are defined for each $y \in \mathcal{N}$ by setting $P^\top_{\mathcal{N}}(y)$ and $P^\perp_{\mathcal{N}}(y)$ as the orthogonal projections on $T_y\mathcal{N}$ and $(T_y\mathcal{N})^\perp$, identified as linear subspaces of $\mathbb{R}^\nu$, then for every $y \in \mathcal{N}_{\delta_{\mathcal{N}}}$ and $v \in \mathbb{R}^\nu$,*

$$|D\,\mathrm{dist}_{\mathcal{N}}(y)[v]|^2 \le |P^\perp_{\mathcal{N}}(\Pi_{\mathcal{N}}(y))[v]|^2 \tag{2.1}$$

*and*

$$\Big(1 - \frac{\mathrm{dist}_{\mathcal{N}}(y)}{\delta_{\mathcal{N}}}\Big)|D\Pi_{\mathcal{N}}(y)[v]|^2 \le |P^\top_{\mathcal{N}}(\Pi_{\mathcal{N}}(y))[v]|^2 \le C|D\Pi_{\mathcal{N}}(y)[v]|^2, \tag{2.2}$$

*for some constant $C \in (0, +\infty)$ depending on $\mathcal{N}$ and $\nu$ only.*

In the particular case of the sphere $\mathcal{N} = \mathbb{S}^n$, one has $\Pi_{\mathcal{N}}(y) = y/|y|$ if $y \in \mathbb{R}^{n+1} \setminus \{0\}$, $D\Pi_{\mathcal{N}}(y)[v] = (v|y|^2 - y(y \cdot v))/|y|^3$, and thus $|D\Pi_{\mathcal{N}}(y)[v]|^2 = |v|^2/|y|^2 - (y \cdot v)^2/|y|^4$ for $v \in \mathbb{R}^{n+1}$. Moreover $\mathrm{dist}_{\mathbb{S}^n}(y) = ||y| - 1|$ and $|D\,\mathrm{dist}_{\mathbb{S}^n}(y)[v]| = |v \cdot y|/|y|$ for $y \in \mathbb{R}^{n+1} \setminus \{0\}$ and $v \in \mathbb{R}^{n+1}$. Besides, if $z \in \mathbb{S}^n$ and $v \in \mathbb{R}^{n+1}$: $P^\perp_{\mathbb{S}^n}(z)[v] = z(z \cdot v)$ and $P^\top_{\mathbb{S}^n}(z)[v] = v - z(z \cdot v)$, so that in this case Lemma 2.1 is a consequence of the formulae

$$|D\,\mathrm{dist}_{\mathbb{S}^n}(y)[v]|^2 = |P^\perp_{\mathbb{S}^n}(\Pi_{\mathbb{S}^n}(y))[v]|^2 \qquad \text{and} \qquad |y|^2|D\Pi_{\mathcal{N}}(y)[v]|^2 = |P^\top_{\mathbb{S}^n}(\Pi_{\mathbb{S}^n}(y))[v]|^2,$$

for $y \in \mathbb{R}^{n+1} \setminus \{0\}$ and $v \in \mathbb{R}^{n+1}$.

The smoothness of the nearest point retraction is classical [27]. For related computations on the distance function to embedded manifolds, we refer the reader to [3, 26]. For every $y \in \mathcal{N}_{\delta_{\mathcal{N}}}$ and $v \in \mathbb{R}^\nu$, we have by orthogonality $|P^\perp_{\mathcal{N}}(\Pi_{\mathcal{N}}(y))[v]|^2 + |P^\top_{\mathcal{N}}(\Pi_{\mathcal{N}}(y))[v]|^2 = |v|^2$, and thus by Lemma 2.1

$$|D\,\mathrm{dist}_{\mathcal{N}}(y)[v]|^2 + \Big(1 - \frac{\mathrm{dist}_{\mathcal{N}}(y)}{\delta_{\mathcal{N}}}\Big)|D\Pi_{\mathcal{N}}(y)[v]|^2 \le |v|^2. \tag{2.3}$$

In the proof of Lemma 2.1 and throughout this work we will use the following facts about the nearest point projection:

$$\text{for all } y \in \mathcal{N}_{\delta_{\mathcal{N}}}, \quad y - \Pi_{\mathcal{N}}(y) \in (T_{\Pi_{\mathcal{N}}(y)}\mathcal{N})^\perp, \tag{2.4}$$

$$\text{for all } y \in \mathcal{N}, \quad D\Pi_{\mathcal{N}}(y) \text{ is the orthogonal projection onto } T_y\mathcal{N} \text{ i.e., } D\Pi_{\mathcal{N}}(y) = P^\top_{\mathcal{N}}(y), \tag{2.5}$$

$$\text{for all } y \in \mathcal{N}, \quad -D^2\Pi_{\mathcal{N}}(y) : T_y\mathcal{N} \otimes T_y\mathcal{N} \to (T_y\mathcal{N})^\perp \quad \text{is the second fundamental form of } \mathcal{N} \subset \mathbb{R}^\nu \text{ at } y. \tag{2.6}$$

Point (2.4) follows from the characterization of the $\Pi_{\mathcal{N}}$. For (2.5) we refer to [43, Lemma 3.1]. We denote by $B_x : T_x\mathcal{N} \otimes T_x\mathcal{N} \to (T_x\mathcal{N})^\perp$ the second fundamental form of $\mathcal{N}$ at $x \in \mathcal{N}$ and we refer to [20, definition 6.2.2] for the definition. We observe that, for $y \in \mathcal{N}$, $D^2\Pi_{\mathcal{N}}(y)_{|T_y\mathcal{N} \otimes T_y\mathcal{N}} = DP^\top_{\mathcal{N}}(y)_{|T_y\mathcal{N} \otimes T_y\mathcal{N}}$ and we refer to [43, Lemma 3.2] for (2.6).

*Proof of Lemma 2.1.* It is well-known that when $\delta > 0$ is small enough, the nearest point retraction $\Pi_{\mathcal{N}}$ is well-defined on $\mathcal{N}_\delta$. For every $y \in \mathcal{N}_\delta$, by using (2.4) we find

$$P_{\mathcal{N}}^{\top}(\Pi_{\mathcal{N}}(y))[\Pi_{\mathcal{N}}(y) - y] = 0.$$

Differentiating this identity with respect to $y$ by using the chain rule and the Leibniz rule, we find for every $y \in \mathcal{N}_\delta$ and $v \in \mathbb{R}^\nu$,

$$P_{\mathcal{N}}^{\top}(\Pi_{\mathcal{N}}(y))\big[D\Pi_{\mathcal{N}}(y)[v] - v\big] + (DP_{\mathcal{N}}^{\top}(\Pi_{\mathcal{N}}(y))[D\Pi_{\mathcal{N}}(y)[v]])\big[\Pi_{\mathcal{N}}(y) - y\big] = 0.$$

Noting that $D\Pi_{\mathcal{N}}(y)[v] \in T_{\Pi_{\mathcal{N}}(y)}\mathcal{N}$, that for every $z \in \mathcal{N}$, $P_{\mathcal{N}}^{\top}(z) + P_{\mathcal{N}}^{\perp}(z) = \mathrm{id}$ so that $DP_{\mathcal{N}}^{\top}(z)[w] = -DP_{\mathcal{N}}^{\perp}(z)[w]$ whenever $w \in T_z\mathcal{N}$, we infer

$$P_{\mathcal{N}}^{\top}(\Pi_{\mathcal{N}}(y))\big[D\Pi_{\mathcal{N}}(y)[v] - v\big] - (DP_{\mathcal{N}}^{\perp}(\Pi_{\mathcal{N}}(y))[D\Pi_{\mathcal{N}}(y)[v]])\big[\Pi_{\mathcal{N}}(y) - y\big] = 0. \tag{2.7}$$

We observe that for every $w \in \mathbb{R}^\nu$, $x \in \mathcal{N} \mapsto P_{\mathcal{N}}^{\perp}(x)[w] \in T_x^{\perp}\mathcal{N}$ is a smooth map, and therefore we have [20, proposition 6.2.3] if $x \in \mathcal{N}$, $w, z \in T_x\mathcal{N}$ and $u \in (T_x\mathcal{N})^{\perp}$,

$$z \cdot (DP_{\mathcal{N}}^{\perp}(x)[w])[u] = -u \cdot B_x(z, w),$$

where $u \cdot B_x$ is the second fundamental form of the submanifold $\mathcal{N}$ along the normal vector $u$ [20, definition 6.2.2]. Moreover, since for every $y \in \mathcal{N}_\delta$, $v \in \mathbb{R}^\nu$, $D\Pi_{\mathcal{N}}(y)[v] \in T_{\Pi_{\mathcal{N}}(y)}\mathcal{N}$, we have

$$D\Pi_{\mathcal{N}}(y)[v] \cdot P_{\mathcal{N}}^{\top}(\Pi_{\mathcal{N}}(y))[D\Pi_{\mathcal{N}}(y)[v] - v] = |D\Pi_{\mathcal{N}}(y)[v]|^2 - D\Pi_{\mathcal{N}}(y)[v] \cdot P_{\mathcal{N}}^{\top}(\Pi_{\mathcal{N}}(y))[v].$$

Therefore, we have, by testing (2.7) against the vector $D\Pi_{\mathcal{N}}(y)[v]$,

$$\begin{aligned} |D\Pi_{\mathcal{N}}(y)[v]|^2 + (\Pi_{\mathcal{N}}(y) - y) \cdot B_{\Pi_{\mathcal{N}}(y)}[D\Pi_{\mathcal{N}}(y)[v], D\Pi_{\mathcal{N}}(y)[v]] \\ = P_{\mathcal{N}}^{\top}(\Pi_{\mathcal{N}}(y))[v] \cdot D\Pi_{\mathcal{N}}(y)[v]. \end{aligned} \tag{2.8}$$

Hence, if $\delta_{\mathcal{N}} \in (0, \delta)$ satisfies $\frac{1}{\delta_{\mathcal{N}}} \geq \sup\{|B_y(z, w)| \;:\; y \in \mathcal{N}, z, w \in T_y\mathcal{N}, |z| \leq 1,\, |w| \leq 1\}$, we have for every $y \in \mathcal{N}_{\delta_{\mathcal{N}}}$ and $v \in \mathbb{R}^\nu$,

$$\Big(1 - \frac{1}{\delta_{\mathcal{N}}}|\Pi_{\mathcal{N}}(y) - y|\Big)|D\Pi_{\mathcal{N}}(y)[v]| \leq |P_{\mathcal{N}}^{\top}(\Pi_{\mathcal{N}}(y))[v]|, \tag{2.9}$$

which is the first inequality in (2.2). In particular, $\ker P_{\mathcal{N}}^{\top}(\Pi_{\mathcal{N}}(y)) \subset \ker D\Pi_{\mathcal{N}}(y)$ and moreover $\ker D\Pi_{\mathcal{N}}(y) = \ker P_{\mathcal{N}}^{\top}(\Pi_{\mathcal{N}}(y))$ since $D\Pi_{\mathcal{N}}(y)$ and $P_{\mathcal{N}}^{\top}(\Pi_{\mathcal{N}}(y))$ are onto from $\mathbb{R}^\nu$ to $T_{\Pi_{\mathcal{N}}(y)}\mathcal{N}$. This yields the second inequality in (2.2).

The first estimate, (2.1) follows from the fact that for every $y \in \mathcal{N}_{\delta_{\mathcal{N}}} \setminus \mathcal{N}$ and $v \in \mathbb{R}^\nu$

$$D\,\mathrm{dist}_{\mathcal{N}}(y)[v] = \frac{v \cdot (y - \Pi_{\mathcal{N}}(y))}{|y - \Pi_{\mathcal{N}}(y)|} = \frac{P_{\mathcal{N}}^{\perp}(\Pi_{\mathcal{N}}(y))[v] \cdot (y - \Pi_{\mathcal{N}}(y))}{|y - \Pi_{\mathcal{N}}(y)|}. \qquad \square$$

2.2. **Non-degeneracy of the penalising potential.** We first show that if $F$ satisfies the following first order non-degeneracy condition,

$$\begin{aligned} &F \in \mathcal{C}^1(\mathbb{R}^\nu, [0, +\infty)) \text{ and there exist } \delta_F \in (0, \delta_{\mathcal{N}}) \text{ and } m_F, M_F \in (0, +\infty), \\ &m_F \,\mathrm{dist}(z, \mathcal{N})^2 \leq DF(z)[z - \Pi_{\mathcal{N}}(z)] \leq M_F \,\mathrm{dist}(z, \mathcal{N})^2 \quad \text{for every } z \in \mathcal{N}_{\delta_F}, \end{aligned} \tag{2.10}$$

then it satisfies our zero order non-degeneracy assumption (1.4). This fact will be useful in Section 9.4.

**Lemma 2.2.** *If $F \in \mathcal{C}^1(\mathbb{R}^\nu, [0, +\infty))$ with $F = 0$ on $\mathcal{N}$ and if* (2.10) *holds, then* (1.4) *holds.*

*Proof.* By (2.10), we have for every $z \in \mathcal{N}_{\delta_F}$ and $t \in [0, 1]$,

$$m_F \, t \,\mathrm{dist}(z, \mathcal{N})^2 \leq DF((1 - t)\Pi_{\mathcal{N}}(z) + tz)[z - \Pi_{\mathcal{N}}(z)] \leq M_F \, t \,\mathrm{dist}(z, \mathcal{N})^2$$

and the conclusion follows by integration over $[0, 1]$ since $F = 0$ on $\mathcal{N}$. $\square$

A more explicit condition on $F$ that implies (2.10) is given by the second order condition:

$$F \in \mathcal{C}^2(\mathbb{R}^\nu, [0, +\infty)) \text{ and for every } y \in \mathcal{N} \text{ and } v \in (T_y\mathcal{N})^{\perp} \setminus \{0\},\; D^2F(y)[v, v] > 0. \tag{2.11}$$

**Lemma 2.3.** *If $F \in \mathcal{C}^2(\mathbb{R}^\nu, [0, +\infty))$ with $F = 0$ on $\mathcal{N}$ and if* (2.11) *holds, then* (2.10) *holds.*

*Proof.* By compactness of $\mathcal{N}$, by continuity of $D^2F$ and by (2.11), there exist $\delta_F \in (0, \delta_{\mathcal{N}})$ and $m_F, M_F \in (0, +\infty)$ such that for every $z \in \mathcal{N}_{\delta_F}$ and $v \in (T_{\Pi_{\mathcal{N}}(z)}\mathcal{N})^\perp$,

$$m_F|v|^2 \leq D^2F(z)[v, v] \leq M_F|v|^2.$$

In particular, since $z - \Pi_{\mathcal{N}}(z) \in (T_{\Pi_{\mathcal{N}}(z)}\mathcal{N})^\perp$, we have for every $t \in [0, 1]$,

$$m_F \operatorname{dist}(z, \mathcal{N})^2 \leq D^2F((1-t)\Pi_{\mathcal{N}}(z) + tz)[z - \Pi_{\mathcal{N}}(z), z - \Pi_{\mathcal{N}}(z)] \leq M_F \operatorname{dist}(z, \mathcal{N})^2,$$

and the conclusion follows by integration over $[0, 1]$ since $DF \equiv 0$. □

*Remark* 2.4. Many potentials $F$satisfy the condition (2.11), the most canonical being $F(z) := \operatorname{dist}(z, \mathcal{N})^2$ in an neighbourhood of $\mathcal{N}$: we have for every $z \in \mathcal{N}_{\delta_{\mathcal{N}}}$ and $v \in \mathbb{R}^\nu$,

$$DF(z)[v] = 2(z - \Pi_{\mathcal{N}}(z)) \cdot v,$$

and for every $v_1, v_2 \in \mathbb{R}^\nu$,

$$D^2F(z)[v_1, v_2] = 2(v_1 - D\Pi_{\mathcal{N}}(z)[v_1]) \cdot v_2,$$

so that, in particular, $D^2F(z)[v, v] = 2|v|^2$ if $z \in \mathcal{N}$ and $v \in (T_y\mathcal{N})^\perp$, since then $D\Pi_{\mathcal{N}}(z)$ is the orthogonal projection on $T_z\mathcal{N}$.

*Remark* 2.5. In the previous example of the squared distance function, we have $|\nabla F|^2 = 4F$. In general, if $F \in \mathcal{C}^3(\mathbb{R}^\nu, [0, +\infty))$ vanishes on $\mathcal{N}$ and satisfies (2.11), then the function $G$, defined by $G(y) = |\nabla F(y)|^2$, vanishes on $\mathcal{N}$ and satisfies (2.11). Indeed, for every $y \in \mathcal{N}$ and $v \in T_y\mathcal{N}^\perp \setminus \{0\}$, we have $D^2G(y)[v, v] = 2|D^2F(y)[v]|^2 > 0$.

## 3. Renormalised energies and renormalisable harmonic maps

3.1. **Topological resolution of the boundary datum.** Following our previous work[39], we describe here the resolution of obstructions of the boundary data that are responsible for asymptotical singularities for Ginzburg–Landau type functionals.

Given an open set $\Omega \subset \mathbb{R}^2$, an integer $k \in \mathbb{N}$ and a family of distinct points $a_1, \dots, a_k \in \Omega$, we define

$$\begin{aligned}\bar{\rho}(a_1, \dots, a_k) &\\ := \sup\{\rho > 0 \ : \ &\bar{B}_\rho(a_i) \cap \bar{B}_\rho(a_j) = \emptyset \text{ for each } i, j \in \{1, \dots, k\} \text{ such that } i \neq j \\ &\text{and } \bar{B}_\rho(a_i) \subset \Omega \text{ for each } i \in \{1, \dots, k\}\}.\end{aligned} \tag{3.1}$$

**Definition 3.1.** Given $\Omega \subset \mathbb{R}^2$ a domain with a Lipschitz boundary, $k \in \mathbb{N}_*$, $k$ maps $\gamma_1, \dots, \gamma_k \in \mathrm{VMO}(\mathbb{S}^1, \mathcal{N})$ and $g \in \mathrm{VMO}(\partial\Omega, \mathcal{N})$, we say that $(\gamma_1, \dots, \gamma_k)$ is a *topological resolution* of $g$ whenever there exist points $a_1, \dots, a_k \in \Omega$, a radius $\rho \in (0, \bar{\rho}(a_1, \dots, a_k))$, and a continuous map $u \in \mathcal{C}(\bar{\Omega} \setminus \bigcup_{i=1}^k B_\rho(a_i), \mathcal{N})$ such that $u|_{\partial\Omega}$ is homotopic to $g$ in $\mathrm{VMO}(\partial\Omega, \mathcal{N})$ and for each $i \in \{1, \dots, k\}$, $u(a_i + \rho\cdot)|_{\mathbb{S}^1}$ is homotopic to $\gamma_i$ in $\mathrm{VMO}(\mathbb{S}^1, \mathcal{N})$.

Definition 3.1 is invariant under changes of the positions of points and of the radius, andunder homotopies of $g$ in $\mathrm{VMO}(\partial\Omega, \mathcal{N})$ and of $\gamma_1, \dots, \gamma_k$ in $\mathrm{VMO}(\mathbb{S}^1, \mathcal{N})$. If $g, \gamma_1, \dots, \gamma_k$ are continuous, then we can assume in the definition that $g = u|_{\partial\Omega}$ and $u(a_i + \rho\cdot)|_{\mathbb{S}^1} = \gamma_i$ everywhere [13, 14]. Topological resolutions can be characterized algebraically in the fundamental group $\pi_1(\mathcal{N})$ by conjugacy classes [39].

3.2. **Singular energy.** The minimal length in the homotopy class of $\gamma \in \mathrm{VMO}(\mathbb{S}^1, \mathcal{N})$ is defined as

$$\inf\Big\{\int_{\mathbb{S}^1} |\tilde{\gamma}'|^2 \ : \ \tilde{\gamma} \in \mathcal{C}^1(\mathbb{S}^1, \mathcal{N}) \text{ and } \gamma \text{ are homotopic}\Big\} =: \frac{\lambda(\gamma)^2}{2\pi}, \tag{3.2}$$

and equality is achieved if $\gamma$ is a minimising geodesic. The quantity $\lambda(\gamma)$ is invariant by homotopy; if the elements of a subset $A \subset \mathrm{VMO}(\mathbb{S}^1, \mathcal{N})$ are all homotopic to each other – for instance, if $A$ is the homotopy class of a given curve – we will denote by $\lambda(A)$ the common value of the quantities $\lambda(\gamma)$ with $\gamma \in A$.

The *systole* of the manifold $\mathcal{N}$ is the length of the shortest closed non-trivial geodesic on $\mathcal{N}$:

$$\operatorname{sys}(\mathcal{N}) = \inf\big\{\lambda(\gamma) \ : \ \gamma \in \mathcal{C}^1(\mathbb{S}^1, \mathcal{N}) \text{ is not homotopic to a constant}\big\}. \tag{3.3}$$

In particular, for every $\gamma \in \mathrm{VMO}(\mathbb{S}^1, \mathcal{N})$, we have $\lambda(\gamma) \in \{0\} \cup [\mathrm{sys}(\mathcal{N}), +\infty)$. When $\mathcal{N}$ is compact, $\mathrm{sys}(\mathcal{N}) > 0$.

**Proposition 3.2.** *If $\mathcal{N}$ is compact, then the set $\{\lambda(\gamma) \;:\; \gamma \in \mathrm{VMO}(\mathbb{S}^1, \mathcal{N})\}$ is discrete.*

*Proof.* By homotopy invariance of $\lambda(\gamma)$ and thanks to the existence of geodesics in each homotopy class, we can assume that the maps $\gamma$ are taken to be minimising geodesics. We consider thus a sequence $(\gamma_n)_{n\in\mathbb{N}}$ in $\mathcal{C}^1(\mathbb{S}^1, \mathcal{N})$ of minimising closed geodesics such that the sequence of numbers $(\lambda(\gamma_n))_{n\in\mathbb{N}}$ converges. In view of (3.2) and the Ascoli–Arzelá compactness criterion, there is a subsequence of $(\gamma_n)_{n\in\mathbb{N}}$ that converges uniformly and hence up to a further subsequence all the maps in the sequence $(\gamma_n)_{n\in\mathbb{N}}$ are homotopic and thus $(\lambda(\gamma_n))_{n\in\mathbb{N}}$ is constant, which implies that the set $\{\lambda(\gamma) \;:\; \gamma \in \mathrm{VMO}(\mathbb{S}^1, \mathcal{N})\}$ is discrete. □

The first key quantity in the asymptotics for Ginzburg–Landau type functional is the following.

**Definition 3.3.** If $\Omega \subset \mathbb{R}^2$ is a Lipschitz bounded domain and $g \in \mathrm{VMO}(\partial\Omega, \mathcal{N})$, we define its *singular energy* to be

$$\mathcal{E}^{\mathrm{sg}}(g) := \inf\left\{ \sum_{i=1}^{k} \frac{\lambda(\gamma_i)^2}{4\pi} \;:\; k \in \mathbb{N}_* \text{ and } (\gamma_1, \dots, \gamma_k) \text{ is a topological resolution of } g \right\}.$$

The singular energy $\mathcal{E}^{\mathrm{sg}}$ is invariant under homotopies. For every $\gamma \in \mathrm{VMO}(\mathbb{S}^1, \mathcal{N})$, we have $\mathcal{E}^{\mathrm{sg}}(\gamma) \le \frac{\lambda(\gamma)^2}{4\pi}$ (where in the definition of $\mathcal{E}^{\mathrm{sg}}(\gamma)$, the circle $\mathbb{S}^1$ is thought as the boundary of $\Omega = B_1$) and for every$g \in \mathrm{VMO}(\partial\Omega, \mathcal{N})$,

$$\mathcal{E}^{\mathrm{sg}}(g) \in \{0\} \cup \Big[\frac{\mathrm{sys}(\mathcal{N})^2}{4\pi}, +\infty\Big). \tag{3.4}$$

We say that $(\gamma_1, \dots, \gamma_k)$ is a *minimal topological resolution* of $g$ whenever it is a topological resolution of $g$ such that $\mathcal{E}^{\mathrm{sg}}(g) = \sum_{i=1}^{k} \frac{\lambda(\gamma_i)^2}{4\pi}$ and for every $i \in \{1, \dots, k\}$, $\lambda(\gamma_i) > 0$. For example, if $g \in \mathrm{VMO}(\partial\Omega, \mathbb{S}^1)$ and $\deg(g) = d \in \mathbb{Z}$ then $\mathcal{E}^{\mathrm{sg}}(g) = \pi|d|$, and a minimal topological resolution is given by $|d|$ maps of degree $1$ if $d > 0$, and $|d|$ maps of degree $-1$ if $d < 0$. However, in general, minimal topological resolutions are not necessarily unique.

A closed curve $\gamma \in \mathcal{C}(\mathbb{S}^1, \mathcal{N})$ is said to be *atomic* whenever $\gamma$ is a minimal topological resolution of $\gamma$. In particular, if $\lambda(\gamma) = \mathrm{sys}(\mathcal{N})$, then $\gamma$ is atomic. Atomicity does not exclude the existence of an alternative minimal topological resolution into several maps, this is the case for the manifold $\mathcal{N}$ arising as quotient of $SU(2) \times SU(2)$ in models of superfluid $^3$He [39, section 9.3.5].

## 3.3. Synharmony between geodesics.

The notion of synharmony between geodesics which quantifies how homotopic mappings can be connected through a harmonic map [39].

**Definition 3.4.** The *synharmonicity* between two given maps $\gamma, \beta \in W^{1/2,2}(\mathbb{S}^1, \mathcal{N})$, is defined as

$$d_{\mathrm{synh}}(\gamma, \beta) := \inf\left\{ \int_{\mathbb{S}^1\times[0,L]} \frac{|Du|^2}{2} - \frac{L}{4\pi}\lambda(\gamma)^2 \;:\; L \in (0, +\infty), u \in W^{1,2}(\mathbb{S}^1 \times [0, L], \mathcal{N}), \right.$$
$$\left. \operatorname{tr}_{\mathbb{S}^1\times\{0\}} u = \gamma \text{ and } \operatorname{tr}_{\mathbb{S}^1\times\{L\}} u = \beta \text{ on } \mathbb{S}^1 \right\}.$$

The synharmonicity is an extended pseudo-distance which is continuous with respect to the strong topology in $W^{1/2,2}(\mathbb{S}^1, \mathcal{N})$. Bounded sets in $W^{1/2,2}(\mathbb{S}^1, \mathcal{N})$ which contain only homotopic maps have bounded synharmonicity [39].

Two maps $\gamma, \beta \in W^{1/2,2}(\mathbb{S}^1, \mathcal{N})$ are *synharmonic* whenever $d_{\mathrm{synh}}(\gamma, \beta) = 0$. The synharmony between minimising geodesics is an equivalence relation, partitioning each homotopy class of minimising geodesics into synharmony classes. If $\gamma, \beta \in W^{1/2,2}(\mathbb{S}^1, \mathcal{N})$ and $d_{\mathrm{synh}}(\gamma, \beta) = 0$, then either $\gamma = \beta$ almost everywhere in $\mathbb{S}^1$ or both $\beta$ and $\gamma$ are minimising geodesics. Minimising geodesics that are homotopic through minimising geodesics are synharmonic; this covers in particular $\gamma \circ R$ and $\gamma$ where $R \in SO(2)$. Although homotopic minimising closed geodesics on a manifold are not synharmonic in general, this is the case on examples that motivate in physics and geometry the use of Ginzburg–Landau type energies.

3.4. **Renormalised energies of configurations of points.** Given a bounded open set $\Omega \subset \mathbb{R}^2$ with Lipschitz boundary $\partial\Omega$, a map $g \in W^{1/2,2}(\partial\Omega, \mathcal{N}) \subset \mathrm{VMO}(\partial\Omega, \mathcal{N})$, $k \in \mathbb{N}_*$ and $k$ closed minimizing geodesics $\gamma_1, \dots, \gamma_k \in W^{1/2,2}(\mathbb{S}^1, \mathcal{N})$ that form a topological resolution of $g$, we consider the *geometrical renormalised energy* defined on the configuration space of $\Omega$,

$$\mathrm{Conf}_k\,\Omega \coloneqq \{(a_1, \dots, a_k) \in \Omega^k \ : \ a_i \neq a_j \text{ if } i \neq j\},$$

by setting for every $(a_1, \dots, a_k) \in \mathrm{Conf}_k\,\Omega$,

$$\begin{aligned}\mathcal{E}^{\mathrm{geom}}_{g,\gamma_1,\dots,\gamma_k}(a_1, \dots, a_k) &\coloneqq \lim_{\rho\to 0} \mathcal{E}^{\mathrm{geom},\rho}_{g,\gamma_1,\dots,\gamma_k}(a_1, \dots, a_k) - \sum_{i=1}^{k} \frac{\lambda(\gamma_i)^2}{4\pi} \log\frac{1}{\rho}\\ &= \inf_{\rho\in(0,\bar\rho(a_1,\dots,a_k))} \mathcal{E}^{\mathrm{geom},\rho}_{g,\gamma_1,\dots,\gamma_k}(a_1, \dots, a_k) - \sum_{i=1}^{k} \frac{\lambda(\gamma_i)^2}{4\pi} \log\frac{1}{\rho},\end{aligned} \tag{3.5}$$

where for a radius $\rho \in (0, \bar\rho(a_1, \dots, a_k))$, we have set

$$\begin{aligned}\mathcal{E}^{\mathrm{geom},\rho}_{g,\gamma_1,\dots,\gamma_k}(a_1, \dots, a_k) = \inf\Big\{\int_{\Omega\setminus\bigcup_{i=1}^k \bar B_\rho(a_i)} \frac{|Du|^2}{2} \ :\ u \in W^{1,2}(\Omega \setminus \textstyle\bigcup_{i=1}^k \bar B_\rho(a_i), \mathcal{N}),\\ \mathrm{tr}_{\partial\Omega}\, u = g \text{ on } \partial\Omega \text{ and } \mathrm{tr}_{\mathbb{S}^1}\, u(a_i + \rho\cdot) = \gamma_i\Big\}.\end{aligned} \tag{3.6}$$

The function $\mathcal{E}^{\mathrm{geom}}_{g,\gamma_1,\dots,\gamma_k} : \mathrm{Conf}_k\,\Omega \to \mathbb{R}$ is locally Lipschitz-continuous. If $(\gamma_1, \dots, \gamma_k)$ is a minimal topological resolution of $g$, then the function $\mathcal{E}^{\mathrm{geom}}_{g,\gamma_1,\dots,\gamma_k}$ is bounded from below on $\mathrm{Conf}_k\,\Omega$; moreover if $\limsup_{n\to\infty} \mathcal{E}^{\mathrm{geom}}_{g,\gamma_1,\dots,\gamma_k}(a_1^n, \dots, a_k^n) < +\infty$, then the singularities $a_1^n, \dots, a_k^n$ always stay away from the boundary and from each other unless their recombination yields another minimal topological resolution of the boundary data $g$. (In our motivating examples this does not happen, but occurs for instance for the torus $\mathbb{S}^1 \times \mathbb{S}^1$ which exhibits a decoupling of the renormalised energy.)

The quantity $\mathcal{E}^{\mathrm{geom}}_{g,\gamma_1,\dots,\gamma_k}(a_1, \dots, a_k)$ depends on the curves $\gamma_i$ only up to synharmonicity: if for each $i$, the curves $\gamma_i$ and $\tilde\gamma_i$ are synharmonic, then

$$\mathcal{E}^{\mathrm{geom}}_{g,\gamma_1,\dots,\gamma_k}(a_1, \dots, a_k) = \mathcal{E}^{\mathrm{geom}}_{g,\tilde\gamma_1,\dots,\tilde\gamma_k}(a_1, \dots, a_k); \tag{3.7}$$

if $\boldsymbol{\gamma}_i$ stands for the synharmony class of $\gamma_i$, we will write

$$\mathcal{E}^{\mathrm{geom}}_{g,\boldsymbol{\gamma}_1,\dots,\boldsymbol{\gamma}_k}(a_1, \dots, a_k) \coloneqq \mathcal{E}^{\mathrm{geom}}_{g,\gamma_1,\dots,\gamma_k}(a_1, \dots, a_k).$$

For the proofs of the above-mentioned facts we refer to [39, section 3 and section 4].

3.5. **Renormalised energy of renormalisable maps.** A last notion from [39] that we will be using is the notion of renormalisable singular mapping and their renormalised energy.

**Definition 3.5.** Let $\Omega \subset \mathbb{R}^2$ be a bounded Lipschitz domain. A mapping $u : \Omega \to \mathcal{N}$ is *renormalisable* whenever there exists a finite set $\{a_1, \dots, a_k\} \subset \Omega$ such that if $\rho > 0$ is small enough, $u \in W^{1,2}(\Omega \setminus \bigcup_{i=1}^k \bar B_\rho(a_i), \mathcal{N})$ and its *renormalised energy* is finite:

$$\mathcal{E}^{\mathrm{ren}}(u) \coloneqq \liminf_{\rho\to 0} \int_{\Omega\setminus\bigcup_{i=1}^k \bar B_\rho(a_i)} \frac{|Du|^2}{2} - \sum_{i=1}^{k} \frac{\lambda(\mathrm{tr}_{\partial B_\rho(a_i)}\, u)^2}{4\pi} \log\frac{1}{\rho} < +\infty.$$

The set of renormalisable mappings is denoted by $W^{1,2}_{\mathrm{ren}}(\Omega, \mathcal{N})$. For every $u \in W^{1,2}_{\mathrm{ren}}(\Omega, \mathcal{N})$ one has

$$\begin{aligned}\mathcal{E}^{\mathrm{ren}}(u) &= \lim_{\rho\to 0} \int_{\Omega\setminus\bigcup_{i=1}^k \bar B_\rho(a_i)} \frac{|Du|^2}{2} - \sum_{i=1}^{k} \frac{\lambda(\mathrm{tr}_{\partial B_\rho(a_i)}\, u)^2}{4\pi} \log\frac{1}{\rho}\\ &= \sup_{\rho\in(0,\bar\rho(a_1,\dots,a_k))} \int_{\Omega\setminus\bigcup_{i=1}^k \bar B_\rho(a_i)} \frac{|Du|^2}{2} - \sum_{i=1}^{k} \frac{\lambda(\mathrm{tr}_{\partial B_\rho(a_i)}\, u)^2}{4\pi} \log\frac{1}{\rho}.\end{aligned} \tag{3.8}$$

The structure of renormalisable mappings is described in the following:

**Proposition 3.6.** *Let $\Omega \subset \mathbb{R}^2$ be a bounded Lipschitz domain. If $u \in W^{1,2}_{\mathrm{ren}}(\Omega, \mathcal{N})$, then either one has $u \in W^{1,2}(\Omega, \mathcal{N})$ or there exist $k \in \mathbb{N}_*$, $(a_1, \dots, a_k) \in \mathrm{Conf}_k\,\Omega$ and $\gamma_1, \dots, \gamma_k \in \mathcal{C}^1(\mathbb{S}^1, \mathcal{N})$ such that*

*(i)* $(\gamma_1, \dots, \gamma_k)$ *is a topological resolution of* $\operatorname{tr}_{\partial\Omega} u$*,*
*(ii) for each* $i \in \{1, \dots, k\}$*,* $\gamma_i$ *is a non-trivial minimising closed geodesic,*
*(iii) for each* $i \in \{1, \dots, k\}$*, there exists a sequence* $(\rho_\ell)_{\ell\in\mathbb{N}}$ *converging to* $0$ *such that the sequence* $(\operatorname{tr}_{\mathbb{S}^1} u(a_i + \rho_\ell \cdot))_{\ell\in\mathbb{N}}$ *converges strongly to* $\gamma_i$ *in* $W^{1,2}(\mathbb{S}^1, \mathcal{N})$*,*
*(iv) for each* $i \in \{1, \dots, k\}$*,* $\lim_{\rho\to 0} d_{\mathrm{synh}}(\operatorname{tr}_{\mathbb{S}^1} u(a_i + \rho\,\cdot), \gamma_i) = 0$*,*
*(v)* $\mathcal{E}^{\mathrm{ren}}(u) \geq \mathcal{E}^{\mathrm{geom}}_{g,\gamma_1,\dots,\gamma_k}(a_1, \dots, a_k)$*.*

*In this case, we denote the set of singularities by* $\operatorname{sing}(u) = \{(a_1, \boldsymbol{\gamma}_1), \dots, (a_k, \boldsymbol{\gamma}_k)\}$*, where* $\boldsymbol{\gamma}_i := \{\gamma \;:\; d_{\mathrm{synh}}(\gamma, \gamma_i) = 0\}$ *is the* synharmony class *of* $\gamma_i$*; in the case where* $u \in W^{1,2}(\Omega, \mathcal{N})$*, we set* $\operatorname{sing}(u) = \emptyset$*.*

Given $u \in W^{1,2}_{\mathrm{ren}}(\Omega, \mathcal{N})$, $a \in \Omega$ and a synharmony class of minimising geodesics $\boldsymbol{\gamma}$, we have that $(a, \boldsymbol{\gamma}) \in \operatorname{sing}(u)$ if and only if $Du$ is not square-integrable near $a$ and if each $\gamma \in \boldsymbol{\gamma}$ satisfies $\lim_{\rho\to 0} d_{\mathrm{synh}}(\operatorname{tr}_{\mathbb{S}^1} u(a_i + \rho\,\cdot), \gamma) = 0$. In particular, the set $\operatorname{sing}(u)$ is well-defined.

## 4. Minimal energy on balls with boundary conditions

We recall that $F$ denotes the Ginzburg–Landau penalisation which satisfies $F \in \mathcal{C}(\mathbb{R}^\nu, [0, +\infty))$ and $F^{-1}(\{0\}) = \mathcal{N}$. For every every radius $R \in (0, +\infty)$ and every curve $\gamma \in W^{1/2,2}(\mathbb{S}^1, \mathbb{R}^\nu)$, we set

$$\mathcal{Q}^R_{F,\gamma} := \inf \left\{ \int_{B_R} \frac{|Du|^2}{2} + F(u) \;:\; u \in W^{1,2}(B_R, \mathbb{R}^\nu) \text{ s.t. } \operatorname{tr}_{\partial B_R} u = \gamma(R\cdot) \right\}, \tag{4.1}$$

where $B_R \subset \mathbb{R}^2$ is the disk of radius $R$ centred at the origin $0 \in \mathbb{R}^2$. By scaling, we have for every $\varepsilon, R \in (0, +\infty)$

$$\inf \left\{ \int_{B_R} \frac{|Du|^2}{2} + \frac{F(u)}{\varepsilon^2} \;:\; u \in W^{1,2}(B_R, \mathbb{R}^\nu) \text{ and } \operatorname{tr}_{\partial B_R} u = \gamma(R\cdot) \right\} = \mathcal{Q}^{R/\varepsilon}_{F,\gamma}. \tag{4.2}$$

**Proposition 4.1.** *If* $\gamma \in \mathcal{C}^1(\mathbb{S}^1, \mathcal{N})$ *is a minimising geodesic, then the map*

$$R \in (0, +\infty) \mapsto \mathcal{Q}^R_{F,\gamma} - \frac{\lambda(\gamma)^2}{4\pi} \log R$$

*is non-increasing.*

By Proposition 4.1, for every *minimising closed geodesic* $\gamma \in \mathcal{C}^1(\mathbb{S}^1, \mathcal{N})$, we can define

$$\mathcal{Q}_{F,\gamma} := \lim_{R\to+\infty} \left( \mathcal{Q}^R_{F,\gamma} - \frac{\lambda(\gamma)^2}{4\pi} \log R \right) \in [-\infty, +\infty). \tag{4.3}$$

When $\mathcal{N} = \mathbb{S}^1$, Proposition 4.1 is due to Bethuel, Brezis and Hélein [9, Lemma III.1].

*Remark* 4.2. We shall see in Section 6 that $\mathcal{Q}_{F,\gamma} > -\infty$ if $\gamma$ is an atomic minimising geodesic, i.e. $\mathcal{E}^{\mathrm{sg}}(\gamma) = \frac{\lambda(\gamma)^2}{4\pi}$ (see Corollary 6.8).

*Proof of Proposition 4.1.* Given $0 < R < S < +\infty$, we consider a map $u \in W^{1,2}(B_R, \mathbb{R}^\nu)$ such that $\operatorname{tr}_{\mathbb{S}^1} u(R\cdot) = \gamma$ on $\mathbb{S}^1$ and we define the map $v \in W^{1,2}(B_S, \mathbb{R}^\nu)$ for $x \in B_R$ by

$$v(x) = \begin{cases} u(x) & \text{if } x \in B_R, \\ \gamma\left(\frac{x}{|x|}\right) & \text{if } x \in B_S \setminus B_R. \end{cases}$$

Since $\gamma$ is by assumption a minimising geodesic, we have

$$\begin{aligned} \int_{B_S} \frac{|Dv|^2}{2} + F(v) &\leq \int_{B_R} \left( \frac{|Du|^2}{2} + F(u) \right) + \int_{\mathbb{S}^1} \frac{|\gamma'|^2}{2} \int_R^S \frac{\mathrm{d}r}{r} \\ &= \int_{B_R} \left( \frac{|Du|^2}{2} + F(u) \right) + \frac{\lambda(\gamma)^2}{4\pi} \log \frac{S}{R}. \end{aligned}$$

By minimising over $u$ and by definition (4.3) of $\mathcal{Q}^R_{F,\gamma}$, we get

$$\mathcal{Q}^S_{F,\gamma} - \frac{\lambda(\gamma)^2}{4\pi} \log S \leq \mathcal{Q}^R_{F,\gamma} - \frac{\lambda(\gamma)^2}{4\pi} \log R. \qquad \square$$

**Proposition 4.3.** *If $\gamma, \tilde{\gamma} \in W^{1/2,2}(\mathbb{S}^1, \mathcal{N})$, then for every $R \in (0, +\infty)$, we have*

$$\inf_{S \ge R} \Big( \mathcal{Q}^S_{F,\tilde{\gamma}} - \frac{\lambda(\tilde{\gamma})^2}{4\pi} \log S \Big) \le \mathcal{Q}^R_{F,\gamma} - \frac{\lambda(\gamma)^2}{4\pi} \log R + d_{\mathrm{synh}}(\gamma, \tilde{\gamma}).$$

In particular, if $\gamma$ and $\tilde{\gamma}$ are minimising geodesics, then $\mathcal{Q}_{F,\tilde{\gamma}} \le \mathcal{Q}_{F,\gamma} + d_{\mathrm{synh}}(\gamma, \tilde{\gamma})$. If moreover the maps $\gamma$ and $\tilde{\gamma}$ are synharmonic, then $\mathcal{Q}_{F,\gamma} = \mathcal{Q}_{F,\tilde{\gamma}}$; if $\boldsymbol{\gamma}$ is the synharmony class of some minimising geodesic $\gamma$, we will denote by

$$\mathcal{Q}_{F,\boldsymbol{\gamma}} \text{ the common value of } \mathcal{Q}_{F,\gamma} \text{ for } \gamma \in \boldsymbol{\gamma}. \tag{4.4}$$

*Proof of Proposition 4.3.* We can assume that the maps $\gamma$ and $\tilde{\gamma}$ are homotopic and, in particular, that $\lambda(\gamma) = \lambda(\tilde{\gamma})$ since otherwise $d_{\mathrm{synh}}(\gamma, \tilde{\gamma}) = +\infty$.

We take $R \in (0, +\infty)$, $u \in W^{1,2}(B_R, \mathbb{R}^\nu)$ such that $\operatorname{tr}_{\partial B_R} u = \gamma(R\cdot)$, $L > 0$ and $H \in W^{1,2}(\mathbb{S}^1 \times [0, L], \mathcal{N})$ such that $H(\cdot, 0) = \gamma$, $H(\cdot, L) = \tilde{\gamma}$. We define $v \in W^{1,2}(B_{e^L R}, \mathbb{R}^\nu)$ by

$$v(x) = \begin{cases} u(x) & \text{if } x \in B_R, \\ H\Big(\frac{x}{|x|}, \log \frac{|x|}{R}\Big) & \text{if } x \in B_{e^L R} \setminus B_R. \end{cases}$$

By taking the infimum with respect to $u$ in the energy of $v$ we obtain,

$$\mathcal{Q}^{e^L R}_{F,\tilde{\gamma}} - \frac{\lambda(\tilde{\gamma})^2}{4\pi} \log(e^L R) \le \mathcal{Q}^R_{F,\gamma} - \frac{\lambda(\gamma)^2}{4\pi} \log R + \int_{\mathbb{S}^1 \times [0,L]} \frac{|DH|^2}{2} - \frac{L}{4\pi} \lambda(\gamma)^2,$$

and thus by definition of synharmonicity (Definition 3.4),

$$\inf_{S \ge R} \Big( \mathcal{Q}^S_{F,\tilde{\gamma}} - \frac{\lambda(\tilde{\gamma})^2}{4\pi} \log S \Big) \le \mathcal{Q}^R_{F,\gamma} - \frac{\lambda(\gamma)^2}{4\pi} \log R + d_{\mathrm{synh}}(\gamma, \tilde{\gamma}). \qquad \square$$

Let $u \in W^{1,2}_{\mathrm{ren}}(\Omega \setminus \{a_1, \dots, a_k\}, \mathcal{N})$, let $\operatorname{sing}(u) =: \{(a_1, \boldsymbol{\gamma}_1), \dots, (a_k, \boldsymbol{\gamma}_k)\}$ be given by Proposition 3.6. We define

$$\mathcal{Q}_F(u) := \sum_{i=1}^k \mathcal{Q}_{F,\boldsymbol{\gamma}_i}, \tag{4.5}$$

where $\mathcal{Q}_{F,\boldsymbol{\gamma}}$ is defined in (4.4). Finally if $\gamma$ does not takes its value in $\mathcal{N}$ but is still close to it, the difference between $\mathcal{Q}^R_{F,\gamma}$ and $\mathcal{Q}^R_{F,\Pi_{\mathcal{N}}(\gamma)}$ can be estimated as follows.

**Proposition 4.4.** *If $F \in \mathcal{C}(\mathbb{R}^\nu, [0, +\infty))$ satisfies $F^{-1}(\{0\}) = \mathcal{N}$ and* (1.4)*, and if $\gamma \in W^{1,2}(\mathbb{S}^1, \mathbb{R}^\nu)$ satisfies $\operatorname{dist}_{\mathcal{N}}(\gamma(\cdot)) < \delta_{\mathcal{N}}/2$ on $\mathcal{N}$, then for every $R \ge 2$,*

$$|\mathcal{Q}^R_{F,\gamma} - \mathcal{Q}^R_{F,\Pi_{\mathcal{N}} \circ \gamma}| \le C \int_{\mathbb{S}^1} \Big( \frac{|\gamma'|^2}{R} + R F(\gamma) \Big).$$

*Proof.* Given $u \in W^{1,2}(B_R, \mathbb{R}^\nu)$ such that $\operatorname{tr}_{\partial B_R}(u) = \gamma(R\cdot)$, we define $v : B_R \to \mathbb{R}^\nu$ by setting for each $x \in B_R$,

$$v(x) = \begin{cases} u(\frac{R}{R-1} x) & \text{if } |x| \le R - 1, \\ (R - |x|)\gamma(\frac{x}{|x|}) + (|x| - (R-1))\Pi_{\mathcal{N}}(\gamma(\frac{x}{|x|})) & \text{if } R - 1 \le |x| \le R. \end{cases}$$

We compute that $Dv(x) = \frac{R}{R-1} Du(\frac{R}{R-1} x)$ if $|x| \le R - 1$ and if $R - 1 \le |x| \le R$

$$\begin{split} |Dv(x)|^2 = \big|\Pi_{\mathcal{N}}\big(\gamma\big(\tfrac{x}{|x|}\big)\big) - \gamma\big(\tfrac{x}{|x|}\big)\big|^2 \\ + \big|(R - |x|)\gamma'\big(\tfrac{x}{|x|}\big) - (|x| - (R-1)) D\Pi_{\mathcal{N}}(\gamma\big(\tfrac{x}{|x|}\big)\big[\gamma'\big(\tfrac{x}{|x|}\big)\big]\big|^2 \end{split}$$

In view of (1.4) we estimate

$$\big|\Pi_{\mathcal{N}}\big(\gamma\big(\tfrac{x}{|x|}\big)\big) - \gamma\big(\tfrac{x}{|x|}\big)\big|^2 = \operatorname{dist}(\gamma\big(\tfrac{x}{|x|}\big), \mathcal{N})^2 \le C_1 F\big(\gamma\big(\tfrac{x}{|x|}\big)\big)$$

and

$$
\begin{aligned}
F\Big(\Pi_{\mathcal{N}}\big(\gamma\big(\tfrac{x}{|x|}\big)\big) + (R-|x|)\,\Big(\gamma\big(\tfrac{x}{|x|}\big) - \Pi_{\mathcal{N}}\big(\gamma\big(\tfrac{x}{|x|}\big)\big)\Big)\Big)&\\
\le C_2 \operatorname{dist}\Big(\Pi_{\mathcal{N}}\big(\gamma\big(\tfrac{x}{|x|}\big)\big) + (R-|x|)\,\Big(\gamma\big(\tfrac{x}{|x|}\big) - \Pi_{\mathcal{N}}\big(\gamma\big(\tfrac{x}{|x|}\big)\big)\Big)\, s, \mathcal{N}\Big)&\\
\le C_3 \big|\Pi_{\mathcal{N}}\big(\gamma\big(\tfrac{x}{|x|}\big)\big) - \gamma\big(\tfrac{x}{|x|}\big)\big| \le C_4 F\big(\gamma\big(\tfrac{x}{|x|}\big)\big)&.
\end{aligned}
$$

By smoothness and compactness the derivatives of $\Pi_{\mathcal{N}}$ are bounded in $\mathcal{N}_{\delta_{\mathcal{N}}/2}$ and we have

$$
\big|(R-|x|)\gamma'\big(\tfrac{x}{|x|}\big) - (|x|-(R-1))D\Pi_{\mathcal{N}}\big(\gamma\big(\tfrac{x}{|x|}\big)\big)\big[\gamma'\big(\tfrac{x}{|x|}\big)\big]\big|^2 \le C\big|\gamma'\big(\tfrac{x}{|x|}\big)\big|^2,
$$

by using a change of variables and integration in polar coordinates we arrive at

$$
\int_{B_R} |Dv|^2 + F(v) \le \int_{B_R} |Du|^2 + F(u) + C_5 \bigg(\int_{\mathbb{S}^1} \frac{|\gamma'|^2}{R} + \int_{\mathbb{S}^1} R\, F \circ \gamma\bigg).
$$

It follows thus that

$$
\mathcal{Q}^R_{F,\gamma} - \mathcal{Q}^R_{F,\Pi_{\mathcal{N}}\circ\gamma} \le C_5 \bigg(\int_{\mathbb{S}^1} \frac{|\gamma'|^2}{R} + R\, F(\gamma)^2\bigg).
$$

The proof of the converse inequality is similar. □

## 5. Upper bound on the energy of minimizers

Thanks to the singular and renormalised energies presented in §3 and the minimal energy on ball developed in §4, we establish an upper bound on the Ginzburg–Landau energy $\mathcal{E}^\varepsilon_F(u)$, defined in (1.3). In this section $\Omega$ is a Lipschitz bounded domain and $F \in \mathcal{C}(\mathbb{R}^\nu, [0,+\infty))$ satisfies $F^{-1}(\{0\}) = \mathcal{N}$ and (1.4).

We first give an upper bound on the infimum of the energy with given Dirichlet boundary data in terms of the infimum of the geometric renormalised energy.

**Proposition 5.1.** *Let $g \in W^{1/2,2}(\partial\Omega, \mathcal{N})$, $k \in \mathbb{N}_*$, $a_1, \dots, a_k$ be distinct points in $\Omega$, and let $(\gamma_1, \dots, \gamma_k)$ be a minimal topological resolution of $g$, then, as $\varepsilon \to 0$,*

$$
\begin{aligned}
\inf\{\mathcal{E}^\varepsilon_F(u) \;:\; u \in W^{1,2}(\Omega, \mathbb{R}^\nu) \text{ and } \operatorname{tr}_{\partial\Omega} u = g\}&\\
\le \mathcal{E}^{\mathrm{sg}}(g) \log \frac{1}{\varepsilon} + \mathcal{E}^{\mathrm{geom}}_{g,\gamma_1,\dots,\gamma_k}(a_1, \dots, a_k) + \sum_{i=1}^k \mathcal{Q}_{F,\gamma_i} + o(1)&.
\end{aligned}
$$

When $\mathcal{N} = \mathbb{S}^1$, Proposition 5.1 is due to Bethuel, Brezis and Hélein [9, Lemma VIII.1].

*Proof of Proposition 5.1.* For every $\rho \in (0, \bar\rho(a_1, \dots, a_k))$, we consider a map $u_* \in W^{1,2}(\Omega \setminus \bigcup_{i=1}^k \bar B_\rho(a_i), \mathcal{N})$ such $\operatorname{tr}_{\partial\Omega} u_* = g$ and $\operatorname{tr}_{\mathbb{S}^1} u_*(a_i + \rho\cdot) = \gamma_i$ for every $i \in \{1, \dots, k\}$ and maps $u_1, \dots, u_k \in W^{1,2}(B_\rho, \mathbb{R}^\nu)$ such that $\operatorname{tr}_{\mathbb{S}^1} u_i(\rho\cdot) = \gamma_i$. We then set

$$
u(x) \coloneqq \begin{cases} u_*(x) & \text{if } x \in \Omega \setminus \bigcup_{i=1}^k B_\rho(a_i),\\ u_i(x - a_i) & \text{if } x \in B_\rho(a_i) \text{ for some } i \in \{1, \dots, k\},\end{cases}
$$

and we have, since $F(u_*) = 0$ in $\Omega \setminus \bigcup_{i=1}^k B_\rho(a_i)$,

$$
\mathcal{E}^\varepsilon_F(u) = \int_\Omega \frac{|Du|^2}{2} + \frac{F(u)}{\varepsilon^2} = \int_{\Omega\setminus\bigcup_{i=1}^k B_\rho(a_i)} \frac{|Du_*|^2}{2} + \sum_{i=1}^n \int_{B_\rho(a_i)} \frac{|Du_i|^2}{2} + \frac{F(u_i)}{\varepsilon^2}.
$$

By taking the infimum over $u_*, u_1, \dots, u_k$, we obtain by (3.6) and (4.2),

$$
\inf\big\{\mathcal{E}^\varepsilon_F(u) \;:\; u \in W^{1,2}(\Omega, \mathbb{R}^\nu) \text{ and } \operatorname{tr}_{\partial\Omega} u = g\big\} \le \mathcal{E}^{\mathrm{geom},\rho}_{g,\gamma_1,\dots,\gamma_k}(a_1, \dots, a_k) + \sum_{i=1}^k \mathcal{Q}^{\rho/\varepsilon}_{F,\gamma_i}.
$$

By choosing now $\rho = \sqrt{\varepsilon}$, we obtain

$$\begin{aligned}\inf\{\mathcal{E}^{\varepsilon}_F(u) \;:\; u \in W^{1,2}(\Omega,\mathbb{R}^\nu) \text{ and } \operatorname{tr}_{\partial\Omega} u = g\} - \sum_{i=1}^{k} \frac{\lambda(\gamma_i)^2}{4\pi} \log\frac{1}{\varepsilon} \\ \le \mathcal{E}^{\mathrm{geom},\sqrt{\varepsilon}}_{g,\gamma_1,\dots,\gamma_k}(a_1,\dots,a_k) - \sum_{i=1}^{k} \frac{\lambda(\gamma_i)^2}{4\pi}\log\frac{1}{\sqrt{\varepsilon}} + \sum_{i=1}^{k} \mathcal{Q}^{\frac{1}{\sqrt{\varepsilon}}}_{F,\gamma_i} - \sum_{i=1}^{k}\frac{\lambda(\gamma_i)^2}{4\pi}\log\frac{1}{\sqrt{\varepsilon}},\end{aligned}$$

and the conclusion follows by letting $\varepsilon \to 0$ from the definition (3.5) of $\mathcal{E}^{\mathrm{geom}}_{g,\gamma_1,\dots,\gamma_k}(a_1,\dots,a_k)$ and the definition (4.3) of $\mathcal{Q}_{F,\gamma_i}$. □

We also have an upper bound around singularities for renormalisable maps.

**Proposition 5.2.** *For every $u \in W^{1,2}_{\mathrm{ren}}(\Omega,\mathcal{N})$, if $\operatorname{sing}(u) = \{(a_1,\gamma_1),\dots,(a_k,\gamma_k)\}$, then for every $\rho \in (0,\bar\rho(a_1,\dots,a_k))$, as $\varepsilon \to 0$,*

$$\begin{aligned}\inf\{\mathcal{E}^{\varepsilon}_F(v) \;:\; v \in W^{1,2}(\Omega,\mathbb{R}^\nu) \textit{ and } v = u \textit{ in } \Omega\setminus \textstyle\bigcup_{i=1}^{k} B_\rho(a_i)\;\} \\ \le \sum_{i=1}^{k}\frac{\lambda(\gamma_i)^2}{4\pi}\log\frac{1}{\varepsilon} + \mathcal{E}^{\mathrm{ren}}(u) + \mathcal{Q}_F(u) + o(1).\end{aligned}$$

The quantity $\mathcal{Q}_F(u)$ has been defined in (4.5).

*Proof of Proposition 5.2.* For every $u_1,\dots,u_k \in W^{1,2}(B_\rho,\mathbb{R}^\nu)$ such that $\operatorname{tr}_{\mathbb{S}^1} u_i(\rho\cdot) = u(a_i+\rho\cdot)$ on $\mathbb{S}^1$ for each $i \in \{1,\dots,k\}$, if we define the function $v : \Omega \to \mathbb{R}^\nu$ for $x \in \Omega$ by

$$v(x) = \begin{cases} u(x) & \text{if } x \in \Omega\setminus\bigcup_{i=1}^k \bar B_\rho(a_i),\\ u_i(x-a_i) & \text{if for some } i\in\{1,\dots,k\},\, x \in B_\rho(a_i),\end{cases}$$

then we have

$$\mathcal{E}^{\varepsilon}_F(v) = \int_{\Omega\setminus\bigcup_{i=1}^k B_\rho(a_i)} \frac{|Du|^2}{2} + \sum_{i=1}^{k}\int_{B_{\rho(a_i)}} \frac{|Du_i|^2}{2} + \frac{F(u_i)}{\varepsilon^2},$$

and thus by taking the infimum over $u_1,\dots,u_k$, we obtain by (4.2),

$$\begin{aligned}\inf\{\mathcal{E}^{\varepsilon}_F(v) \;:\; v \in W^{1,2}(\Omega,\mathbb{R}^\nu) \text{ and } v = u \text{ in } \Omega\setminus\textstyle\bigcup_{i=1}^k B_\rho(a_i)\;\} - \sum_{i=1}^{k}\frac{\lambda(\gamma_i)^2}{4\pi}\log\frac{1}{\varepsilon} \\ \le \int_{\Omega\setminus\bigcup_{i=1}^k B_\rho(a_i)} \frac{|Du|^2}{2} - \sum_{i=1}^{k}\frac{\lambda(\gamma_i)^2}{4\pi}\log\frac{1}{\rho} + \sum_{i=1}^{k}\Bigl(\mathcal{Q}^{\rho/\varepsilon}_{F,\operatorname{tr}_{\mathbb{S}^1} u(a_i+\rho\cdot)} - \frac{\lambda(\gamma_i)^2}{4\pi}\log\frac{\rho}{\varepsilon}\Bigr).\end{aligned}$$

We conclude by the definition (3.8) of $\mathcal{E}^{\mathrm{ren}}(u)$, by the definitions (4.3) and (4.5) of the quantities $\mathcal{Q}_{F,\gamma_i}(u)$ and $\mathcal{Q}_F(u)$, and by Proposition 4.3 and (iv) in Proposition 3.6. □

## 6. Lower bounds on the energy

We derive a lower bound for the Ginzburg–Landau energy $\mathcal{E}^{\varepsilon}_F(u)$, defined in (1.3), of maps $u$ in $W^{1,2}(\Omega,\mathbb{R}^\nu)$ with given boundary datum $\operatorname{tr}_{\partial\Omega} u = g$ that matches the upper bound of Proposition 5.2. We first prove in Section 6.1 a lower bound of the form $\mathcal{E}^{\mathrm{sg}}(g)\log\frac{1}{\varepsilon} - C$ for maps in $W^{1,2}(\Omega,\mathbb{R}^\nu)$ and for the Ginzburg–Landau energy. This lower bound along with a localisation of the energy argument allows us to prove boundedness of sequences which have their energies bounded by $\mathcal{E}^{\mathrm{sg}}(g)\log\frac{1}{\varepsilon} + C$ in Section 6.2. We have seen in the previous section that such a bound is satisfied by minimisers of (1.3). With the help of the compactness of minimisers we are able to improve the lower bound and obtain the desired result in Section 7.1.

In this section $\Omega$ is a Lipschitz bounded domain and $F \in \mathcal{C}(\mathbb{R}^\nu,[0,+\infty))$ satisfies $F^{-1}(\{0\}) = \mathcal{N}$ and (1.4).

6.1. **Global lower bound.** The global lower bound depends on the tubular neighbourhood extension energy.

**Definition 6.1.** Let $\Omega \subset \mathbb{R}^2$ be a bounded Lipschitz domain and $g \in W^{1/2,2}(\partial\Omega, \mathcal{N})$. We define the *tubular neighbourhood extension energy* of $g$ to be

$$\mathcal{E}^{\mathrm{ext}}(g) \coloneqq \inf\Big\{\int_{\partial\Omega\times[0,1]} \frac{|Dv|^2}{2} \;:\; v \in W^{1,2}(\partial\Omega \times [0,1], \mathcal{N}) \text{ and } \operatorname{tr}_{\partial\Omega\times\{0\}} v = g\Big\}.$$

**Proposition 6.2.** *There exists a constant $C \in (0, +\infty)$, depending only on $\Omega$ and $F$, such that for every $\varepsilon > 0$ and every $u \in W^{1,2}(\Omega, \mathbb{R}^\nu)$ with $g \coloneqq \operatorname{tr}_{\partial\Omega} u$ satisfying $g \in \mathcal{N}$ almost everywhere on $\partial\Omega$, we have*

$$\begin{aligned}\mathcal{E}^{\varepsilon}_F(u) + C\mathcal{E}^{\mathrm{ext}}(g) - \mathcal{E}^{\mathrm{sg}}(g) \log \frac{1}{C\varepsilon\mathcal{E}^{\mathrm{sg}}(g)}& \\ \ge \frac{1}{C}\bigg(\frac{|D(\operatorname{dist}_{\mathcal{N}} \circ u)|^2}{2} &+ \frac{F(u)}{\varepsilon^2} + \sup_{t>0} t^2 \mathcal{L}^2\big(|Du|^{-1}([t, +\infty))\big)\bigg),\end{aligned}$$

*where the last term on the right-hand side is understood to vanish when $\mathcal{E}^{\mathrm{sg}}(g) = 0$.*

When $\mathcal{N} = \mathbb{S}^1$, Proposition 6.2, without the weak estimate on the gradient, is due to Sandier [48, Theorem 2], the corresponding weak estimate being due to Serfaty and Tice [51, Theorem 2]. In the general case, the fact that the right-hand side is non-negative is due to Canevari [15].

Proposition 6.2 will follow from a slightly refined result for smooth maps, see Lemma 6.7 , together with an approximation argument. The proof of Lemma 6.7 follows Sandier's strategy [48, Proof of Theorem 2] by an application of the coarea formula and the lower estimates for the Dirichlet energy outside a compact set of maps into a manifold, which depends on the one-dimensional Hausdorff content, whose definition and properties we recall now.

**Definition 6.3.** The *one-dimensional Hausdorff content* of a compact set $K \subset \mathbb{R}^2$ is defined as

$$\mathcal{H}^1_\infty(K) \coloneqq \inf\Big\{\sum_{B\in\mathcal{B}} \operatorname{diam}(B) \;:\; K \subset \bigcup_{B\in\mathcal{B}} B \text{ and } \mathcal{B} \text{ is a finite collection of closed balls}\Big\}.$$

The one-dimensional Hausdorff content is an outer measure and is bounded from above by the Hausdorff measure:

$$\mathcal{H}^1_\infty(K) \le \mathcal{H}^1(K). \tag{6.1}$$

We also recall the following lemma which will be used repeatedly to transform a covering of some set by balls into a covering by closed balls with disjoint closure (see Lemma 4.1 in [49]).

**Lemma 6.4.** *For every finite set $\mathcal{B}$ of balls of $\mathbb{R}^2$, there exists a finite set $\mathcal{B}'$ of disjoint non-empty closed balls of $\mathbb{R}^2$ such that*

$$\mathcal{B} = \bigcup_{B'\in\mathcal{B}'} \{B \in \mathcal{B} \;:\; B \subseteq B'\},$$

*and*

$$\sum_{B'\in\mathcal{B}'} \operatorname{diam}(B') = \sum_{B\in\mathcal{B}} \operatorname{diam}(B).$$

We finally rely on the equality between the one-dimensional Hausdorff content of a compact set and of its boundary.

**Lemma 6.5.** *If $K \subseteq \mathbb{R}^2$ is compact, then $\mathcal{H}^1_\infty(K) = \mathcal{H}^1_\infty(\partial K)$.*

Lemma 6.5 does not hold for the *Haudorff measure*; the proof of Lemma 6.5 can be seen to work when $K \subseteq \mathbb{R}^n$ is compact and $n \ge 2$; the equality fails when $n = 1$ and $K = [0, 1] \subset \mathbb{R}$.

*Proof of Lemma 6.5.* By monotonicity of the Hausdorff content, we have $\mathcal{H}^1_\infty(K) \ge \mathcal{H}^1_\infty(\partial K)$. It remains thus to establish the converse inequality.

We fix $\eta > 0$. By definition of the Hausdorff content, there exist points $a_1, \dots, a_k \in \mathbb{R}^2$ and radii $\rho_1, \dots, \rho_k \in (0, +\infty)$ such that $\partial K \subseteq \bigcup_{i=1}^k B_{\rho_i}(a_i)$ and $\sum_{i=1}^k 2\rho_i \le \mathcal{H}^1_\infty(\partial K) + \eta$. By Lemma 6.4, we can assume that $\bar{B}_{\rho_i}(a_i) \cap \bar{B}_{\rho_j}(a_j) = \emptyset$ if $i, j \in \{1, \dots, k\}$ with $i \ne j$. We claim that $K \subset \bigcup_{i=1}^k B_{\rho_i}(a_i)$. Indeed, assume by contradiction that there exists a point $x \in K \setminus \bigcup_{i=1}^k B_{\rho_i}(a_i)$.

Since the balls $\bar{B}_{\rho_1}(a_1), \dots, \bar{B}_{\rho_k}(a_k)$ are pairwise disjoint by construction, the set $\mathbb{R}^2 \setminus \bigcup_{i=1}^k B_{\rho_i}(a_i)$ is path-connected. Since the set $K$ is compact, we have $\mathbb{R}^2 \setminus (K \cup \bigcup_{i=1}^k B_{\rho_i}(a_i)) \neq \emptyset$ and there exists thus a continuous map $\gamma \in \mathcal{C}([0,1], \mathbb{R}^2 \setminus \bigcup_{i=1}^k B_{\rho_i}(a_i))$ such that $\gamma(0) = x$ and $\gamma(1) \notin K$. Since the map $\gamma$ is continuous, there exists some $t_* \in [0,1]$ such that $\gamma(t_*) \in \partial K$ and we would thus have $\partial K \setminus \bigcup_{i=1}^k B_{\rho_i}(a_i) \neq \emptyset$, which is a contradiction. We have thus

$$\mathcal{H}^1_\infty(K) \leq 2 \sum_{i=1}^k \rho_i \leq \mathcal{H}^1_\infty(\partial K) + \eta;$$

we conclude by letting $\eta \to 0$. □

We will use the lower estimate on the Dirichlet energy of maps into a manifold proved in [39].

**Theorem 6.6** ([39, Theorem 5.1])**.** *For every Lipschitz bounded domain $\Omega \subset \mathbb{R}^2$, every compact set $K \subset \Omega$ such that $\mathcal{H}^1_\infty(K) > 0$ and every map $v \in W^{1,2}(\Omega \setminus K, \mathcal{N})$, we have*

$$\int_{\Omega \setminus K} \frac{|Dv|^2}{2} \geq \mathcal{E}^{\mathrm{sg}}(\operatorname{tr}_{\partial\Omega} v) \log \frac{\operatorname{dist}(K, \partial\Omega)}{2\mathcal{H}^1_\infty(K)}. \tag{6.2}$$

*More precisely, there exists a constant $C > 0$ such that*

$$\sup_{t>0} t^2 \mathcal{L}^2\big(\{x \in \Omega \setminus K \ : \ |Dv| \geq t\}\big) \leq C\bigg(\int_{\Omega \setminus K} \frac{|Dv|^2}{2} - \mathcal{E}^{\mathrm{sg}}(\operatorname{tr}_{\partial\Omega} v) \log \frac{\operatorname{dist}(K, \partial\Omega)}{2\mathcal{H}^1_\infty(K)}\bigg). \tag{6.3}$$

The left-hand side of (6.3) is the weak-$L^2$ quasi-norm of $|Du|$. Theorem 6.6 has its roots in a corresponding estimate for maps outside a finite collection of balls [9, Corollary II.1].

We are now ready to state a slightly refined version of Proposition 6.2 in the smooth setting:

**Lemma 6.7.** *There exist constants $C \in (0 + \infty)$ and $\delta \in (0, +\infty)$ depending only on $\Omega$ and $F$, such that for every $\varepsilon > 0$ and every map $u \in \mathcal{C}^2(\bar{\Omega}, \mathbb{R}^\nu)$ with $g := \operatorname{tr}_{\partial\Omega} u$ satisfying $g(\partial\Omega) \subseteq \mathcal{N}$ and $\mathcal{E}^{\mathrm{sg}}(g) > 0$, we have*

$$\begin{aligned}
&\mathcal{E}^\varepsilon_F(u) + C\mathcal{E}^{\mathrm{ext}}(g) - \mathcal{E}^{\mathrm{sg}}(g) \log \frac{1}{C\varepsilon\mathcal{E}^{\mathrm{sg}}(g)} \\
&\quad \geq \frac{1}{C}\bigg(\int_\Omega \bigg(\frac{|D(\operatorname{dist}_{\mathcal{N}} \circ u)|^2}{2} + \frac{F(u)}{\varepsilon^2}\bigg) + \sup_{t>0} t^2 \mathcal{L}^2\big(\{x \in \Omega \ : \ |Du(x)| \geq t\}\big) \\
&\qquad\qquad + \mathcal{E}^{\mathrm{sg}}(g)\, \frac{1}{\delta} \int_0^\delta \Psi\bigg(\frac{\mathcal{H}^1_\infty(K_s)\, s}{C\varepsilon\mathcal{E}^{\mathrm{sg}}(g)}\bigg) \,\mathrm{d}s\bigg),
\end{aligned}$$

*where the function $\Psi : (0, +\infty) \to \mathbb{R}_+$ is defined by $\Psi(\tau) := \tau - 1 - \log \tau$ for each $\tau \in (0, +\infty)$, and where the sets $K_s$ are defined for every $s \in (0, +\infty)$ by*

$$K_s := \{x \in \Omega \ : \ \operatorname{dist}(u(x), \mathcal{N}) \geq s\}.$$

Before proving Lemma 6.7 we extend maps in $u \in W^{1,2}(\Omega, \mathbb{R}^\nu)$ in the following way. In view of Definition 6.1, there exists $\delta_{\partial\Omega} > 0$ such that if we set

$$\Omega_{\delta_{\partial\Omega}} := \{x \in \mathbb{R}^2 \ : \ \operatorname{dist}(x, \partial\Omega) < \delta_{\partial\Omega}\}, \tag{6.4}$$

then, we can extend the function $u \in W^{1,2}(\Omega, \mathbb{R}^\nu)$ to a function $u \in W^{1,2}(\Omega_{\delta_{\partial\Omega}}, \mathbb{R}^\nu)$ in such a way that $u \in \mathcal{N}$ almost everywhere in $\Omega_{\delta_{\partial\Omega}} \setminus \Omega$ and

$$\int_{\Omega_{\delta_{\partial\Omega}} \setminus \Omega} \frac{|Du|^2}{2} \leq C_1 \mathcal{E}^{\mathrm{ext}}(g), \tag{6.5}$$

for some constant $C_1$ depending only on $\partial\Omega$. *In the rest of the paper we will always assume that maps $u \in W^{1,2}(\Omega, \mathbb{R}^\nu)$ are extended to the larger domain $\Omega_{\delta_{\partial\Omega}}$ as explained above.*

*Proof of Lemma 6.7.* We proceed in several steps:

*Step 1. Splitting normal and tangential derivatives.* We set for $x \in \Omega \setminus K_{\delta_{\mathcal{N}}}$,

$$D^\top u(x) := P^\top_{\mathcal{N}}(\Pi_{\mathcal{N}}(u(x)))[Du(x)] \qquad \text{and} \qquad D^\perp u(x) := P^\perp_{\mathcal{N}}(\Pi_{\mathcal{N}}(u(x)))[Du(x)],$$

with the nearest point retraction $\Pi_{\mathcal{N}}$ and the projections $P^{\top}_{\mathcal{N}}$ and $P^{\perp}_{\mathcal{N}}$ being defined in Lemma 2.1; there holds in particular, within the set $\Omega \setminus K_{\delta_{\mathcal{N}}}$,

$$\left(1 - \frac{\operatorname{dist}_{\mathcal{N}} \circ u}{\delta_{\mathcal{N}}}\right) |D(\Pi_{\mathcal{N}} \circ u)|^2 \le |D^{\top} u|^2 \le C_2 |D(\Pi_{\mathcal{N}} \circ u)|^2, \tag{6.6}$$

and

$$|D(\operatorname{dist}_{\mathcal{N}} \circ u)|^2 \le |D^{\perp} u|^2. \tag{6.7}$$

We also let $\delta_F \in (0, \delta_{\mathcal{N}})$ be a constant as in Lemma 2.3 so that for all $y \in \mathcal{N}_{\delta_F}$, we have

$$F(y) \ge \frac{m_F}{2} \operatorname{dist}_{\mathcal{N}}(y)^2. \tag{6.8}$$

By orthogonality between $P^{\perp}_{\mathcal{N}}$ and $P^{\top}_{\mathcal{N}}$, we have for every $\delta \in (0, \delta_F]$

$$\begin{aligned}\mathcal{E}^{\varepsilon}_F(u) &= \int_{\Omega \setminus K_\delta} \left(\frac{|D^{\perp} u|^2}{2} + \frac{F \circ u}{\varepsilon^2}\right) + \int_{\Omega \setminus K_\delta} \frac{|D^{\top} u|^2}{2} + \int_{K_\delta} \left(\frac{|Du|^2}{2} + \frac{F \circ u}{\varepsilon^2}\right)\\ &=: (\mathbf{I}) + (\mathbf{II}) + (\mathbf{III}).\end{aligned} \tag{6.9}$$

*Step 2. Estimate of* $(\mathbf{I})$ *from below.* Since $u \in \mathcal{C}^2(\Omega, \mathbb{R}^{\nu})$, by Sard's lemma and by the implicit function theorem, for almost every $s \in (0, +\infty)$, the set $K_s \subset \Omega$ has a $\mathcal{C}^2$ boundary and

$$\partial K_s = \Sigma_s \coloneqq \{x \in \Omega \ : \ \operatorname{dist}(u(x), \mathcal{N}) = s\}.$$

Hence, using successively (6.7), Young's inequality, (6.8) and the coarea formula, we obtain

$$\begin{aligned}(\mathbf{I}) = \int_{\Omega \setminus K_\delta} \left(\frac{|D^{\perp} u|^2}{2} + \frac{F \circ u}{\varepsilon^2}\right) &\ge \int_{\Omega \setminus K_\delta} \left(\frac{|D(\operatorname{dist}_{\mathcal{N}} \circ u)|^2}{2} + \frac{F \circ u}{\varepsilon^2}\right)\\ &\ge \int_{\Omega \setminus K_\delta} \frac{1}{\varepsilon} |D(\operatorname{dist}_{\mathcal{N}} \circ u)| \sqrt{2F \circ u}\\ &\ge \sqrt{m_F} \int_{\Omega \setminus K_\delta} \frac{1}{\varepsilon} |D(\operatorname{dist}_{\mathcal{N}} \circ u)| (\operatorname{dist}_{\mathcal{N}} \circ u)\\ &= \sqrt{m_F} \int_0^{\delta} \frac{\mathcal{H}^1(\Sigma_s)\, s}{\varepsilon} \,\mathrm{d}s.\end{aligned}$$

But, by Lemma 6.5 and (6.1), we have for almost every $s > 0$, $\mathcal{H}^1_{\infty}(K_s) = \mathcal{H}^1_{\infty}(\Sigma_s) \le \mathcal{H}^1(\Sigma_s)$; hence

$$(\mathbf{I}) \ge \sqrt{m_F} \int_0^{\delta} \frac{\mathcal{H}^1_{\infty}(K_s)\, s}{\varepsilon} \,\mathrm{d}s.$$

Moreover, by Chebychev's inequality, we have also

$$(\mathbf{I}) \ge \int_{\Omega \setminus K_\delta} \frac{|D^{\perp} u|^2}{2} \ge \sup_{t > 0} \frac{t^2}{8} \mathcal{L}^2\big(\{x \in \Omega \setminus K_\delta \ : \ |D^{\perp} u(x)| \ge t/2\}\big).$$

We have thus proved that there exists a constant $C_3 > 0$ such that

$$\begin{aligned}(\mathbf{I}) \ge \frac{1}{C_3} \Bigg(\int_0^{\delta} \frac{\mathcal{H}^1_{\infty}(K_s)\, s}{\varepsilon} \,\mathrm{d}s + \int_{\Omega \setminus K_\delta} \frac{|D(\operatorname{dist}_{\mathcal{N}} \circ u)|^2}{2} + \frac{F \circ u}{\varepsilon^2}\\ + \sup_{t > 0} t^2 \mathcal{L}^2\big(\{x \in \Omega \setminus K_\delta \ : \ |D^{\perp} u(x)| \ge t/2\}\big)\Bigg).\end{aligned} \tag{6.10}$$

*Step 3. Estimate of* $(\mathbf{II})$ *from below.* By (6.6) and Fubini's theorem, we have

$$\begin{aligned}(\mathbf{II}) = \int_{\Omega \setminus K_\delta} \frac{|D^{\top} u|^2}{2} &\ge \int_{\Omega \setminus K_\delta} \left(1 - \frac{\operatorname{dist}_{\mathcal{N}} \circ u}{\delta_{\mathcal{N}}}\right) \frac{|D(\Pi_{\mathcal{N}} \circ u)|^2}{2}\\ &= \int_{\Omega \setminus K_\delta} \left(\frac{1}{\delta} \int_{\operatorname{dist}_{\mathcal{N}} \circ u}^{\delta} \mathrm{d}s\right) \frac{|D(\Pi_{\mathcal{N}} \circ u)|^2}{2}\\ &= \frac{1}{\delta} \int_0^{\delta} \left(\int_{\Omega \setminus K_s} \frac{|D(\Pi_{\mathcal{N}} \circ u)|^2}{2}\right) \mathrm{d}s.\end{aligned} \tag{6.11}$$

Then, by (6.5), we have for every $s \in (0, \delta)$,

$$\int_{\Omega \setminus K_s} \frac{|D(\Pi_{\mathcal{N}} \circ u)|^2}{2} \geq \int_{\Omega_{\delta_{\partial\Omega}} \setminus K_s} \frac{|D(\Pi_{\mathcal{N}} \circ u)|^2}{2} - C_1 \mathcal{E}^{\mathrm{ext}}(g),$$

while by the lower estimate on the Dirichlet energy of mappings Theorem 6.6, since $\Pi_{\mathcal{N}} \circ u \in W^{1,2}(\Omega_{\delta_{\partial\Omega}} \setminus K_\delta, \mathcal{N})$, for every $t > 0$,

$$\begin{aligned}\int_{\Omega_{\delta_{\partial\Omega}} \setminus K_s} \frac{|D(\Pi_{\mathcal{N}} \circ u)|^2}{2} \geq \mathcal{E}^{\mathrm{sg}}(g) \log \frac{\delta_{\partial\Omega}}{2\mathcal{H}^1_\infty(K_s)} \\ + \frac{1}{C_4} t^2 \mathcal{L}^2(\{x \in \Omega \setminus K_s \;:\; |D(\Pi_{\mathcal{N}} \circ u)| \geq t\}),\end{aligned}$$

for some constant $C_4 > 0$. Since by (6.6), we have $|D^\top u| \leq \sqrt{C_2} |D(\Pi_{\mathcal{N}} \circ u)|$, we have also

$$\mathcal{L}^2(\{x \in \Omega \setminus K_s \;:\; |D(\Pi_{\mathcal{N}} \circ u)| \geq t\}) \geq \mathcal{L}^2(\{x \in \Omega \setminus K_s \;:\; |D^\top u| \geq \sqrt{C_2}\, t\}).$$

We thus arrive at

$$\begin{aligned}\int_{\Omega \setminus K_s} \frac{|D(\Pi_{\mathcal{N}} \circ u)|^2}{2} \geq \mathcal{E}^{\mathrm{sg}}(g) \log \frac{\delta_{\partial\Omega}}{2\mathcal{H}^1_\infty(K_s)} - C_1 \mathcal{E}^{\mathrm{ext}}(g) \\ + \frac{1}{C_4} t^2 \mathcal{L}^2(\{x \in \Omega \setminus K_s \;:\; |D^\top u| \geq \sqrt{C_2}\, t\}).\end{aligned}$$

By integration with respect to $s$ over $(0, \delta)$, we obtain in view of (6.11),

$$\begin{aligned}(\mathbf{II}) \geq \mathcal{E}^{\mathrm{sg}}(g) \frac{1}{\delta} \int_0^\delta \log \frac{\delta_{\partial\Omega}}{2\mathcal{H}^1_\infty(K_s)} \,\mathrm{d}s - C_1 \mathcal{E}^{\mathrm{ext}}(g) \\ + \frac{1}{C_4} t^2 \frac{1}{\delta} \int_0^\delta \mathcal{L}^2(\{x \in \Omega \setminus K_s \;:\; |D^\top u| \geq \sqrt{C_2}\, t\}) \,\mathrm{d}s.\end{aligned}$$

By Fubini's theorem, we compute

$$\begin{aligned}\frac{1}{\delta} \int_0^\delta \mathcal{L}^2(\{x \in \Omega \setminus K_s \;:\; |D^\top u| \geq \sqrt{C_2}\, t\}) \,\mathrm{d}s &= \int_{\{x \in \Omega \setminus K_\delta \,:\, |D^\top u| \geq \sqrt{C_2}\, t\}} \Bigl(1 - \frac{\mathrm{dist}_{\mathcal{N}} \circ u}{\delta}\Bigr) \\ &\geq \frac{1}{2} \mathcal{L}^2\bigl(\{x \in \Omega \setminus K_{\delta/2} \;:\; |D^\top u| \geq \sqrt{C_2}\, t\}\bigr)\end{aligned}$$

and by the change of variable $s = 2\sqrt{C_2}\, t$,

$$\sup_{t > 0} t^2 \mathcal{L}^2\bigl(\{x \in \Omega \setminus K_{\delta/2} \;:\; |D^\top u| \geq \sqrt{C_2}\, t\}\bigr) \geq \sup_{s > 0} \frac{s^2}{4C_2} \mathcal{L}^2\bigl(\{x \in \Omega \setminus K_{\delta/2} \;:\; |D^\top u| \geq s/2\}\bigr).$$

Hence, we have proved

$$\begin{aligned}(\mathbf{II}) \geq \mathcal{E}^{\mathrm{sg}}(g) \frac{1}{\delta} \int_0^\delta \log \frac{\delta_{\partial\Omega}}{2\mathcal{H}^1_\infty(K_s)} \,\mathrm{d}s - C_1 \mathcal{E}^{\mathrm{ext}}(g) \\ + \frac{1}{8C_2C_4} \sup_{t > 0} t^2 \mathcal{L}^2\bigl(\{x \in \Omega \setminus K_{\delta/2} \;:\; |D^\top u| \geq t/2\}\bigr).\end{aligned} \tag{6.12}$$

*Step 4. Estimate of* $(\mathbf{III})$ *from below.* By (6.7) and Chebychev's inequality, we have

$$\begin{aligned}(\mathbf{III}) &= \int_{K_\delta} \Bigl(\frac{|Du|^2}{2} + \frac{F \circ u}{\varepsilon^2}\Bigr) \geq \frac{1}{2} \Bigl(\int_{K_\delta} \Bigl(\frac{|D(\mathrm{dist}_{\mathcal{N}} \circ u)|^2}{2} + \frac{F \circ u}{\varepsilon^2}\Bigr) + \int_{K_\delta} \frac{|Du|^2}{2}\Bigr) \\ &\geq \frac{1}{8} \Bigl(\int_{K_\delta} \Bigl(\frac{|D(\mathrm{dist}_{\mathcal{N}} \circ u)|^2}{2} + \frac{F \circ u}{\varepsilon^2}\Bigr) + \sup_{t > 0} t^2 \mathcal{L}^2(\{x \in K_\delta \;:\; |Du(x)| \geq t\})\Bigr).\end{aligned} \tag{6.13}$$

*Step 5. Putting things together.* We first observe that for every $t > 0$,

$$\begin{aligned}t^2 \mathcal{L}^2(\{x \in K_\delta \cup \Omega \setminus K_{\delta/2} \;:\; |Du(x)| \geq t\}) \leq t^2 \mathcal{L}^2(\{x \in K_\delta \;:\; |Du(x)| \geq t\}) \\ + t^2 \mathcal{L}^2(\{x \in \Omega \setminus K_{\delta/2} \;:\; |D^\perp u(x)| \geq t/2\}) \\ + t^2 \mathcal{L}^2(\{x \in \Omega \setminus K_{\delta/2} \;:\; |D^\top u(x)| \geq t/2\}),\end{aligned}$$

which in view of (6.9), by adding (6.10), (6.12) and (6.13), gives the existence of a constant $C_5 > 0$ such that

$$\begin{aligned}(6.14)\quad \mathcal{E}^{\varepsilon}_F(u) \geq \mathcal{E}^{\mathrm{sg}}(g)\frac{1}{\delta}\int_0^{\delta}\left(\frac{\delta\mathcal{H}^1_\infty(K_s)\,s}{C_3\varepsilon\mathcal{E}^{\mathrm{sg}}(g)} + \log\frac{\delta_{\partial\Omega}}{2\mathcal{H}^1_\infty(K_s)}\right)\mathrm{d}s - C_1\mathcal{E}^{\mathrm{ext}}(g)\\ + \frac{1}{C_5}\biggl(\int_\Omega \frac{|D(\mathrm{dist}_{\mathcal{N}}\circ u)|^2}{2} + \frac{F\circ u}{\varepsilon^2}\\ + \sup_{t>0} t^2\mathcal{L}^2\bigl(\{x \in K_\delta \cup \Omega\setminus K_{\delta/2}\ :\ |Du(x)| \geq t\}\bigr)\biggr).\end{aligned}$$

Applying the identity $\tau = 1 + \log\tau + \psi(\tau)$ to $\tau = \frac{\delta\mathcal{H}^1_\infty(K_s)\,s}{C_3\varepsilon\mathcal{E}^{\mathrm{sg}}(g)}$, we obtain

$$\frac{\delta\mathcal{H}^1_\infty(K_s)\,s}{C_3\varepsilon\mathcal{E}^{\mathrm{sg}}(g)} + \log\frac{\delta_{\partial\Omega}}{2\mathcal{H}^1_\infty(K_s)} = 1 + \log\frac{\delta\delta_{\partial\Omega}s}{2C_3\varepsilon\mathcal{E}^{\mathrm{sg}}(g)} + \Psi\left(\frac{\delta\mathcal{H}^1_\infty(K_s)\,s}{C_3\varepsilon\mathcal{E}^{\mathrm{sg}}(g)}\right),$$

and we compute that

$$\frac{1}{\delta}\int_0^{\delta}\left(1 + \log\frac{\delta\delta_{\partial\Omega}s}{2C_3\varepsilon\mathcal{E}^{\mathrm{sg}}(g)}\right)\mathrm{d}s = \log\frac{\delta^2\delta_{\partial\Omega}}{2C_3\varepsilon\mathcal{E}^{\mathrm{sg}}(g)}.$$

Hence, there exists a constant $C > 0$ such that

$$\begin{aligned}(6.15)\quad &\mathcal{E}^{\varepsilon}_F(u) + C\mathcal{E}^{\mathrm{ext}}(g) - \mathcal{E}^{\mathrm{sg}}(g)\log\frac{1}{C\varepsilon\mathcal{E}^{\mathrm{sg}}(g)}\\ &\geq \frac{1}{C}\biggl(\int_\Omega \frac{|D(\mathrm{dist}_{\mathcal{N}}\circ u)|^2}{2} + \frac{F\circ u}{\varepsilon^2} + \sup_{t>0} t^2\mathcal{L}^2\bigl(\{x \in K_\delta \cup \Omega\setminus K_{\delta/2}\ :\ |Du(x)| \geq t\}\bigr)\\ &\qquad + \mathcal{E}^{\mathrm{sg}}(g)\frac{1}{\delta}\int_0^{\delta}\Psi\left(\frac{\mathcal{H}^1_\infty(K_s)\,s}{C\varepsilon\mathcal{E}^{\mathrm{sg}}(g)}\right)\mathrm{d}s\biggr).\end{aligned}$$

Since we have

$$\begin{aligned}\mathcal{L}^2\bigl(\{x \in \Omega\ :\ |Du(x)| \geq t\}\bigr) \leq &\mathcal{L}^2\bigl(\{x \in K_{\delta_F} \cup \Omega\setminus K_{\delta_F/2}\ :\ |Du(x)| \geq t\}\bigr)\\ &+ \mathcal{L}^2\bigl(\{x \in K_{\delta_F/2} \cup \Omega\setminus K_{\delta_F/4}\ :\ |Du(x)| \geq t\}\bigr),\end{aligned}$$

the desired estimate follows by taking the average of (6.15) for $\delta \in \{\delta_F, \frac{\delta_F}{2}\}$. □

*Proof of Proposition 6.2.* If the Ginzburg–Landau functional is continuous with respect to the $W^{1,2}$ strong convergence, the conclusion follows from Lemma 6.7 and an approximation argument.

If the Ginzburg–Landau functional is not continuous, we consider a non-decreasing sequence $(F^\ell)_{\ell\in\mathbb{N}}$ of bounded and continuous functions coinciding with $F$ in a neighbourhood of $\mathcal{N}$ and converging to $F$ a.e. The theorem holds for each of these functions, since $\mathcal{E}^{\varepsilon}_{F_\ell}$ is continuous for the $W^{1,2}$ strong convergence by Lebesgue's dominated convergence. The conclusion then holds by Lebesgue's monotone convergence theorem. □

As a consequence of Proposition 6.2, we obtain the finiteness of the quantity $\mathcal{Q}_{F,\gamma}$ defined in (4.3), when $\gamma$ is an atomic minimizing geodesic, i.e. when $\mathcal{E}^{\mathrm{sg}}(\gamma) = \lambda(\gamma)^2/(4\pi)$.

**Corollary 6.8.** *Let $F \in \mathcal{C}(\mathbb{R}^\nu, [0, +\infty))$. If $F^{-1}(\{0\}) = \mathcal{N}$, if $F$ satisfies* (1.4) *and if $\gamma \in \mathcal{C}^1(\mathbb{S}^1, \mathcal{N})$ is an atomic minimising geodesic, then $\mathcal{Q}_{F,\gamma} > -\infty$.*

*Proof.* By Proposition 6.2 applied to $\Omega = B_1$, the unit ball with center 0 in $\mathbb{R}^2$, there exists a constant $C_1 \in (0, +\infty)$ such that for every $\varepsilon > 0$ and every $u \in W^{1,2}(B_1, \mathbb{R}^\nu)$ we have

$$\int_{B_1} \frac{|Du|^2}{2} + \frac{F(u)}{\varepsilon^2} \geq \mathcal{E}^{\mathrm{sg}}(\mathrm{tr}_{\partial\Omega} u)\log\frac{1}{C_1\varepsilon\mathcal{E}^{\mathrm{sg}}(\mathrm{tr}_{\partial\Omega} u)} - C_1\mathcal{E}^{\mathrm{ext}}(\mathrm{tr}_{\partial\Omega} u).$$

By taking the infimum over $u$ such that $\mathrm{tr}_{\partial B_1} u = \gamma$, we obtain, in view of (4.2), with $\rho = \frac{1}{\varepsilon}$,

$$\mathcal{Q}^{\rho}_{F,\gamma} - \mathcal{E}^{\mathrm{sg}}(\gamma)\log\rho \geq \mathcal{E}^{\mathrm{sg}}(\gamma)\log\frac{1}{C_1\mathcal{E}^{\mathrm{sg}}(\gamma)} - C_1\mathcal{E}^{\mathrm{ext}}(\gamma).$$

The claim follows from (4.3) since, by assumption, $\mathcal{E}^{\mathrm{sg}}(\gamma) = \frac{\lambda(\gamma)^2}{4\pi}$. □

6.2. **Localised lower bound on the energy.** The next proposition provides some information on the localisation of the energy of mapping satisfying a logarithmic bound.

**Proposition 6.9.** *There exists $C \in (0, +\infty)$ such that for every $\kappa \in (0, +\infty)$, $\eta \in (0, 1/C)$, $\gamma \in (0, 1)$, $\varepsilon \in (0, +\infty)$ and $g \in W^{1/2,2}(\partial\Omega, \mathcal{N})$ such that $\mathcal{E}^{\mathrm{sg}}(g) > 0$,*

$$Ce^{C\gamma\kappa}(\mathcal{E}^{\mathrm{sg}}(g)\varepsilon)^{1-\gamma} \le \gamma\eta, \qquad C\mathcal{E}^{\mathrm{sg}}(g)\varepsilon \le \gamma\eta, \qquad C\varepsilon\kappa \le 1 \qquad \textit{and} \qquad \mathcal{E}^{\mathrm{ext}}(g) \le \kappa,$$

*if $u \in W^{1,2}(\Omega, \mathbb{R}^\nu)$ satisfies $\operatorname{tr}_{\partial\Omega} u = g$ and*

$$\int_\Omega \frac{|Du|^2}{2} + \frac{F(u)}{\varepsilon^2} \le \mathcal{E}^{\mathrm{sg}}(g) \log \frac{1}{\varepsilon\mathcal{E}^{\mathrm{sg}}(g)} + \kappa, \tag{6.16}$$

*and if we still denote by $u$ the extension to $\Omega_{\delta_{\partial\Omega}}$ satisfying* (6.5)*, then there exists a collection of balls $\mathcal{B}$ in $\mathbb{R}^2$ with*

*(i) for every $B \in \mathcal{B}$, $\operatorname{diam}(B) \le 2\eta$ and $\bar{B} \subset \Omega_{\delta_{\partial\Omega}}$,*
*(ii) for every $B \in \mathcal{B}$, $\operatorname{dist}_{\mathcal{N}} \circ \operatorname{tr}_{\partial B} u < \delta_{\mathcal{N}}$, the map $\Pi_{\mathcal{N}} \circ \operatorname{tr}_{\partial B} u$ is not homotopic to a constant and the maps $(\Pi_{\mathcal{N}} \circ \operatorname{tr}_{\partial B} u)_{B\in\mathcal{B}}$ are a topological resolution of $g = \operatorname{tr}_{\partial\Omega} u$,*
*(iii) for every subset $\mathcal{B}' \subset \mathcal{B}$,*

$$\int\limits_{\Omega\cap\bigcup_{B\in\mathcal{B}'} B} \frac{|Du|^2}{2} + \frac{F(u)}{\varepsilon^2} \ge \sum_{B\in\mathcal{B}'} \mathcal{E}^{\mathrm{sg}}(\Pi_{\mathcal{N}} \circ \operatorname{tr}_{\partial B} u) \log \frac{\gamma\eta}{C\mathcal{E}^{\mathrm{sg}}(g)\varepsilon} - C(\kappa + \mathcal{E}^{\mathrm{sg}}(g)),$$

*(iv) one has*

$$\frac{\operatorname{sys}(\mathcal{N})^2}{4\pi} \#\mathcal{B} \le \sum_{B\in\mathcal{B}} \mathcal{E}^{\mathrm{sg}}(\Pi_{\mathcal{N}} \circ \operatorname{tr}_{\partial B} u) \le \mathcal{E}^{\mathrm{sg}}(g) + \frac{(\log \frac{C}{\gamma\eta} + C\varepsilon)\mathcal{E}^{\mathrm{sg}}(g) + (1 + C\varepsilon)\kappa}{\log \frac{\gamma\eta}{C\varepsilon\mathcal{E}^{\mathrm{sg}}(g)}}.$$

In the previous statement, $\operatorname{sys}(\mathcal{N})$ denotes the systole of the manifold $\mathcal{N}$ defined in (3.3) as the shortest length of a closed geodesic which is not homotopic to a constant.

Proposition 6.9 has its roots in lower bounds for minimisers of the Ginzburg–Landau energy for $\mathcal{N} = \mathbb{S}^1$ [9, Theorem V.2]; localised lower bounds Proposition 6.9 for $\mathcal{N} = \mathbb{S}^1$ are originally due to Sandier [48, Theorem 3′] and Jerrard [35, Theorem 1.2].

We follow in our proof the Jerrard's strategy [35] (see also the recent work by Ignat and Jerrard [33]). As a first tool to prove Proposition 6.9, we have a Sobolev type embedding theorem with dependence on $\varepsilon$ for maps defined on $\mathbb{S}^1_r := \partial B_r$, the circle of radius $r$ centered at the origin in $\mathbb{R}^2$ (see also [35, Lemma 2.3]).

**Lemma 6.10.** *There exists a constant $C > 0$ such that for every $r > 0$, every $h \in W^{1,2}(\mathbb{S}^1_r, \mathbb{R})$ and every $\varepsilon \in (0, r]$,*

$$\|h\|^2_{L^\infty(\mathbb{S}^1_r)} \le C \int_{\mathbb{S}^1_r} \varepsilon|h'|^2 + \frac{1}{\varepsilon}h^2.$$

*Proof.* By Morrey–Sobolev embedding , the function $h$ is continuous on $\mathbb{S}^1_r$. By the mean value theorem, there exists $a \in \mathbb{S}^1_r$ such that $h^2(a) = \frac{1}{2\pi r}\int_{\mathbb{S}^1_r} h^2$. By the fundamental theorem of calculus we can write

$$h(x)^2 = h(a)^2 + \int_{t_a}^{t_x} \big((h \circ \gamma)^2\big)' \,\mathrm{d}t$$

where $\gamma$ is a smooth path on $\mathbb{S}^1_r$ such that $\gamma(t_a) = a$ and $\gamma(t_x) = x$. Thus, for any $C > 0$, by using Young's inequality $2|h'h| \le C\varepsilon|h'|^2 + \frac{|h|^2}{C\varepsilon}$ and by recalling that $\varepsilon \le r$ we find

$$\|h^2\|_{L^\infty(\mathbb{S}^1_r)} \le \frac{1}{2\pi r}\int_{\mathbb{S}^1_r} h^2 + \int_{\mathbb{S}^1_r} |(h^2)'| \le \int_{\mathbb{S}^1_r} C\varepsilon|h'|^2 + \Big(\frac{1}{2\pi} + \frac{1}{C}\Big)\frac{h^2}{\varepsilon}.$$

The conclusion follows by taking $C = \frac{1+\sqrt{1+16\pi^2}}{4\pi}$ which solves $C = \frac{1}{2\pi} + \frac{1}{C}$. □

The next tool for the proof of Proposition 6.9, is a lower bound on the Ginzburg–Landau energy on circles at scales larger than $\varepsilon$. (When $\mathcal{N} = \mathbb{S}^1$, see [35, Proof of Proposition 3.1, Claim 1.]).

**Lemma 6.11.** *There exists a constant $c_1 > 0$, such that for every $r > 0$, for every $u \in W^{1,2}(\mathbb{S}^1_r, \mathbb{R}^\nu)$ such that $\operatorname{dist}(u, \mathcal{N}) < \delta_{\mathcal{N}}$ almost everywhere in $\mathbb{S}^1_r$ and for every $\varepsilon < r$, one has*

$$\int_{\mathbb{S}^1_r} \frac{|u'|^2}{2} + \frac{F(u)}{\varepsilon^2} \geq \frac{1}{\frac{\varepsilon}{c_1} + \frac{4\pi r}{\lambda(\Pi_{\mathcal{N}} \circ u)^2}}.$$

We remark that the right-hand side in the inequality of Lemma 6.11 is an increasing function of $c_1$. The proof of Lemma 6.11 relies on the following elementary inequality.

**Lemma 6.12.** *For every $z \in [0, 1]$ and $\alpha \in (0, +\infty)$, one has*

$$\frac{1 - z}{\alpha} + z^2 \geq \frac{1}{\alpha + 1}$$

*Proof.* If $\alpha \geq \frac{1}{2}$, then the left-hand side in the desired inequality is minimal for $z = \frac{1}{2\alpha} \in [0, 1]$ and we thus obtain that for every $z \in [0, 1]$, $\frac{1-z}{\alpha} + z^2 \geq \frac{1}{\alpha} - \frac{1}{4\alpha^2} \geq \frac{1}{\alpha+1}$; if $\alpha < \frac{1}{2}$, we have $\frac{1-z}{\alpha} + z^2 \geq 2(1 - z) + z^2 \geq 1 + (1 - z)^2 \geq 1 \geq \frac{1}{\alpha+1}$. □

*Proof of Lemma 6.11.* Since by assumption $\operatorname{dist}(u, \mathcal{N}) < \delta_{\mathcal{N}}$ almost everywhere on $\mathbb{S}^1_r$ and since the function $F$ satisfies the non-degeneracy assumption (1.4), we have by Lemma 2.1 and by (2.3),

$$\frac{|u'|^2}{2} + \frac{F(u)}{\varepsilon^2} \geq \left(1 - \frac{\operatorname{dist}_{\mathcal{N}} \circ u}{\delta_{\mathcal{N}}}\right) \frac{|(\Pi_{\mathcal{N}} \circ u)'|^2}{2} + \frac{|(\operatorname{dist}_{\mathcal{N}} \circ u)'|^2}{2} + \frac{m_F}{2\varepsilon^2} (\operatorname{dist}_{\mathcal{N}} \circ u)^2,$$

almost everywhere on $\mathbb{S}^1_r$. If we set $\theta \coloneqq \|\operatorname{dist}_{\mathcal{N}} \circ u\|_{L^\infty(\mathbb{S}^1_r)} \in [0, \delta_{\mathcal{N}}]$, we have on the one hand, by definition of $\theta$ and by the characterisation of $\lambda(\Pi_{\mathcal{N}} \circ u)$ (see (3.2)),

$$\int_{\mathbb{S}^1_r} \left(1 - \frac{\operatorname{dist}_{\mathcal{N}} \circ u}{\delta_{\mathcal{N}}}\right) \frac{|(\Pi_{\mathcal{N}} \circ u)'|^2}{2} \geq \left(1 - \frac{\theta}{\delta_{\mathcal{N}}}\right) \frac{\lambda(\Pi_{\mathcal{N}} \circ u)^2}{4\pi r}, \tag{6.17}$$

and on the other hand, by Lemma 6.10,

$$\int_{\mathbb{S}^1_r} \frac{|(\operatorname{dist}_{\mathcal{N}} \circ u)'|^2}{2} + \frac{m_F}{2\varepsilon^2} (\operatorname{dist}_{\mathcal{N}} \circ u)^2 \geq \frac{\theta^2}{C_1 \varepsilon}, \tag{6.18}$$

for some constant $C_1 > 0$. It follows thus from (6.17) and (6.18) that if $c_1 \leq \delta_{\mathcal{N}}^2 / C_1$, by applying Lemma 6.12 with $z = \frac{\theta}{\delta_{\mathcal{N}}}$, since $\theta \leq \delta_{\mathcal{N}}$

$$\begin{aligned}
\int_{\mathbb{S}^1_r} \frac{|u'|^2}{2} + \frac{F(u)}{\varepsilon^2} &\geq \left(1 - \frac{\theta}{\delta_{\mathcal{N}}}\right) \frac{\lambda(\Pi_{\mathcal{N}} \circ u)^2}{4\pi r} + \frac{\theta^2}{C_1 \varepsilon} \\
&\geq \frac{c_1}{\varepsilon} \left( \left(1 - \frac{\theta}{\delta_{\mathcal{N}}}\right) \frac{\lambda(\Pi_{\mathcal{N}} \circ u)^2 \varepsilon}{4\pi r c_1} + \left(\frac{\theta}{\delta_{\mathcal{N}}}\right)^2 \right) \\
&\geq \frac{c_1}{\varepsilon \left(\frac{4\pi r c_1}{\lambda(\Pi_{\mathcal{N}} \circ u)^2 \varepsilon} + 1\right)} = \frac{1}{\frac{\varepsilon}{c_1} + \frac{4\pi r}{\lambda(\Pi_{\mathcal{N}} \circ u)^2}}.
\end{aligned}$$

□

A last tool is the following lower bound on the energy inside the ball $B_r$ of radius $r$ centered at the origin, with non-trivial boundary conditions.

**Lemma 6.13.** *There exists a constant $c_2 > 0$, such that if $r > 0$, if $u \in W^{1,2}(B_r, \mathbb{R}^\nu)$ satisfies $\|\operatorname{dist}(\operatorname{tr}_{\partial B_r} u(\cdot), \mathcal{N})\|_{L^\infty(\partial B_r)} < \delta_{\mathcal{N}}$ and if $\int_{B_r} |Du|^2 \leq c_2$, then the map $\Pi_{\mathcal{N}} \circ \operatorname{tr}_{\partial B_r} u$ is homotopic to a constant map.*

*Proof.* We have by the trace theorem

$$\|\Pi_{\mathcal{N}} \circ \operatorname{tr}_{\partial B_r} u\|_{\dot{W}^{1/2,2}(\partial B_r)} \leq C_1 \|\operatorname{tr}_{\partial B_r} u\|_{\dot{W}^{1/2,2}(\partial B_r)} \leq C_2 \|Du\|_{L^2(B_r)}.$$

On the other hand, if $\|\Pi_{\mathcal{N}} \circ \operatorname{tr}_{\mathbb{S}^1_r} u\|_{\dot{W}^{1/2,2}(\mathbb{S}^1_r)}$ is small enough, then $\Pi_{\mathcal{N}} \circ \operatorname{tr}_{\mathbb{S}^1_r} u$ is homotopic in $\mathrm{VMO}(\partial B_r, \mathcal{N})$ to a constant map see [13, Lemma A.19]. □

*Proof of Proposition 6.9.* We first consider the case where $u \in \mathcal{C}^2(\bar{\Omega})$ with $\operatorname{tr}_{\partial\Omega} u = g$. We recall that, in view of Definition 6.1, we have assumed that the function $u$ is extended to a function $u \in W^{1,2}(\Omega_{\delta_{\partial\Omega}}, \mathbb{R}^\nu)$ in such a way that $u \in \mathcal{N}$ almost everywhere in $\Omega_{\delta_{\partial\Omega}} \setminus \Omega$ and

$$\int_{\Omega_{\delta_{\partial\Omega}} \setminus \Omega} \frac{|Du|^2}{2} \leq C_1 \mathcal{E}^{\mathrm{ext}}(g).$$

By Lemma 6.7, there exist constants $C_2$ and $\delta \in (0, \delta_{\mathcal{N}})$, depending on $F$ and $\Omega$ only, such that

$$\mathcal{E}^{\mathrm{sg}}(g)\,\frac{1}{\delta}\int_0^\delta \Psi\left(\frac{\mathcal{H}^1_\infty(K_s)\,s}{C_2\varepsilon\mathcal{E}^{\mathrm{sg}}(g)}\right)\mathrm{d}s \le C_2(\kappa + \mathcal{E}^{\mathrm{sg}}(g)).$$

Since for every $\tau \in (0, +\infty)$, one has $\Psi(\tau) = \tau - 1 - \log \tau \ge \frac{\tau}{2} - \log 2$, we deduce that

$$\int_0^\delta \frac{\mathcal{H}^1_\infty(K_s)\,s}{2C_2\varepsilon}\,\mathrm{d}s \le C_2(\kappa + \mathcal{E}^{\mathrm{sg}}(g)) + \mathcal{E}^{\mathrm{sg}}(g)\log 2 \le (C_2 + \log 2)(\kappa + \mathcal{E}^{\mathrm{sg}}(g)) \tag{6.19}$$

and then, by monotonicity of the Hausdorff content, that

$$\mathcal{H}^1_\infty(K_\delta) \le \frac{2}{\delta^2}\int_0^\delta \mathcal{H}^1_\infty(K_s)\,s\,\mathrm{d}s \le C_3\varepsilon(\kappa + \mathcal{E}^{\mathrm{sg}}(g)). \tag{6.20}$$

Since the set $K_\delta \subset \Omega_{\delta_{\partial\Omega}}$ is compact, by definition of the Hausdorff content (Definition 6.3) and by Lemma 6.4, there exists a family of disks $\mathcal{B}_0$ with disjoint closures such that $K_\delta \subset \bigcup_{B_\rho(a)\in\mathcal{B}_0} \bar{B}_\rho(a)$, and

$$\sum_{B_\rho(a)\in\mathcal{B}_0} 2\rho \le 2\mathcal{H}^1_\infty(K_\delta) \le 2C_3\varepsilon(\kappa + \mathcal{E}^{\mathrm{sg}}(g)). \tag{6.21}$$

In particular, if we assume that

$$2C_3\varepsilon(\kappa + \mathcal{E}^{\mathrm{sg}}(g)) \le \delta_{\partial\Omega}/2, \tag{6.22}$$

and if, without loss of generality, all the disks of $\mathcal{B}_0$ intersect $K_\delta$, then the disks of $\mathcal{B}_0$ are all contained in $\Omega_{\delta_{\partial\Omega}/2}$. We define

$$\bar{s} := \sup\left\{s \in [0, +\infty) \;:\; \frac{s}{\log(1+s)}\varepsilon\left(\frac{\mathcal{E}^{\mathrm{sg}}(g)}{c_0}\log\frac{C_4}{\mathcal{E}^{\mathrm{sg}}(g)\varepsilon} + C_5\kappa\right) \le \frac{\delta_{\partial\Omega}}{4}\right\},$$

where $c_0 = \min\{c_1, c_2\}$ with $c_1, c_2$ defined in Lemma 6.11, Lemma 6.13,

$$C_4 = e^{2C_3c_0} \qquad \text{and} \qquad C_5 := \frac{1}{c_0} + 2C_3. \tag{6.23}$$

We claim that for every $s \in [0, \bar{s})$, there exists a collection of disks $\mathcal{B}(s)$ such that

(a) the closure of the disks in $\mathcal{B}(s)$ are disjoint and contained in $\Omega_{\delta_{\partial\Omega}}$,
(b) if $t \in [0, s)$, then $\bigcup_{B_\sigma(b)\in\mathcal{B}(t)} B_\sigma(b) \subset \bigcup_{B_\rho(a)\in\mathcal{B}(s)} B_\rho(a)$,
(c) for every $B_\rho(a) \in \mathcal{B}(s)$, $\operatorname{dist}_{\mathcal{N}} \circ \operatorname{tr}_{\partial B_\rho(a)} u < \delta_{\mathcal{N}}$ and

$$\rho \ge \frac{\varepsilon s}{c_0}\mathcal{E}^{\mathrm{sg}}(\operatorname{tr}_{\partial B_\rho(a)} u),$$

(d) for every $B_\rho(a) \in \mathcal{B}(s)$,

$$\int_{B_\rho(a)} \frac{|Du|^2}{2} + \frac{F(u)}{\varepsilon^2} \ge \frac{c_0}{\varepsilon}\Bigg(\rho\frac{\log(1+s)}{s} - \sum_{\substack{B_\sigma(b)\in\mathcal{B}_0\\ B_\sigma(b)\subset B_\rho(a)}} \sigma\Bigg).$$

In order to construct this collection of balls $\mathcal{B}(s)$ for $s \in [0, \bar{s})$ we first set $\mathcal{B}(0) := \mathcal{B}_0$. We have showed that (a) holds for $s = 0$ provided (6.22) holds; the assertion (b) hold trivially when $s = 0$ and (c) holds since every connected components of $K_\delta$ is contained in a unique ball of $\mathcal{B}_0$. Finally for (d), we observe that when $s \to 0$ the limit of the right-hand side vanishes. By continuity, we can take $\mathcal{B}(s) = \mathcal{B}(0)$ for $s > 0$ close enough from 0. We assume now that the assertions (a), (b), (c) and (d) are satisfied for some $s_* \in (0, \bar{s})$. We define then the set of disks

$$\mathcal{B}_* := \{B_\rho(a) \in \mathcal{B}(s_*) \;:\; \text{equality holds in (c)}\}.$$

These disks are referred to as minimising disks.

The first step consists in an expansion phase: we let the radii of the minimising disks grow in the following way. We define, for $s \ge s_*$

$$\mathcal{B}(s) := \{B_{\rho s/s_*}(a) \;:\; B_\rho(a) \in \mathcal{B}_*\} \cup (\mathcal{B}(s_*) \setminus \mathcal{B}_*)$$

and the number

$$
\begin{aligned}
s^* := \sup\bigl\{\sigma \in [s_*, \bar{s}] \ : \ &\text{for each } s \in [s_*, \sigma) \text{ (a) holds,}\\
&\text{strict inequality holds in (c) for each } B_\rho(a) \in \mathcal{B}(s_*) \setminus \mathcal{B}_*\\
&\text{and } \varepsilon \notin (\rho, \bar{s}\rho/s_*) \text{ for each } B_\rho(a) \in \mathcal{B}(s)\bigr\}.
\end{aligned}
$$

We check that for $s_* \le s \le \bar{s}$ the families of balls $\mathcal{B}(s)$ satisfy (a), (b) , (c), (d). By construction, the assertions (a) and (c) hold for every $s \in [s_*, s^*)$. Property (b) is also satisfied. We now prove (d). If $B_\rho(a) \in \mathcal{B}(s_*) \setminus \mathcal{B}_*$, (d) is true by assumption. If $B_\rho(a) \in \mathcal{B}_*$, we first consider the case where $\rho s^*/s_* \le \varepsilon$. Then since equality holds in (c) and since maps homotopic to a constant have zero singular energy $\mathcal{E}^{\mathrm{sg}}$, the map $\operatorname{tr}_{\partial B_\rho(a)} \Pi_{\mathcal{N}} \circ u$ is not homotopic to a constant. Hence for every $s \in [s_*, s^*]$, the map $\operatorname{tr}_{\partial B_{\rho s/s_*}(a)} \Pi_{\mathcal{N}} \circ u$ is not homotopic to a constant and satisfies by Lemma 6.13, since $\rho s/s_* \le \varepsilon$,

$$
\int_{B_{\rho s/s_*}(a)} |Du|^2 + \frac{F(u)}{\varepsilon^2} \ge \int_{B_{\rho s/s_*}(a)} |Du|^2 \ge c_0 \ge \frac{c_0 \rho s}{\varepsilon s^*} \ge \frac{c_0}{\varepsilon}\Biggl(\frac{\rho s}{s_*}\frac{\log(1+s)}{s} - \sum_{\substack{B_\sigma(b) \in \mathcal{B}_0\\ B_\sigma(b) \subset B_{s\rho/s_*}(a)}} \sigma\Biggr).
$$

In the last inequality we have used that $\log(1+s)/s \le 1$. If $B_\rho(a) \in \mathcal{B}_*$ and $\rho s^*/s_* > \varepsilon$ we have, from the definition of $s^*$, that $\rho > \varepsilon$. Then, if $B_\rho(a) \in \mathcal{B}_*$, we apply Lemma 6.11, since $\operatorname{dist}_{\mathcal{N}} \circ u < \delta$ on $B_{\rho s/s_*}(a) \setminus B_\rho(a) \subset \Omega_{\delta_{\partial\Omega}} \setminus K_{\delta_{\mathcal{N}}}$ and since $\mathcal{E}^{\mathrm{sg}}(\Pi_{\mathcal{N}} \circ \operatorname{tr}_{\partial B_t(a)} u) = \frac{\rho c_0}{s_* \varepsilon}$ for $t \in (\rho, \frac{\rho s}{s_*})$

$$
\begin{aligned}
\int_{B_{\rho s/s_*}(a) \setminus B_\rho(a)} \frac{|Du|^2}{2} + \frac{F(u)}{\varepsilon^2} &\ge \int_\rho^{\rho s/s_*} \frac{1}{\frac{\varepsilon}{c_0} + \frac{t}{\mathcal{E}^{\mathrm{sg}}(\Pi_{\mathcal{N}} \circ \operatorname{tr}_{\partial B_t(a)} u)}} \,\mathrm{d}t\\
&= \frac{c_0 \rho}{\varepsilon s_*} \int_\rho^{\rho s/s_*} \frac{1}{\frac{\rho}{s_*} + t} \,\mathrm{d}t = \frac{c_0 \rho}{\varepsilon s_*} \log \frac{1+s}{1+s_*}\\
&\ge \frac{c_0 \rho}{\varepsilon}\left(\frac{\log(1+s)}{s} - \frac{\log(1+s_*)}{s_*}\right).
\end{aligned}
\tag{6.24}
$$

We use that from (d), for any $B_\rho(a) \in \mathcal{B}(s_*)$ we have

$$
\rho \frac{\log(1+s_*)}{s_*} \le \sum_{\substack{B_\sigma(b) \in \mathcal{B}_0\\ B_\sigma(b) \subset B_\rho(a)}} \sigma + \frac{\varepsilon}{c_0}\left(\int_{B_\rho(a)} \frac{|Du|^2}{2} + \frac{F(u)}{\varepsilon^2}\right)
\tag{6.25}
$$

and therefore by (6.24) and (6.25)

$$
\int_{B_{\rho s/s_*}(a)} \frac{|Du|^2}{2} + \frac{F(u)}{\varepsilon^2} \ge \frac{c_0}{\varepsilon}\Biggl(\rho\frac{\log(1+s)}{s} - \sum_{\substack{B_\sigma(b) \in \mathcal{B}_0\\ B_\sigma(b) \subset B_\rho(a)}} \sigma\Biggr).
$$

Moreover, we deduce from (d), from our assumption (6.16) and from (6.21) that

$$
\begin{aligned}
\sum_{B_\rho(a) \in \mathcal{B}(s)} \rho &\le \frac{s}{\log(1+s)}\Biggl(\frac{\varepsilon}{c_0}\int_\Omega \frac{|Du|^2}{2} + \frac{F(u)}{\varepsilon^2} + \sum_{B_\sigma(b) \in \mathcal{B}_0} \sigma\Biggr)\\
&\le \frac{s}{\log(1+s)}\varepsilon\left(\frac{\mathcal{E}^{\mathrm{sg}}(g)}{c_0} \log \frac{C_4}{\mathcal{E}^{\mathrm{sg}}(g)\varepsilon} + C_5 \kappa\right),
\end{aligned}
\tag{6.26}
$$

in view of the definition of $C_4$ and $C_5$ in (6.23). Thus we find a collection the desired collection of disks $\mathcal{B}(s)$ for $0 \le s \le s^*$. In order to define $\mathcal{B}(s^*)$, we set

$$
\mathcal{B}^* := \{B_{\rho s^*/s_*}(a) \ : \ B_\rho(a) \in \mathcal{B}_*\} \cup \mathcal{B}(s_*) \setminus \mathcal{B}_*.
$$

We first note that by (6.26), since $s < \bar{s}$, we have $\bigcup_{B_\rho(a) \in \mathcal{B}^*} B_\rho(a) \subset \Omega_{\delta_{\mathcal{N}}/2}$. We also note that the family $\mathcal{B}^*$ satisfies all the desired properties except that some disks can have intersect boundaries.

If this is the case we perform then a disk merging procedure by Lemma 6.4 and we define $\mathcal{B}(s^*)$ to be the resulting disk collection. By (c), for every $B_\rho(a) \in \mathcal{B}(s^*)$, we have

$$\rho = \sum_{\substack{B_\sigma(b)\in\mathcal{B}^*\\ B_\sigma(b)\subset B_\rho(a)}} \sigma \geq \sum_{\substack{B_\sigma(b)\in\mathcal{B}^*\\ B_\sigma(b)\subset B_\rho(a)}} \frac{\varepsilon s^*}{c_0}\mathcal{E}^{\mathrm{sg}}(\operatorname{tr}_{\partial B_\sigma(b)} v) \geq \frac{\varepsilon s^*}{c_0}\mathcal{E}^{\mathrm{sg}}(\operatorname{tr}_{\partial B_\rho(b)} v),$$

so that assertion (c) still holds for the modified collection of disks. We also have, since $\mathcal{B}^*$ satisfies (d),

$$\begin{aligned}\int_{B_\rho(a)} \frac{|Du|^2}{2} + \frac{F(u)}{\varepsilon^2} &\geq \sum_{\substack{B_\sigma(b)\in\mathcal{B}^*\\ B_\sigma(b)\subset B_\rho(a)}} \int_{B_\sigma(b)} \frac{|Du|^2}{2} + \frac{F(u)}{\varepsilon^2}\\ &\geq \sum_{\substack{B_\sigma(b)\in\mathcal{B}^*\\ B_\sigma(b)\subset B_\rho(a)}} \frac{c_0}{\varepsilon}\Bigg(\sigma\frac{\log(1+s)}{s} - \sum_{\substack{B_\tau(c)\in\mathcal{B}_0\\ B_\tau(c)\subseteq B_\sigma(b)}} \tau\Bigg)\\ &= \frac{c_0}{\varepsilon}\Bigg(\rho\frac{\log(1+s)}{s} - \sum_{\substack{B_\tau(c)\in\mathcal{B}_0\\ B_\tau(c)\subseteq B_\rho(a)}} \tau\Bigg),\end{aligned}$$

and hence assertion (d) also holds for the modified collection of disks. We can then continue alternatively with expansion phases and merging steps. Since at each step either the number of disks decreases or the number of disks with equality in (c) increases, we fill the full announced interval of $[0, \bar{s})$ in a finite number of steps.

In order to conclude if

$$\eta \geq 2\varepsilon\mathcal{E}^{\mathrm{sg}}(g)/(\gamma c_0), \tag{6.27}$$

we set

$$\tilde{s} \coloneqq \frac{c_0\gamma\eta}{\varepsilon\mathcal{E}^{\mathrm{sg}}(g)} - 1 \geq 1$$

so that,

$$\begin{aligned}&\frac{\tilde{s}}{\log(1+\tilde{s})}\varepsilon\bigg(\frac{\mathcal{E}^{\mathrm{sg}}(g)}{c_0}\log\frac{C_4}{\mathcal{E}^{\mathrm{sg}}(g)\varepsilon} + C_5\kappa\bigg)\\ &\qquad\leq \frac{\frac{c_0\eta\gamma}{\varepsilon\mathcal{E}^{\mathrm{sg}}(g)} - 1}{\log\frac{c_0\gamma\eta}{\varepsilon\mathcal{E}^{\mathrm{sg}}(g)}}\varepsilon\bigg(\frac{\mathcal{E}^{\mathrm{sg}}(g)}{c_0}\log\frac{C_4}{\mathcal{E}^{\mathrm{sg}}(g)\varepsilon} + C_5\kappa\bigg) \leq \gamma\eta\frac{\log\frac{C_4e^{C_5\kappa}}{\mathcal{E}^{\mathrm{sg}}(g)\varepsilon}}{\log\frac{c_0\gamma\eta}{\mathcal{E}^{\mathrm{sg}}(g)\varepsilon}} \leq \eta,\end{aligned}$$

provided

$$(C_4e^{C_5\kappa})^\gamma(\mathcal{E}^{\mathrm{sg}}(g)\varepsilon)^{1-\gamma} \leq c_0\gamma\eta. \tag{6.28}$$

It follows that if $\eta \leq \delta_{\partial\Omega}/4$, $\tilde{s} \in [0, \bar{s}]$. Moreover, we have by (6.26),

$$\sum_{B_\rho(a)\in\mathcal{B}(\tilde{s})} \rho \leq \eta. \tag{6.29}$$

We now define the collection $\mathcal{B} \coloneqq \{B_\rho(a) \in \mathcal{B}(\tilde{s}) \;:\; \mathcal{E}^{\mathrm{sg}}(\Pi_{\mathcal{N}} \circ \operatorname{tr}_{\partial B_r(a)} u) > 0\}$. We then have for every $B_\rho(a) \in \mathcal{B}$, by (c) and by (d),

$$\int_{B_\rho(a)} \frac{|Du|^2}{2} + \frac{F(u)}{\varepsilon^2} \geq \mathcal{E}^{\mathrm{sg}}(\Pi_{\mathcal{N}} \circ \operatorname{tr}_{\partial B_\rho(a)} u)\log\left(\frac{c_0\gamma\eta}{\varepsilon\mathcal{E}^{\mathrm{sg}}(g)}\right) - \sum_{\substack{B_\sigma(b)\in\mathcal{B}_0\\ B_\sigma(b)\subseteq B_\rho(a)}} \sigma.$$

Hence, for every subcollection of disks $\mathcal{B}' \subset \mathcal{B}$, by summing and by (6.21), we obtain

$$\begin{aligned}&\int_{\bigcup_{B_\rho(a)\in\mathcal{B}'} B_\rho(a)} \frac{|Du|^2}{2} + \frac{F(u)}{\varepsilon^2}\\ &\qquad\geq \sum_{B_\rho(a)\in\mathcal{B}'} \mathcal{E}^{\mathrm{sg}}(\Pi_{\mathcal{N}} \circ \operatorname{tr}_{\partial B_\rho(a)} u)\log\left(\frac{c_0\gamma\eta}{\varepsilon\mathcal{E}^{\mathrm{sg}}(g)}\right) - 2C_3(\mathcal{E}^{\mathrm{sg}}(g) + \kappa)\varepsilon.\end{aligned} \tag{6.30}$$

By (3.4), our assumption (6.16) and by (6.30), we deduce that

$$\begin{aligned}\frac{\operatorname{sys}(\mathcal{N})^2}{4\pi}\#\mathcal{B} &\le \sum_{B_\rho(a)\in\mathcal{B}} \mathcal{E}^{\mathrm{sg}}(\Pi_{\mathcal{N}}\circ \operatorname{tr}_{\partial B_\rho(a)} u)\\ &\le \mathcal{E}^{\mathrm{sg}}(g) + \frac{(2C_3\varepsilon + \log\frac{1}{c_0\gamma\eta})\mathcal{E}^{\mathrm{sg}}(g) + (2C_3\varepsilon+1)\kappa}{\log\frac{c_0\gamma\eta}{\varepsilon\mathcal{E}^{\mathrm{sg}}(g)}}.\end{aligned}\tag{6.31}$$

The proposition is proved when $u\in\mathcal{C}^2(\bar\Omega)$, with (i) following from (6.29), with a constant $C$ in the conditions coming from the conditions (6.22), (6.27) and (6.28); the conclusion (iii) follows from (6.30) and (iv) from (6.31).

In the general case where $u\in W^{1,2}(\Omega,\mathbb{R}^\nu)$, we first consider the case where the function $F$ is bounded and continuous, so that the Ginzburg–Landau functional is continuous for the strong convergence in $W^{1,2}$. We consider a sequence $(u_n)_{n\in\mathbb{N}}$ in $\mathcal{C}^2(\bar\Omega)$ converging strongly to $u$ in $W^{1,2}(\Omega,\mathbb{R}^\nu)$. We apply the proposition to $u_n$ and let $\mathcal{B}_n$ be the associated disks. By (iv), up to a subsequence, $(\Pi_{\mathcal{N}}\circ\operatorname{tr}_{\partial B_\rho(a)} u_n)_{B_\rho(a)\in\mathcal{B}_n}$ can be chosen to remain in the same homotopy class and $\#\mathcal{B}_n$ can be chosen to be constant.

If $F$ is not bounded, then we apply the proposition to a sequence of bounded functions $\tilde F_l\in\mathcal{C}(\mathbb{R}^\nu,[0,+\infty))$ such that $\tilde F_l\le F$, $\tilde F_l$ converges to $F$ everywhere in $\mathbb{R}^\nu$ and $\tilde F_l = F$ on a neighbourhood of $\mathcal{N}$. The conclusion follows by Leguesgue's monotone convergence theorem. □

## 7. Energy convergence

We investigate first in §7.1 the convergence of sequences whose Ginzburg–Landau energy satisfies a logarithmic bound. This bound being satisfied for minimisers in view of Proposition 5.1, we apply this result to minimisers and get additional properties in §7.3.

In this section $\Omega$ is a Lipschitz bounded domain and $F\in\mathcal{C}(\mathbb{R}^\nu,[0,+\infty))$ satisfies $F^{-1}(\{0\}) = \mathcal{N}$ and (1.4).

### 7.1. **Convergence of bounded sequences.**

The main result about convergence of sequences whose Ginzburg–Landau energy satisfies a logarithmic bound is

**Theorem 7.1.** *Let $g\in W^{1/2,2}(\partial\Omega,\mathcal{N})$, $(u_n)_{n\in\mathbb{N}}$ be a sequence in $W^{1,2}(\Omega,\mathbb{R}^\nu)$ with $\operatorname{tr}_{\partial\Omega} u_n = g$ and $(\varepsilon_n)_{n\in\mathbb{N}}$ be a sequence in $(0,+\infty)$ converging to $0$ such that*

$$\sup_{n\in\mathbb{N}}\int_\Omega \frac{|Du_n|^2}{2} + \frac{F(u_n)}{\varepsilon_n^2} - \mathcal{E}^{\mathrm{sg}}(g)\log\frac{1}{\varepsilon_n} < +\infty.\tag{7.1}$$

*Then up to a subsequence, there exists a map $u_*\in W^{1,2}_{\mathrm{ren}}(\Omega,\mathcal{N})$, such that if we write $\operatorname{sing}(u_*) = \{(a_1,\gamma_1),\dots,(a_k,\gamma_k)\}$, we have*

*(i) the sequence $(u_n)_{n\in\mathbb{N}}$ converges to $u_*$ weakly in $W^{1,2}_{loc}(\Omega\setminus\{a_1,\dots,a_k\},\mathbb{R}^\nu)$ and almost everywhere in $\Omega$,*

*(ii) $\mathcal{E}^{\mathrm{sg}}(g) = \sum_{i=1}^k \frac{\lambda(\gamma_i)^2}{4\pi}$,*

*(iii) $\displaystyle\sup_{n\in\mathbb{N}}\int_\Omega \frac{|D(\operatorname{dist}_{\mathcal{N}}\circ u_n)|^2}{2} + \frac{F(u_n)}{\varepsilon_n^2} + \sup_{t>0} t^2\mathcal{L}^2(|Du_n|^{-1}([t,+\infty))) < +\infty$,*

*(iv) one has, weakly as measures on $\Omega$,*

$$\frac{|Du_n|^2}{2\log\frac{1}{\varepsilon_n}} \underset{n\to\infty}{\rightharpoonup} \sum_{i=1}^k \frac{\lambda(\gamma_i)^2}{4\pi}\delta_{a_i},$$

*(v) $\displaystyle\mathcal{E}^{\mathrm{ren}}(u_*) + \mathcal{Q}_F(u_*) \le \liminf_{n\to\infty}\int_\Omega \frac{|Du_n|^2}{2} + \frac{F(u_n)}{\varepsilon_n^2} - \mathcal{E}^{\mathrm{sg}}(g)\log\frac{1}{\varepsilon_n}$,*

*(vi) for every $\rho\in(0,\bar\rho(a_1,\dots,a_k))$,*

$$\mathcal{E}^{\mathrm{ren}}(u_*)+\mathcal{Q}_F(u_*) \le \int_{\Omega\setminus\bigcup_{i=1}^k B_\rho(a_i)} \frac{|Du^*|^2}{2} + \liminf_{n\to\infty}\int_{\bigcup_{i=1}^k B_\rho(a_i)} \frac{|Du_n|^2}{2} + \frac{F(u_n)}{\varepsilon_n^2} - \mathcal{E}^{\mathrm{sg}}(g)\log\frac{1}{\varepsilon_n}.$$

Theorem 7.1 follows from Proposition 6.9 as in [35, 48].

*Proof of Theorem 7.1.* The boundedness assertion (iii) follows immediately from the lower bound for the Ginzburg-Landau energy of Proposition 6.2. Since $g \in W^{1/2,2}(\partial\Omega, \mathcal{N})$, there exists a map $w \in W^{1,2}(\Omega_{\delta_{\partial\Omega}}, \mathcal{N})$, with $\Omega_{\delta_{\partial\Omega}}$ defined in (6.4) , such that $\operatorname{tr}_{\partial\Omega} w = g$. For each $n \in \mathbb{N}$, we define the function $\bar{u}_n \in W^{1,2}(\Omega_{\delta_{\partial\Omega}}, \mathbb{R}^\nu)$ in such a way that $\bar{u}_n|_\Omega = u_n$ and $\bar{u}_n|_{\Omega_{\delta_{\partial\Omega}} \setminus \Omega} = w$.

We let $C_1 \in (0, +\infty)$ be a constant as in Proposition 6.9 and we consider a sequence $(\eta_p)_{p\in\mathbb{N}}$ in $(0, +\infty)$ converging to 0. Since whatever the constants $\kappa \in (0, +\infty)$, $\gamma \in (0, 1)$, and for each $p \in \mathbb{N}$, there exists $n_p \in \mathbb{N}$ such that for every $n \geq n_p$,

$$C_1 e^{\gamma C_1 \kappa} (\mathcal{E}^{\mathrm{sg}}(g)\varepsilon_n)^{1-\gamma} \leq \gamma\eta_p, \qquad C_1 \mathcal{E}^{\mathrm{sg}}(g)\varepsilon_n \leq \gamma\eta_p, \qquad \text{and } C_1\kappa\varepsilon_n \leq 1,$$

we have by Proposition 6.9 a finite collection $\mathcal{B}_{n,p}$ of disjoint disks of radii less than $\eta_p$ such that for every $B \in \mathcal{B}_{n,p}$ we have $\bar{B} \subset \Omega_{\delta_{\partial\Omega}}$ and for every $B \in \mathcal{B}_{n,p}$ and every $n \geq n_p$, we have $\operatorname{dist}_{\mathcal{N}} \circ \operatorname{tr}_{\partial B} \bar{u}_n < \delta_{\mathcal{N}}$,

$$\int_{\Omega_{\delta_{\partial\Omega}} \cap \bigcup_{B\in\mathcal{B}_{n,p}} B} \frac{|D\bar{u}_n|^2}{2} + \frac{F(\bar{u}_n)}{\varepsilon_n^2} \geq \sum_{B\in\mathcal{B}_{n,p}} \mathcal{E}^{\mathrm{sg}}(\Pi_{\mathcal{N}} \circ \operatorname{tr}_{\partial B} \bar{u}_n) \log \frac{\eta_p}{\varepsilon_n} + C_2, \tag{7.2}$$

and the maps $(\Pi_{\mathcal{N}} \circ \operatorname{tr}_{\partial B} \bar{u}_n)_{B\in\mathcal{B}_{n,p}}$ form a topological resolution of $g$; in particular,

$$\sum_{B\in\mathcal{B}_{n,p}} \mathcal{E}^{\mathrm{sg}}(\Pi_{\mathcal{N}} \circ \operatorname{tr}_{\partial B} \bar{u}_n) \geq \mathcal{E}^{\mathrm{sg}}(g).$$

It follows thus from the boundedness assumption (7.1) that

$$\sum_{B\in\mathcal{B}_{n,p}} \mathcal{E}^{\mathrm{sg}}(\Pi_{\mathcal{N}} \circ \operatorname{tr}_{\partial B} \bar{u}_n) \log \frac{\eta_p}{\varepsilon_n} \leq \mathcal{E}^{\mathrm{sg}}(g) \log \frac{1}{\varepsilon_n} + C_3. \tag{7.3}$$

Since the manifold $\mathcal{N}$ is compact, in view of Proposition 3.2, the set $\{\lambda(\gamma) \ : \ \gamma \in \mathrm{VMO}(\mathbb{S}^1, \mathcal{N})\}$ is discrete, and thus there exists $\delta > 0$ such that if $(\gamma_1, \dots, \gamma_\ell)$ is a topological resolution of $g$, and $\sum_{i=1}^{\ell} \mathcal{E}^{\mathrm{sg}}(\gamma_i) \leq \mathcal{E}^{\mathrm{sg}}(g) + \delta$, then $\sum_{i=1}^{\ell} \mathcal{E}^{\mathrm{sg}}(\gamma_i) = \mathcal{E}^{\mathrm{sg}}(g)$. By Proposition 6.9 (iv), and taking $n_p$ larger if necessary, we can thus assume that

$$\#\mathcal{B}_{n,p} \frac{\operatorname{sys}(\mathcal{N})^2}{4\pi} \leq \sum_{B\in\mathcal{B}_{n,p}} \mathcal{E}^{\mathrm{sg}}(\Pi_{\mathcal{N}} \circ \operatorname{tr}_{\partial B} \bar{u}_n) \leq \mathcal{E}^{\mathrm{sg}}(g),$$

so that $(\Pi_{\mathcal{N}} \circ \operatorname{tr}_{\partial B} \bar{u}_n)_{B\in\mathcal{B}_{n,p}}$ is a minimal topological resolution of $\operatorname{tr}_{\partial\Omega} u_n = g$. By (7.3) and our assumption (7.1), we have

$$\int_{\Omega_{\delta_{\partial\Omega}} \setminus \bigcup_{B\in\mathcal{B}_{n,p}} B} \frac{|D\bar{u}_n|^2}{2} + \frac{F(\bar{u}_n)}{\varepsilon_n^2} \leq \mathcal{E}^{\mathrm{sg}}(g) \log \frac{1}{\eta_p} + C_4. \tag{7.4}$$

Let $\mathcal{C}_{n,p}$ denote the set of centres of the disks in $\mathcal{B}_{n,p}$. Up to a subsequence in $n$ and by a diagonal argument, we can assume that for each $p \in \mathbb{N}$, the sequence $(\mathcal{C}_{n,p})_{n\in\mathbb{N}}$ converges in Hausdorff distance to a finite set $\mathcal{C}_p$ in $\bar{\Omega}$ of cardinality at most $4\pi\mathcal{E}^{\mathrm{sg}}(g)/\operatorname{sys}(\mathcal{N})^2$. Taking a subsequence, we can assume further that $(\mathcal{C}_p)_{p\in\mathbb{N}}$ converges in Hausdorff distance to a finite set $\mathcal{C} = \{a_1, \dots, a_k\} \subset \bar{\Omega}$, with $k \leq 4\pi\mathcal{E}^{\mathrm{sg}}(g)/\operatorname{sys}(\mathcal{N})^2$. (The sets $\mathcal{C}_{p,n}$, $\mathcal{C}_p$ and $\mathcal{C}$ being possibly empty, we understand that a sequence converges to the empty set in Hausdorff distance whenever it is eventually a constant sequence of empty sets.)

For each $n \in \mathbb{N}$, by the bound (7.4), we have if $\operatorname{dist}_{\mathcal{H}}(\mathcal{C}_{n,p}, \mathcal{C}) \leq \eta_p$,

$$\int_{\Omega_{\delta_{\partial\Omega}} \setminus \bigcup_{i=1}^{k} B_{2\eta_p}(a_i)} \frac{|D\bar{u}_n|^2}{2} + \frac{F(\bar{u}_n)}{\varepsilon_n^2} \leq \mathcal{E}^{\mathrm{sg}}(g) \log \frac{1}{2\eta_p} + C_5, \tag{7.5}$$

with $C_5 \coloneqq C_4 + \mathcal{E}^{\mathrm{sg}}(g) \log 2$. By weak compactness, Rellich's compactness theorem and a diagonal argument, the sequence $(\bar{u}_n)_{n\in\mathbb{N}}$ converges almost everywhere to some $\bar{u}_* : \Omega_{\delta_{\partial\Omega}} \to \mathbb{R}^\nu$ and weakly to $\bar{u}_*$ in $W^{1,2}(\Omega_{\delta_{\partial\Omega}} \setminus \bigcup_{i=1}^{k} \bar{B}_\rho(a_i), \mathbb{R}^\nu)$ for every $\rho > 0$. We have $\bar{u}_* = w$ on $\Omega \setminus \Omega_{\delta_{\partial\Omega}}$. We define $u_* = \bar{u}_*|_\Omega$. Since by Fatou's lemma,

$$\int_{\Omega \setminus \bigcup_{i=1}^{k} B_{2\eta_p}(a_i)} F(u_*) \leq \lim_{n\to\infty} \varepsilon_n^2 \int_{\Omega \setminus \bigcup_{i=1}^{k} B_{2\eta_p}(a_i)} \frac{F(u_n)}{\varepsilon_n^2} = 0,$$

we have $u_* \in \mathcal{N}$ almost everywhere in $\Omega$. Moreover, for every $p \in \mathbb{N}$, we have by (7.5) and by lower semicontinuity

$$\begin{aligned}\int_{\Omega_{\delta_{\partial\Omega}} \setminus \bigcup_{i=1}^k B_{2\eta_p}(a_i)} \frac{|D\bar{u}_*|^2}{2} &\leq \liminf_{n\to\infty} \int_{\Omega_{\delta_{\partial\Omega}} \setminus \bigcup_{i=1}^k B_{2\eta_p}(a_i)} \frac{|D\bar{u}_n|^2}{2} + \frac{F(\bar{u}_n)}{\varepsilon_n^2} \\ &\leq \mathcal{E}^{\mathrm{sg}}(g) \log \frac{1}{2\eta_p} + C_5.\end{aligned} \tag{7.6}$$

By [39, Lemma 6.2], for $p$ large enough so that

$$\begin{aligned}\eta_p < \bar{\rho} \coloneqq \sup\{r > 0 \ : \ &\text{for each } i \in \{1, \dots, k\},\ B_r(a_i) \subset \Omega_{\delta_{\partial\Omega}} \\ &\text{and for each } j \in \{1, \dots, k\} \setminus \{i\},\ B_r(a_i) \cap B_r(a_j) = \emptyset\},\end{aligned}$$

we have

(7.7)
$$\int_{\Omega_{\delta_{\partial\Omega}} \setminus \bigcup_{i=1}^k B_{2\eta_p}(a_i)} \frac{|D\bar{u}_*|^2}{2} \geq \sum_{i=1}^k \frac{\lambda(\operatorname{tr}_{\partial B_{2\eta_p}(a_i)} \bar{u}_*)^2}{4\pi\nu_{\bar{\rho},2\eta_p}(a_i)} \log \frac{\bar{\rho}}{2\eta_p} \left(1 - \Big(\frac{2\pi C_6 \mathcal{E}^{\mathrm{ext}}(\operatorname{tr}_{\partial\Omega} \bar{u}_*)}{\lambda(\operatorname{tr}_{\partial B_{2\eta_p}} u)^2 \log \frac{\bar{\rho}}{2\eta_p}}\Big)^{1/2}\right)^2,$$

where

$$\nu_{\bar{\rho},2\eta_p}(a) \coloneqq \frac{1}{2\pi \log \frac{\bar{\rho}}{2\eta_p}} \int_{(B_{\bar{\rho}}(a) \setminus \bar{B}_{2\eta_p}(a)) \cap \Omega_{\delta_{\partial\Omega}}} \frac{1}{|x-a|^2} \,\mathrm{d}x \leq 1.$$

Since $(\operatorname{tr}_{\partial B_{2\eta_p}(a_i)} \bar{u}_*)_{1 \leq i \leq k}$ is a topological resolution of $\operatorname{tr}_{\partial\Omega_{\delta_{\partial\Omega}}} \bar{u}_*$, and thus of $g$, we have

$$\sum_{i=1}^k \frac{\lambda(\operatorname{tr}_{\partial B_{2\eta_p}(a_i)} \bar{u}_*)^2}{4\pi\nu_{\bar{\rho},2\eta_p}(a_i)} \geq \sum_{i=1}^k \frac{\lambda(\operatorname{tr}_{\partial B_{2\eta_p}(a_i)} \bar{u}_*)^2}{4\pi} \geq \mathcal{E}^{\mathrm{sg}}(g).$$

It thus follows by (7.6) and (7.7) that $\lim_{p\to\infty} \nu_{\bar{\rho},2\eta_p}(a_i) = 1$ which implies that $a_i \in \Omega$ (since $\Omega$ has Lipschitz boundary). It also follows that $(\operatorname{tr}_{\partial B_{2\eta_p}(a_i)} \bar{u}_*)_{1 \leq i \leq k}$ is a minimal topological resolution of $g$. Hence, in view of Definition 3.5 and (7.6), the map $u_*$ is renormalisable. Thus, if we let $\operatorname{sing}(u_*) = \{(a_1, \gamma_1), \dots, (a_k, \gamma_k)\}$ we obtain (i) and (ii).

We now prove (iv). By (i) we deduce that, up to a subsequence,

$$\frac{|Du_n|^2}{2|\log \varepsilon_n|} \rightharpoonup \sum_{i=1}^k \alpha_i \delta_{a_i} \text{ in the sense of measures} \tag{7.8}$$

for some constants $\alpha_1, \dots, \alpha_k \in \mathbb{R}$. Thanks to the upper bound (7.1), up to a subsequence, we can assume

$$\left(\int_\Omega \frac{|Du_n|^2}{2} + \frac{F(u_n)}{\varepsilon_n} - \mathcal{E}^{\mathrm{sg}}(g) \log \frac{1}{\varepsilon_n}\right)_{n\in\mathbb{N}}$$

converges. Using this and and the fact that, by (iii), $\frac{1}{|\log \varepsilon_n|} \int_\Omega \frac{F(u_n)}{\varepsilon_n^2} \to 0$ we obtain that

$$\sum_{i=1}^k \alpha_i = \mathcal{E}^{\mathrm{sg}}(g). \tag{7.9}$$

Now we use (iii) in Proposition 6.9, to obtain that for every $p \in \mathbb{N}$ and for every $0 \leq i \leq k$

$$\int_{B_{2\eta_p}(a_i)} \frac{|D\bar{u}_n|^2}{2} + \frac{F(\bar{u}_n)}{\varepsilon_n^2} \geq \mathcal{E}^{\mathrm{sg}}(\Pi_{\mathcal{N}} \circ \operatorname{tr}_{\partial B_{2\eta_p}(a_i)} \bar{u}_n) \log \frac{2\eta_p}{\varepsilon_n} + C_7. \tag{7.10}$$

By a Fubini type argument we can assume that $\bar{u}_n \rightharpoonup \bar{u}_*$ in $W^{1,2}(\partial B_{2\eta_p}(a_i), \mathbb{R}^\nu)$. By Sobolev embedding and Arzela-Ascoli criterion, we obtain that for $n$ large enough $\operatorname{tr}_{\partial B_{2\eta_p}(a_i)} \bar{u}_n$ and $\operatorname{tr}_{\partial B_{2\eta_p}(a_i)} \bar{u}_*$ are homotopic. This implies that for $n$ and $p$ large enough

$$\mathcal{E}^{\mathrm{sg}}(\Pi_{\mathcal{N}} \circ \operatorname{tr}_{\partial B_{2\eta_p}(a_i)} \bar{u}_n) = \mathcal{E}^{\mathrm{sg}}(\operatorname{tr}_{\partial B_{2\eta_p}(a_i)} \bar{u}_*) = \mathcal{E}^{\mathrm{sg}}(\gamma_i) = \frac{\lambda(\gamma_i)^2}{4\pi}, \tag{7.11}$$

since $(\gamma_1, \dots, \gamma_k)$ is a minimal topological resolution of $g$. By using (7.10) and (7.11) we obtain that $\alpha_i \geq \frac{\lambda(\gamma_i)^2}{4\pi}$ for $1 \leq i \leq k$. By (7.8) and (7.9) we obtain (iv).

The rest of the proof is devoted to assertions (v) and (vi). By (7.5), for almost every $r \in (0, \bar\rho)$, we have for each $i \in \{1, \dots, k\}$,

$$\liminf_{n\to\infty} \int_{\partial B_r(a_i)} \frac{|Du_n|^2}{2} + \frac{F(u_n)}{\varepsilon_n^2} < +\infty.$$

Moreover, for almost every $r \in (0, \bar\rho)$, for each $i \in \{1, \dots, k\}$ and $n \in \mathbb{N}$, $\operatorname{tr}_{\partial B_r(a_i)} u_n = u_n|_{\partial B_r(a_i)}$. Hence if we define $\gamma^r_{i,n} : \mathbb{S}^1 \to \mathbb{R}^\nu$ by $\gamma^r_{i,n}(x) \coloneqq u_n(a_i + rx)$, we have by Fatou's lemma, for almost every $r \in (0, \bar\rho)$

$$\liminf_{n\to\infty} \int_{\mathbb{S}^1} \frac{|(\gamma^r_{i,n})'|^2}{2} + \frac{r^2}{\varepsilon_n^2} F(\gamma^r_{i,n}) < +\infty \tag{7.12}$$

There exists thus a subsequence $(n_k)_{k\in\mathbb{N}}$ (depending on $r$) such that

$$\sup_{k\in\mathbb{N}} \int_{\mathbb{S}^1} \frac{|(\gamma^r_{i,n_k})'|^2}{2} + \frac{r^2}{\varepsilon_{n_k}^2} F(\gamma^r_{i,n_k}) < +\infty. \tag{7.13}$$

By using (7.13), Sobolev embeddings and Arzela-Ascoli's theorem we have that, up to a subsequence, $\gamma^r_{i,n_k}$ converges uniformly when $k$ tends to infinity. On the other hand we also obtain from (7.13) that $F(\gamma^r_{i,n_k}) \to 0$ a.e. up to a subsequence. Hence, by using Lemma 2.3 we have $\operatorname{dist}_{\mathcal{N}} \circ \gamma^r_{i,n_k} \to 0$ a.e. and by uniform convergence of $\gamma^r_{i,n_k}$, if $k$ is large enough then $\operatorname{dist}_{\mathcal{N}} \circ \gamma^r_{i,n_k} < \delta_{\mathcal{N}}$. Thus, by Proposition 4.4, we have

$$\lim_{k\to\infty} \bigl| \mathcal{Q}^{r/\varepsilon_{n_k}}_{F,\gamma^r_{i,n_k}} - \mathcal{Q}^{r/\varepsilon_{n_k}}_{F,\Pi_{\mathcal{N}}\circ\gamma^r_{i,n_k}} \bigr| \le \lim_{k\to\infty} C_8 \frac{r}{\varepsilon_{n_k}} \int_{\mathbb{S}^1} \frac{|(\gamma^r_{i,n_k})'|^2}{2} + \frac{r^2}{\varepsilon_{n_k}^2} F(\gamma^r_{i,n_k}) = 0, \tag{7.14}$$

so that

$$\liminf_{k\to\infty} \mathcal{Q}^{r/\varepsilon_{n_k}}_{F,\gamma^r_{i,n_k}} - \frac{\lambda(\gamma_i)^2}{4\pi} \log \frac{r}{\varepsilon_{n_k}} = \liminf_{k\to\infty} \mathcal{Q}^{r/\varepsilon_{n_k}}_{F,\Pi_{\mathcal{N}}\circ\gamma^r_{i,n_k}} - \frac{\lambda(\gamma_i)^2}{4\pi} \log \frac{r}{\varepsilon_{n_k}}. \tag{7.15}$$

On the other hand, by (7.12) and Sobolev embeddings, the sequence $(\gamma^r_{i,n_k})_{k\in\mathbb{N}}$ converges strongly to $u_*(a_i + r\cdot)$ in $W^{1/2,2}(\mathbb{S}^1, \mathbb{R}^\nu)$, and thus $(\Pi_{\mathcal{N}} \circ \gamma^r_{i,n_k})_{k\in\mathbb{N}}$ converges to $u_*(a_i + r\cdot)$ in $W^{1/2,2}(\mathbb{S}^1, \mathcal{N})$. Hence, by [39, Proposition 3.3], $\lim_{k\to\infty} d_{\mathrm{synh}}(\Pi_{\mathcal{N}} \circ \gamma^r_{i,n_k}, u_*(a_i + r\cdot)) = 0$. Thus by (7.15) and Proposition 4.3

$$\liminf_{k\to\infty} \mathcal{Q}^{r/\varepsilon_{n_k}}_{F,\gamma^r_{i,n_k}} - \frac{\lambda(\gamma_i)^2}{4\pi} \log \frac{r}{\varepsilon_{n_k}} = \liminf_{k\to\infty} \mathcal{Q}^{r/\varepsilon_{n_k}}_{F,u_*(a_i+r\cdot)} - \frac{\lambda(\gamma_i)^2}{4\pi} \log \frac{r}{\varepsilon_{n_k}}. \tag{7.16}$$

Finally by Proposition 4.3 again, we have in view of (7.16)

$$\liminf_{k\to\infty} \mathcal{Q}^{r/\varepsilon_{n_k}}_{F,\gamma^r_{i,n_k}} - \frac{\lambda(\gamma_i)^2}{4\pi} \log \frac{r}{\varepsilon_{n_k}} \ge \mathcal{Q}_{F,\gamma_i} - d_{\mathrm{synh}}(u_*(a_i + r\cdot), \gamma_i).$$

It follows thus that

$$\begin{aligned}
&\lim_{k\to\infty} \int_\Omega \frac{|Du_{n_k}|^2}{2} + \frac{F(u_{n_k})}{\varepsilon_{n_k}^2} - \mathcal{E}^{\mathrm{sg}}(g) \log \frac{1}{\varepsilon_{n_k}} \\
&\quad\ge \liminf_{k\to\infty} \int_{\Omega\setminus\bigcup_{i=1}^k B_r(a_i)} \frac{|Du_{n_k}|^2}{2} + \frac{F(u_n)}{\varepsilon_n^2} + \sum_{i=1}^k \liminf_{k\to\infty} \int_{B_r(a_i)} \frac{|Du_{n_k}|^2}{2} + \frac{F(u_{n_k})}{\varepsilon_{n_k}^2} \\
&\qquad - \mathcal{E}^{\mathrm{sg}}(g) \log \frac{1}{\varepsilon_{n_k}} \\
&\quad\ge \int_{\Omega\setminus\bigcup_{i=1}^k B_r(a_i)} \frac{|Du_*|^2}{2} - \mathcal{E}^{\mathrm{sg}}(g) \log \frac{1}{r} + \sum_{i=1}^k \liminf_{k\to\infty} \mathcal{Q}^{r/\varepsilon_{n_k}}_{F,\gamma^r_{i,n_k}} - \frac{\lambda(\gamma_i)^2}{4\pi} \log \frac{r}{\varepsilon_{n_k}} \\
&\quad\ge \int_{\Omega\setminus\bigcup_{i=1}^k B_r(a_i)} \frac{|Du_*|^2}{2} - \mathcal{E}^{\mathrm{sg}}(g) \log \frac{1}{r} + \mathcal{Q}_F(u_*) - \sum_{i=1}^k d_{\mathrm{synh}}(u_*(a_i + r\cdot), \gamma_i),
\end{aligned}$$

We reach the conclusion (v) by letting $r \to 0$.

The proof of (vi) proceeds as the proof of (v) in order to reach

$$\lim_{n\to\infty}\sum_{i=1}^{k}\int_{B_\rho(a_i)}\frac{|Du_n|^2}{2}+\frac{F(u_{n_k})}{\varepsilon_{n_k}^2}-\mathcal{E}^{\mathrm{sg}}(g)\log\frac{1}{\varepsilon_{n_k}}$$
$$\geq\sum_{i=1}^{k}\int_{B_\rho(a_i)\setminus B_r(a_i)}\frac{|Du_*|^2}{2}-\mathcal{E}^{\mathrm{sg}}(g)\log\frac{1}{r}+\mathcal{Q}_F(u_*)-\sum_{i=1}^{k}d_{\mathrm{synh}}(u_*(a_i+r\cdot),\gamma_i),$$

the conclusion follows then by letting $\rho\to 0$ and additivity of integrals. □

*Remark* 7.2 (Γ–convergence). For each $g\in W^{1/2,2}(\partial\Omega,\mathcal{N})$ and $\varepsilon\in(0,+\infty)$, we define $E^g_\varepsilon$ on set of measurable functions by setting

$$E^g_\varepsilon(u)=\begin{cases}\displaystyle\int_\Omega\Big(\frac{|Du|^2}{2}+\frac{F(u)}{\varepsilon^2}\Big)-\mathcal{E}^{\mathrm{sg}}(g)\log\frac{1}{\varepsilon} & \text{if } u\in W^{1,2}(\Omega,\mathbb{R}^\nu)\text{ and }\operatorname{tr}_{\partial\Omega}u=g\text{ on }\partial\Omega,\\ +\infty & \text{otherwise,}\end{cases}$$

and we define the limit functional $E^g_0$ on the set of measurable functions by setting

$$E^g_0(u)=\begin{cases}\mathcal{E}^{\mathrm{ren}}(u)+\mathcal{Q}_F(u) & \text{if } u\in\mathcal{A}_g(\Omega,\mathcal{N}),\\ +\infty & \text{otherwise,}\end{cases}\tag{7.17}$$

where $\mathcal{A}_g(\Omega,\mathcal{N})$ is the set of maps $u\in W^{1,2}_{\mathrm{ren}}(\Omega,\mathcal{N})$ such that $\operatorname{tr}_{\partial\Omega}u=g$ and $(\gamma_1,\dots,\gamma_k)$ is a minimal topological resolution of $g$, where $\operatorname{sing}(u)=\{(a_1,\gamma_1),\dots,(a_k,\gamma_k)\}$.

The family of functionals $(E^g_\varepsilon)_{\varepsilon>0}$ Γ-converges as $\varepsilon\to 0$ to $E^g_0$ in $L^p(\Omega,\mathbb{R}^\nu)$ endowed with the strong topology for every $p\in[1,+\infty)$, and in $W^{1,p}(\Omega,\mathbb{R}^\nu)$ endowed with the weak or strong topology for every $p\in[1,2)$. The upper bound follows from the upper bound Proposition 5.2 and the lower bound from Theorem 7.1. For $\mathcal{N}=\mathbb{S}^1$ and for the strong convergence in $W^{1,1}$, a Γ–convergence result at leading order, i.e. the Γ–convergence of $\mathcal{E}^\varepsilon_F/\log\frac{1}{\varepsilon}$, is due to Jerrard and Soner [32, Theorem 4.1]. For $\mathcal{N}=\mathbb{S}^1$, a Γ–convergence type result at next order can be found in [1]: if $Ju=\det\nabla u$ denotes the Jacobian of $u$, the authors show the Γ–convergence of the energy $\inf_{\{u\,:\,Ju=J\}}\mathcal{E}^\varepsilon_F(u)-\mathcal{E}^{\mathrm{sg}}(g)\log\frac{1}{\varepsilon}$ in the Jacobian variable $J\in\mathcal{C}^{0,1}(\Omega)'$ endowed with the convergence in the flat norm. Our framework allows us to state the Γ–convergence of $\mathcal{E}^\varepsilon_F(u)-\mathcal{E}^{\mathrm{sg}}(g)\log\frac{1}{\varepsilon}$ in the variable $u$; this in particular requires to introduce the renormalised energy $\mathcal{E}^{\mathrm{ren}}$ of renormalised maps; to our knowledge, such a Γ–convergence result is new.

7.2. **Convergence of minimisers.** We are now ready to fully state and prove our result about the convergence of minimisers:

**Theorem 7.3.** *Let $g\in W^{1/2,2}(\partial\Omega,\mathcal{N})$, let $(\varepsilon_n)_{n\in\mathbb{N}}$ be a sequence in $(0,+\infty)$ converging to $0$ and for each $n\in\mathbb{N}$ let $u_n\in W^{1,2}(\Omega,\mathbb{R}^\nu)$ be a minimiser of the Ginzburg–Landau energy $\mathcal{E}^\varepsilon_F$ under the Dirichlet boundary condition $\operatorname{tr}_{\partial\Omega}u_n=g$. Then, up to a subsequence, there exists a map $u_*\in W^{1,2}_{\mathrm{ren}}(\Omega,\mathcal{N})$ such that if we write $\operatorname{sing}(u_*)=\{(a_1,\gamma_1),\dots,(a_k,\gamma_k)\}$, we have*

*(i) the sequence $(u_n)_{n\in\mathbb{N}}$ converges almost everywhere to $u_*$ and strongly in $W^{1,2}_{loc}(\bar\Omega\setminus\{a_1,\dots,a_k\})$ and $F(u_{\varepsilon_n})/\varepsilon_n^2\to 0$ in $L^1_{\mathrm{loc}}(\bar\Omega\setminus\{a_1,\dots,a_k\})$,*
*(ii) $\mathcal{E}^{\mathrm{sg}}(g)=\sum_{i=1}^k\frac{\lambda(\gamma_i)^2}{4\pi}$,*
*(iii) $\displaystyle\sup_{n\in\mathbb{N}}\int_\Omega\frac{|D(\operatorname{dist}_{\mathcal{N}}\circ u_n)|^2}{2}+\frac{F(u_n)}{\varepsilon_n^2}+\sup_{t>0}t^2\mathcal{L}^2(|Du_n|^{-1}[0,+\infty))<+\infty$,*
*(iv) one has, weakly as measures on $\Omega$,*

$$\frac{|Du_n|^2}{2\log\frac{1}{\varepsilon_n}}\rightharpoonup\sum_{i=1}^{k}\frac{\lambda(\gamma_i)^2}{4\pi}\delta_{a_i},$$

*(v) $\operatorname{tr}_{\partial\Omega}u_*=g$ and $u_*$ is a minimising renormalisable stationary harmonic map (see Remark 7.5) so that in particular, for every $\rho\in\overline{\rho}(a_1,\dots,a_k)$, $u_*\in\mathcal{C}^\infty(\Omega\setminus\{a_1,\dots,a_k\},\mathcal{N})$ is harmonic minimising in $\Omega\setminus\bigcup_{i=1}^k\bar B_\rho(a_i)$ with respect to its own boundary conditions,*

*(vi) we have the equalities*

$$\begin{aligned}\lim_{n\to\infty} \int_\Omega \frac{|Du_\varepsilon|^2}{2} + \frac{F(u_\varepsilon)}{\varepsilon^2} - \mathcal{E}^{\mathrm{sg}}(g) \log \frac{1}{\varepsilon} \\ = \mathcal{E}^{\mathrm{ren}}(u_*) + \mathcal{Q}_F(u_*) = \inf\{\mathcal{E}^{\mathrm{ren}}(u) + \mathcal{Q}_F(u) \ : \ u \in W^{1,2}_{\mathrm{ren}}(\Omega, \mathcal{N}) \textit{ and } \operatorname{tr}_{\partial\Omega} u = g\} \\ = \mathcal{E}^{\mathrm{geom}}_{g,\gamma_1,\dots,\gamma_k}(a_1, \dots, a_k) + \sum_{i=1}^{k} \mathcal{Q}_{F,\gamma_i} = \mathcal{W}_{\min},\end{aligned}$$

*where*

$$\mathcal{W}_{\min} \coloneqq \inf\Biggl\{\mathcal{E}^{\mathrm{geom}}_{g,\eta_1,\dots,\eta_\ell}(b_1, \dots, b_\ell) + \sum_{i=1}^{\ell} \mathcal{Q}_{F,\eta_i} \ : \ b_1, \dots, b_\ell \in \Omega \textit{ are distinct} \\ \textit{and } (\eta_1, \dots, \eta_\ell) \textit{ is a minimal resolution of } g\Biggr\}. \tag{7.18}$$

When $\mathcal{N} = \mathbb{S}^1$, the weak $L^2$ estimate (iii) on the gradient is due to Serfaty and Tice [51, Proposition 1.3].

*Remark* 7.4. By (i) and (iii), for every $p \in [1, 2)$, the sequence $(u_n)_{n\in\mathbb{N}}$ converges to $u$ strongly in $W^{1,p}(\Omega)$. When $\mathcal{N} = \mathbb{S}^1$, such a convergence was known for smooth data [9, Lemma X.11] and $W^{1/2,2}$ data [10].

*Remark* 7.5. Following [39], the map $u_* \in W^{1,2}_{\mathrm{ren}}(\Omega, \mathcal{N})$ being a minimising renormalisable singular harmonic map means that for every map $v \in W^{1,2}_{\mathrm{ren}}(\Omega, \mathcal{N})$ with $\operatorname{sing}(v) = \{(b_1, \boldsymbol{\gamma}_1), \dots, (b_k, \boldsymbol{\gamma}_k)\}$ (that is $\operatorname{sing}(v)$ differs from $\operatorname{sing}(u^*)$ only by the position of the points, but not by the $\boldsymbol{\gamma}_i$), one has

$$\mathcal{E}_{\mathrm{ren}}(u_*) \le \mathcal{E}_{\mathrm{ren}}(v).$$

In particular $u_*$ is a stationary renormalisable harmonic map, which means that the its stress-energy energy tensor has vanishing flux around every singularity, or equivalently the residue of its Hopf differential vanishes at every singularity, cf. Proposition 7.9 in [39].

When $\mathcal{N} = \mathbb{S}^1$, Theorem 7.3 is essentially due to Bethuel, Brezis and Hélein [9] for star-shaped domains, and to Struwe for simply connected domains [52]. The existence of finitely many singularities and the strong convergence in the case of a general compact manifold $\mathcal{N}$ was proved by Canevari [15] general smooth bounded domains

*Proof of Theorem 7.3.* Since $(u_n)_{n\in\mathbb{N}}$ is a sequence of minimisers, it follows from Proposition 5.1 that we have

$$\limsup_{n\to\infty} \int_\Omega \frac{|Du_n|^2}{2} + \frac{F(u_n)}{\varepsilon_n^2} - \mathcal{E}^{\mathrm{sg}}(g) \log \frac{1}{\varepsilon_n} \le \mathcal{W}_{\min}. \tag{7.19}$$

By (7.19) and by Theorem 7.1, up to a subsequence, there exists a family of points $(a_1, \dots, a_k)$ in $\Omega$ such that $(u_n)_{n\in\mathbb{N}}$ converges weakly in $W^{1,2}_{\mathrm{loc}}(\overline{\Omega} \setminus \{a_1, \dots, a_k\}, \mathbb{R}^\nu)$ to some limit $u_* \in W^{1,2}_{\mathrm{ren}}(\Omega, \mathcal{N})$ and

$$\mathcal{E}^{\mathrm{ren}}(u_*) + \mathcal{Q}_F(u_*) \le \mathcal{W}_{\min}. \tag{7.20}$$

Note that from Theorem 7.1, (ii), (iii) and (iv) hold. Furthermore (i) also holds if the strong convergence is replaced by the weak convergence.

Since the map $u_*$ is renormalisable, by Proposition 3.6 and by (4.5), there exists a topological resolution $(\gamma_1, \dots, \gamma_k)$ of $g$ such that for every $i \in \{1, \dots, \ell\}$, $\lim_{\rho\to 0} d_{\mathrm{synh}}(u_*(a_i + \rho\cdot), \gamma_i) = 0$ and

$$\mathcal{E}^{\mathrm{ren}}(u_*) + \mathcal{Q}_F(u_*) \ge \mathcal{E}^{\mathrm{geom}}_{\gamma_1,\dots,\gamma_k}(a_1, \dots, a_k) + \sum_{i=1}^{\ell} \mathcal{Q}_{F,\gamma_i}; \tag{7.21}$$

and $\operatorname{sing}(u) = \{(a_1, \boldsymbol{\gamma}_1), \dots, (a_k, \boldsymbol{\gamma}_k)\}$ with each $\boldsymbol{\gamma}_i$ being the synharmony class of $\gamma_i$. It follows thus from (7.18), (7.21) and (7.20) that

$$\mathcal{E}^{\mathrm{ren}}(u_*) + \mathcal{Q}_F(u_*) = \mathcal{W}_{\min} = \mathcal{E}^{\mathrm{geom}}_{\gamma_1,\dots,\gamma_k}(a_1, \dots, a_k) + \sum_{i=1}^{\ell} \mathcal{Q}_{F,\gamma_i}. \tag{7.22}$$

By Proposition 5.2, since $(u_n)_{n\in\mathbb{N}}$ is a sequence of minimisers and since $\mathcal{E}^{\mathrm{sg}}(g) = \sum_{i=1}^{k} \frac{\lambda(\gamma_i)^2}{4\pi}$, we have also,

$$\limsup_{n\to\infty} \int_\Omega \frac{|Du_n|^2}{2} + \frac{F(u_n)}{\varepsilon_n^2} - \mathcal{E}^{\mathrm{sg}}(g)\log\frac{1}{\varepsilon_n} \leq \inf\{\mathcal{E}^{\mathrm{ren}}(u) + \mathcal{Q}_F(u) \;:\; u \in W^{1,2}_{\mathrm{ren}}(\Omega,\mathcal{N}) \text{ and } \operatorname{tr}_{\partial\Omega} u = g\}, \tag{7.23}$$

which together with (v) in Theorem 7.1 yields

$$\begin{aligned}\mathcal{E}^{\mathrm{ren}}(u_*) + \mathcal{Q}_F(u_*) &= \lim_{n\to\infty} \int_\Omega \frac{|Du_n|^2}{2} + \frac{F(u_n)}{\varepsilon_n^2} - \mathcal{E}^{\mathrm{sg}}(g)\log\frac{1}{\varepsilon_n}\\ &= \inf\{\mathcal{E}^{\mathrm{ren}}(u) + \mathcal{Q}_F(u) \;:\; u \in W^{1,2}_{\mathrm{ren}}(\Omega,\mathcal{N}) \text{ and } \operatorname{tr}_{\partial\Omega} u = g\}.\end{aligned} \tag{7.24}$$

Thus we have proved (vi).

For every $\rho \in \overline{\rho}(a_1,\dots,a_k)$, if the map $u \in W^{1,2}_{\mathrm{loc}}(\Omega\setminus\{a_1,\dots,a_k\},\mathcal{N})$ is renormalisable and satisfies $\operatorname{tr}_{\partial\Omega} u = g$ on $\partial\Omega$ and $u = u_*$ in $B_\rho(a_i)$, then $\mathcal{Q}_F(u) = \mathcal{Q}_F(u_*)$ and by (7.24),

$$\begin{aligned}\int_{\Omega\setminus\bigcup_{i=1}^k B_\rho(a_i)} \frac{|Du|^2}{2} &= \int_{\Omega\setminus\bigcup_{i=1}^k B_\rho(a_i)} \frac{|Du_*|^2}{2} + \mathcal{E}^{\mathrm{ren}}(u) + \mathcal{Q}_F(u) - (\mathcal{E}^{\mathrm{ren}}(u_*) + \mathcal{Q}_F(u_*))\\ &\geq \int_{\Omega\setminus\bigcup_{i=1}^k B_\rho(a_i)} \frac{|Du_*|^2}{2} \geq 0\end{aligned}$$

so that $u_*$ is harmonic minimising in $\Omega\setminus\bigcup_{i=1}^k \bar{B}_\rho(a_i)$ with respect to its own boundary conditions and, in particular, $u_* \in \mathcal{C}^\infty(\Omega\setminus\{a_1,\dots,a_k\},\mathcal{N})$ by the result of [42]; this proves (v).

Finally, by (7.24) and Theorem 7.1 (vi), we have for every $\rho \in (0,\bar{\rho}(a_1,\dots,a_k))$,

$$\limsup_{n\to\infty} \int_{\Omega\setminus\bigcup_{i=1}^k B_\rho(a_i)} \frac{|Du_n|^2}{2} + \frac{F(u_n)}{\varepsilon_n^2} \leq \int_{\Omega\setminus\bigcup_{i=1}^k B_\rho(a_i)} \frac{|Du_*|^2}{2},$$

which implies the announced strong convergence in (i). □

## 8. An explicit computation of the renormalised energy

Although the geometric renormalised energy of singularities and the renormalised energy of renormalisable maps are defined via a shrinking holes approach and are thus quite implicit, if $\Omega$ is simply connected and $g$ is a reparametrisation of a minimising atomic geodesic in $\mathcal{N}$, the geometric renormalised energy of a single singularity coincides strikingly with $\mathcal{N} = \mathbb{S}^1$ [39, Theorem 10.1]. When $\Omega = B_1$ this geometric renormalised energy can be explicitly computed and this allows one to locate asymptotic singularities for strictly atomic minimizing geodesic boundary conditions, as Bethuel, Brezis and Hélein did for $\mathcal{N} = \mathbb{S}^1$ [9, Theorem 0.4] in response to a question of Matano.

**Theorem 8.1.** *Let $\Omega$ be a Lipschitz bounded domain and let $F \in \mathcal{C}(\mathbb{R}^\nu,[0,+\infty))$ satisfy $F^{-1}(\{0\}) = \mathcal{N}$ and* (1.4). *Assume that*

*(a) $g : \mathbb{S}^1 \to \mathcal{N}$ is a minimising geodesic,*
*(b) if $(\gamma_1,\dots,\gamma_k)$ is a minimal topological resolution of $g$ then $k = 1$ and $\gamma_1$ is homotopic to $g$,*
*(c) every map homotopic to $g$ is synharmonic to $g$.*

*If for each $\varepsilon \in (0,+\infty)$, $u_\varepsilon$ is a minimiser of $\mathcal{E}^\varepsilon_F$ in $W^{1,2}(B_1,\mathbb{R}^\nu)$ under the condition* $\operatorname{tr}_{\partial\Omega} u_\varepsilon = g$, *then*

$$u_\varepsilon \underset{\varepsilon\to 0}{\to} u_* \quad \textit{in } W^{1,2}_{loc}(B_1\setminus\{0\},\mathbb{R}^\nu),$$

*with $u_*(x) = g(x/|x|)$.*

*Proof.* By our assumptions and Theorem 7.3, for every sequence $(\varepsilon_n)_{n\in\mathbb{N}}$ in $(0,+\infty)$ converging to 0, there exists a map $u_* \in W^{1,2}_{\mathrm{ren}}(\Omega,\mathcal{N})$ such that $\operatorname{sing}(u_*) = \{(a,\gamma)\}$ for some $a \in \Omega$ and $(u_{\varepsilon_n})_{n\in\mathbb{N}} \to u_*$ in $W^{1,2}_{\mathrm{loc}}(B\setminus\{a\},\mathbb{R}^\nu)$ with

$$\mathcal{E}^{\mathrm{ren}}(u_*) + Q_{F,\gamma} = \inf\{\mathcal{E}^{\mathrm{geom}}_{g,\gamma}(x) + \mathcal{Q}_{F,\gamma} \;:\; x \in B_1\} = \mathcal{E}^{\mathrm{geom}}_{g,g}(a) + \mathcal{Q}_{F,\gamma}.$$

It follows then by our assumptions, (3.7), Proposition 4.3 and Theorem 10.1 in [39] that for every $x \in B_1$,

$$\mathcal{E}^{\mathrm{geom}}_{g,\gamma}(x) + \mathcal{Q}_{F,\gamma} = \mathcal{E}^{\mathrm{geom}}_{g,g}(x) + \mathcal{Q}_{F,g} = \frac{\lambda(g)^2}{4\pi}\log\frac{1}{1-|x|^2} + \mathcal{Q}_{F,g}. \tag{8.1}$$

The minimum is clearly achieved when $x = 0$, and thus $a = 0$. By the characterisation (3.8) of the renormalised energy of the renormalisable map $u_*$,

$$0 = \mathcal{E}^{\mathrm{ren}}(u_*) = \sup_{\rho \to 0} \frac{1}{2} \int_{B_1 \setminus B_\rho} |Du_*|^2 - \frac{\lambda(g)^2}{4\pi} \log \frac{1}{\rho}, \tag{8.2}$$

which implies that for almost every $x \in B_1$, $u_*(x) = g\big(\frac{x}{|x|}\big)$. Since the limit is independent of the subsequence, the convergence holds for the whole family. □

## 9. Convergence of solutions to the Ginzburg–Landau equation

We consider now solutions to the Ginzburg–Landau equation (1.5) arising at least formally as the Euler–Lagrange equation of the Ginzburg–Landau energy (1.3).

In this section, $\Omega$ is a Lipschitz bounded domain and we assume that $F \in \mathcal{C}^1(\mathbb{R}^\nu, [0, +\infty))$, that $F^{-1}(\{0\}) = \mathcal{N}$ and (1.4).

9.1. **Boundedness and Euler–Lagrange equation for minimisers.** We first show that under fairly general and reasonable conditions, minimisers of the Ginzburg–Landau energy $\mathcal{E}^\varepsilon_F$ are weak solutions to the Ginzburg–Landau equation (1.5).

**Proposition 9.1.** *If $u \in W^{1,2}(\Omega, \mathbb{R}^\nu)$ is a minimiser for the Ginzburg–Landau energy and if $\nabla F(u) \in L^1_{\mathrm{loc}}(\Omega)$, then for every $\varphi \in \mathcal{C}^1_c(\Omega, \mathbb{R}^\nu)$,*

$$\int_\Omega Du \cdot D\varphi + \frac{\nabla F(u)}{\varepsilon^2} \cdot \varphi = 0.$$

The proof of Proposition 9.1 follows a truncation argument due to Bousquet [12; 46, proof of Theorem 4.23].

*Proof of Proposition 9.1.* Let $\theta \in \mathcal{C}^1(\mathbb{R}^+)$ such that $\theta = 1$ on $(0, 1)$ and $\theta = 0$ on $(2, +\infty)$. For every $R > 0$, we consider the function $\eta_R := \theta(|u|/R)$. We have, since $\eta_R \in W^{1,2}(\Omega)$ and since $u$ is bounded on the set $\{\eta_R \neq 0\}$,

$$\begin{aligned} 0 = \lim_{t \to 0} \frac{\mathcal{E}^\varepsilon_F(u + t\eta_R\varphi) - \mathcal{E}^\varepsilon_F(u)}{t} &= \int_\Omega \eta_R Du \cdot D\varphi + Du \cdot (D\eta_R)\varphi + \frac{\nabla F(u)}{\varepsilon^2} \cdot \varphi\eta_R \\ &= \int_\Omega \Big(Du \cdot D\varphi + \frac{\nabla F(u)}{\varepsilon^2} \cdot \varphi\Big)\theta(\tfrac{|u|}{R}) + \frac{|Du|^2\varphi\theta'(\frac{|u|}{R})}{R}. \end{aligned}$$

Letting $R \to +\infty$, we conclude in view of Lebesgue's dominated convergence theorem. □

The condition of Proposition 9.1 can be obtained by establishing an apriori bound on the minimiser.

**Proposition 9.2.** *If there exists a function $\Psi : \mathcal{C}^{0,1}(\mathbb{R}^\nu, \mathbb{R}^\nu)$ such that*
*(a) $\Psi$ is non-expansive in $\mathbb{R}^\nu$, i.e., $|\Psi(x) - \Psi(y)| \le |x - y|$*
*(b) $F \circ \Psi \le F$ in $\mathbb{R}^\nu$,*
*(c) $\Psi = \mathrm{id}$ on $\mathcal{N}$,*
*then for every $g \in W^{1/2,2}(\partial\Omega, \mathcal{N})$, if $u$ is a minimiser for the Ginzburg–Landau energy such that $\operatorname{tr}_{\partial\Omega} u = g$, then $u \in \bar{K}_\Psi$ almost everywhere in $\Omega$, where*

$$K_\Psi := \Big\{x \in \mathbb{R}^\nu \ : \ \limsup_{h \to 0} \frac{|\Psi(x + h) - \Psi(x)|}{|h|} = 1\Big\}.$$

In particular, if $F(Rz/|z|) \le F(z)$ for some $R > 0$ and every $z \in \mathbb{R}^\nu \setminus B_R$, taking

$$\Psi(z) := \begin{cases} z & \text{if } |z| < R \\ Rz/|z| & \text{if } |z| \ge R \end{cases},$$

we conclude that any minimiser of the Ginzburg–Landau energy (1.3) satisfies $\|u\|_{L^\infty(\Omega)} \le R$. This is the case in particular when $\mathcal{N} = \mathbb{S}^1$ and $F(z) = (1 - |z|^2)^2/4$ [8, Proposition 2].

When the set $K_\Psi$ is bounded, Proposition 9.1 also implies that $u$ is a weak solution of the Ginzburg–Landau equation.

*Proof of Proposition 9.2.* If $u$ is a minimiser, we set $v = \Psi \circ u$. By (c), we have $\operatorname{tr}_{\partial\Omega} v = \Psi \circ \operatorname{tr}_{\partial\Omega} u = g$ on $\partial\Omega$. Now, by (b) and since $u$ is a minimiser, we have

$$\int_\Omega \frac{|Du|^2}{2} \le \int_\Omega \frac{|Dv|^2}{2} + \int_\Omega \frac{F(v)}{\varepsilon^2} - \int_\Omega \frac{F(u)}{\varepsilon^2} \le \int_\Omega \frac{|Dv|^2}{2}.$$

By (a) and by the chain rule for distributional derivatives [2], we have $|Dv|^2 \le |Du|^2$ almost everywhere in $\Omega$, and either $|Du|^2 = |Dv|^2 = 0$ or $|Dv|^2 < |Du|^2$ on $u^{-1}(\mathbb{R}^\nu \setminus K_\Psi)$. By optimality, this means that $Du = 0$ a.e. on $u^{-1}(\mathbb{R}^\nu \setminus K_\Psi)$; hence, by the chain rule, $D\left(\operatorname{dist}(u(x), K_\psi)\right) = 0$ a.e. on $u^{-1}(\mathbb{R}^\nu \setminus K_\Psi)$. Since the weak derivative of $\operatorname{dist}(u(\cdot), K_\psi)$ also vanishes a.e. on the zero level set, i.e. on $u^{-1}(\bar{K}_\psi)$, this implies that $D\left(\operatorname{dist}(u(x), K_\psi)\right) = 0$ a.e. in $\Omega$. By (c), $\mathcal{N} \subset K_\psi$ and thus by the trace condition we find $\operatorname{dist}(u, K_\psi) = 0$ a.e. in $\Omega$ which implies the conclusion. □

9.2. **Uniform convergence to the manifold.** Given a boundary data $g \in W^{1/2,2}(\partial\Omega, \mathcal{N})$, we show that the asymptotic vanishing of the penalisation term in the Ginzburg–Landau equation for a sequence of solutions implies that the distance to the manifold vanishes asymptotically uniformly.

**Theorem 9.3.** *If $(\varepsilon_n)_{n\in\mathbb{N}}$ is a sequence in $(0, +\infty)$ converging to $0$ and if for every $n \in \mathbb{N}$, $u_n \in W^{1,2}(\Omega, \mathbb{R}^\nu)$ is a solution to the Ginzburg–Landau equation* (1.5) *and* $\operatorname{tr}_{\partial\Omega} u_n =: g \in W^{1/2,2}(\partial\Omega, \mathcal{N})$, *if $a \in \Omega$ and $\rho > 0$ are such that*

$$\lim_{n\to\infty} \int_{\Omega \cap B_\rho(a)} \frac{F(u_n)}{\varepsilon_n^2} = 0,$$

*and*

$$\sup_{n\in\mathbb{N}} \|Du_n\|_{L^2(\Omega\cap B_\rho(a))} + \|F(u_n)\|_{L^\infty(\Omega\cap B_\rho(a))} < +\infty,$$

*then*

$$\lim_{n\to\infty} \|\operatorname{dist}(u_n, \mathcal{N})\|_{L^\infty(\Omega\cap B_{\rho/2}(a))} = 0.$$

The assumptions of boundedness on $Du_n$ and of convergence of $F(u_n)/\varepsilon_n^2$ hold for sequences of minimisers away from singularities (Theorem 7.3). The uniform bound on $\Delta u_n$ follows from an a priori bound on $u_n$ (see Proposition 9.2) and the local boundedness of $\nabla F$; it could also follow from the global boundedness of $\nabla F$.

The uniform convergence to the vacuum manifold $\mathcal{N}$ away from singularities was known for $\mathcal{N} = \mathbb{S}^1$ [8, Step B.2]. The result is reminiscent of uniform convergence of the modulus of $W^{1,2}$–converging sequences of functions whose Laplacian and whose modulus on the boundary are controlled [11, Lemma 2.13].

The next lemma states that harmonic functions tend uniformly to the image of their trace when we approach the boundary.

**Lemma 9.4.** *If $\Omega$ has a Lipschitz boundary and if $v \in W^{1,2}(\Omega, \mathbb{R}^\nu)$ satisfies $-\Delta v = 0$ in $\Omega$ and if $\operatorname{tr}_{\partial\Omega} v \in \mathcal{N}$ almost everywhere in $\partial\Omega$, then*

$$\lim_{\substack{x\in\Omega \\ \operatorname{dist}(x,\partial\Omega)\to 0}} \operatorname{dist}(v(x), \mathcal{N}) = 0.$$

Lemma 9.4 follows from the corresponding property for harmonic extensions of functions of vanishing mean oscillation (VMO) [14, Theorem A3.2] and the embedding of $W^{1/2,2}(\partial\Omega, \mathbb{R}^\nu)$ in $\mathrm{VMO}(\partial\Omega, \mathbb{R}^\nu)$ (see [11, Lemma 2.12] for $\mathcal{N} = \mathbb{S}^1$). We give a direct proof when $v$ is is $W^{1,2}(\Omega, \mathbb{R}^\nu)$.

*Proof of Lemma 9.4.* Since the function $v$ is harmonic, by the maximum principle, $v$ is bounded. There exists a constant $C_1$ such that for every $y \in \Omega$ (see for example [29, Theorem 2.10])

$$|Dv(y)| \le \frac{C_1 \|v\|_{L^\infty(\Omega)}}{\operatorname{dist}(y, \partial\Omega)}. \tag{9.1}$$

For every $x \in \Omega$, we let $r := \operatorname{dist}(x, \partial\Omega)$. If $0 < \eta < 1$, we have

$$\operatorname{dist}(v(x), \mathcal{N}) \le \fint_{B_{\eta r}(x)} |v(y) - v(x)| \,\mathrm{d}y + \fint_{B_{\eta r}(x)} \operatorname{dist}(v(y), \mathcal{N}) \,\mathrm{d}y. \tag{9.2}$$

In view of (9.1), we have

$$\fint_{B_{\eta r}(x)} |v(y) - v(x)| \,\mathrm{d}y \le \frac{C_1 \|v\|_{L^\infty(\Omega)}}{\frac{1}{\eta} - 1}. \tag{9.3}$$

Next, since the set $\Omega$ has a Lipschitz boundary and $\operatorname{tr}_\Omega \operatorname{dist}(v, \mathcal{N}) = 0$, we have the following Poincaré inequality

$$\int_{B_{2r}(x)\cap\Omega} \operatorname{dist}(v(y), v(\partial\Omega))^2 \le C_2 r^2 \int_{B_{2r}(x)\cap\Omega} |Dv|^2. \tag{9.4}$$

It follows from (9.4) that

$$\fint_{B_{\eta r}(x)} \operatorname{dist}(v(y), \mathcal{N}) \,\mathrm{d}y \le \Bigl(\fint_{B_{\eta r}(x)} \operatorname{dist}(v(y), \mathcal{N})^2 \,\mathrm{d}y\Bigr)^{\frac{1}{2}} \le \frac{C_3}{\eta}\Bigl(\int_{B_{2r}(x)\cap\Omega} |Dv|^2\Bigr)^{\frac{1}{2}}. \tag{9.5}$$

In order to conclude we observe that when $\eta$ is small enough, the first-term in the right-hand side of (9.2) can be made arbitrarily small by (9.3), while for any given $\eta > 0$ the second term in the right-hand side of (9.2) goes to 0 in view of (9.5) and Lebesgue's dominated convergence theorem since $v \in W^{1,2}(\Omega, \mathbb{R}^\nu)$. ☐

**Lemma 9.5** (Regularity estimate). *If $\Omega$ is a Lipschitz bounded domain and if $w \in W^{1,2}(\Omega, \mathbb{R}^\nu)$ is such that $\Delta w \in L^\infty(\Omega, \mathbb{R}^\nu)$ and $\operatorname{tr}_{\partial\Omega} w = 0$, then for $\rho > 0$ and for every $a \in \Omega$,*

$$\|Dw\|_{L^\infty(\Omega\cap B_{\rho/2}(a))} \le C(\rho)\bigl(\|Dw\|_{L^2(\Omega\cap B_\rho(a))} + \|\Delta w\|_{L^\infty(\Omega\cap B_\rho(a))}\bigr)^{\frac{1}{2}} \|Dw\|^{\frac{1}{2}}_{L^2(\Omega\cap B_\rho(a))}.$$

Lemma 9.5 is reminiscent of the $L^\infty$ estimates [8, Lemma A.1].

*Proof of Lemma 9.5.* We fix $p > 2$. Since $\partial\Omega$ has a Lipschitz boundary, there exists $\rho_0 > 0$ such that if $2\operatorname{dist}(x, \partial\Omega) \le r \le \rho_0$, then $B_r(x) \cap \Omega$ is homeomorphic to a half-ball and has uniformly Lipschitz boundary. By a finite covering argument, we can assume that $\rho \le \rho_0$.

By classical Calderón–Zygmund estimates and a scaling argument, we have for every $x \in \Omega$ and $r \in (0, \rho/2) \setminus (\operatorname{dist}(x, \partial\Omega), 2\operatorname{dist}(x, \partial\Omega))$,

$$\|D^2 w\|_{L^p(\Omega\cap B_{r/2}(x))} \le C_1\Bigl(\frac{\|w\|_{L^p(\Omega\cap B_r(x))}}{r^2} + \|\Delta w\|_{L^p(\Omega\cap B_r(x))}\Bigr). \tag{9.6}$$

If $r \in (0, \rho) \cap (2\operatorname{dist}(x, \partial\Omega), +\infty)$, from the Poincaré-Sobolev inequality following from the Poincaré inequality $\|w\|_{L^2(\Omega\cap B_r(b)))} \le C_2 r \|Dw\|^2_{L^2(\Omega\cap B_r(b))}$ combined with the two-dimensional Sobolev inequality, $\|w\|_{L^p(\Omega\cap B_r(b))} \le C_3 r^{2/p}(r^{-1}\|w\|_{L^2(\Omega\cap B_r(b))} + \|Dw\|_{L^2(\Omega\cap B_r(b))})$, we have since $w = 0$ on $\partial\Omega$,

$$\|w\|_{L^p(\Omega\cap B_r(x))} \le C_4 r^{\frac{2}{p}} \|Dw\|_{L^2(\Omega\cap B_r(x))}, \tag{9.7}$$

and thus by (9.6) and (9.7)

$$\|D^2 w\|_{L^p(\Omega\cap B_{r/2}(x))} \le C_5\Bigl(\frac{\|Dw\|_{L^2(\Omega\cap B_r(x))}}{r^{2-\frac{2}{p}}} + \|\Delta w\|_{L^p(\Omega\cap B_r(x))}\Bigr); \tag{9.8}$$

the latter inequality also holds when $r \in (0, \rho/2) \cap (0, \operatorname{dist}(x, \partial\Omega))$ by assuming without loss of generality that $\int_{B_r(x)} w = 0$, since a Poincaré-Sobolev inequality is also at hand in this case.

By the Morrey–Sobolev embedding $W^{1,p} \subset C^{0,1-2/p}$ and the Cauchy–Schwarz inequality, we have for almost every $x \in \Omega$ and $r \in (0, \rho/2) \setminus (2\operatorname{dist}(x, \partial\Omega), 4\operatorname{dist}(x, \partial\Omega))$

$$\begin{aligned} |Dw(x)| &\le \fint_{\Omega\cap B_{r/2}(x)} |Dw(x) - Dw(y)| \,\mathrm{d}y + \fint_{\Omega\cap B_{r/2}(x)} |Dw| \\ &\le C_6\Bigl(\|D^2 w\|_{L^p(\Omega\cap B_{r/2}(x))} r^{1-\frac{2}{p}} + \frac{\|Dw\|_{L^2(\Omega\cap B_{r/2}(x))}}{r}\Bigr), \end{aligned} \tag{9.9}$$

and it follows thus from (9.8), (9.9) and $\|\Delta w\|_{L^p(\Omega\cap B_r(x))} \le (\pi r^2)^{1/p}\|\Delta w\|_{L^\infty(\Omega\cap B_r(x))}$ that if $r \in (0, \rho/2) \setminus (\operatorname{dist}(x, \partial\Omega), 4\operatorname{dist}(x, \partial\Omega))$,

$$|Dw(x)| \le C_7\Bigl(r\|\Delta w\|_{L^\infty(\Omega\cap B_\rho(a))} + \frac{\|Dw\|_{L^2(\Omega\cap B_\rho(a))}}{r}\Bigr), \tag{9.10}$$

since $B_r(x) \subset B_\rho(a)$. We observe now that (9.10) holds also for $r \in [\operatorname{dist}(x, \partial\Omega), 4\operatorname{dist}(x, \partial\Omega)] \cap (0, \rho/2)$ with $4C_7$ instead of $C_7$ and we conclude by taking

$$r := \min\left(\rho/2, \sqrt{\frac{\|Dw\|_{L^2(\Omega\cap B_\rho(a))}}{\|\Delta w\|_{L^\infty(\Omega\cap B_\rho(a))}}}\right). \qquad \square$$

*Proof of Theorem 9.3.* Following [8, Proof of Step B.1], let $v \in W^{1,2}(\Omega, \mathbb{R}^\nu)$ be a solution to the Dirichlet problem

$$\begin{cases} -\Delta v = 0 & \text{in } \Omega, \\ \quad v = g & \text{on } \partial\Omega. \end{cases}$$

For each $n \in \mathbb{N}$, we define the function $w_n := u_n - v$, which satisfies by assumption on $u_n$ and by construction of $v$,

$$\begin{cases} -\Delta w_n = \dfrac{\nabla F(u_n)}{\varepsilon_n^2} & \text{in } \Omega, \\ \quad w_n = 0 & \text{on } \partial\Omega. \end{cases}$$

By Lemma 9.5 and by assumption, we have

(9.11)
$$\begin{aligned} \|Dw_n\|_{L^\infty(\Omega\cap B_{\rho/2}(a))} &\le C_1\big(\|Dw_n\|_{L^2(\Omega\cap B_\rho(a))} + \|\Delta w_n\|_{L^\infty(\Omega\cap B_\rho(a))}\big)^{\frac12}\|Dw_n\|^{\frac12}_{L^2(\Omega\cap B_\rho(a))} \\ &\le \frac{C_2}{\varepsilon_n}. \end{aligned}$$

Let now $\delta \in (0, \frac{\delta_\mathcal{N}}{4})$. By Lemma 9.4, there exists $r > 0$, such that if $\operatorname{dist}(x, \partial\Omega) \le r$, then $\operatorname{dist}(v(x), \mathcal{N}) \le \delta/2$. If moreover $x \in \Omega \cap B_{\rho/2}(a)$ and $\operatorname{dist}(x, \partial\Omega) \le \varepsilon_n\delta/(2C_2) < r$, then, thanks to (9.11), we have $|w_n(x)| \le \delta/2$. Hence for $n$ large enough, as $u_n = w_n + v$,

$$\text{for all } x \in \Omega \cap B_{\rho/2}(a) \text{ such that } \operatorname{dist}(x, \partial\Omega) \le \varepsilon_n\delta/(2C_2),\ \operatorname{dist}(u_n(x), \mathcal{N}) \le \delta. \tag{9.12}$$

We consider now a point $x \in \Omega \cap B_{\rho/2}(a)$ such that $\operatorname{dist}(x, \partial\Omega) > \varepsilon_n\delta/(4C_2)$; we have by classical estimates on harmonic extensions (see (9.1)) and by (9.11)

$$|Du_n(x)| \le \frac{C_3}{\varepsilon_n\delta}. \tag{9.13}$$

We assume now by contradiction that there exists a sequence $(a_n)_{n\in\mathbb{N}}$ in $B_{\rho/2}(a) \cap \Omega$ such that $\operatorname{dist}(u_n(a_n), \mathcal{N}) \ge 2\delta$. By continuity, we can assume that $\operatorname{dist}(u_n(a_n), \mathcal{N}) = 2\delta$. By (9.12), we have in particular $\operatorname{dist}(a_n, \partial\Omega) > \varepsilon_n\delta/(2C_2)$ and so for $n$ large enough,

$$B_{\varepsilon_n\delta/(4C_2)}(a_n) \subset \{x \in \Omega \cap B_\rho(a) \ :\ \operatorname{dist}(x, \partial\Omega) > \varepsilon_n\delta/(4C_2)\}.$$

Since the distance to a closed set is non-expansive, using (9.13), we have if $x \in B_{\varepsilon_n\delta/(4C_2)}(a_n)$,

$$|\operatorname{dist}(u_n(x), \mathcal{N}) - \operatorname{dist}(u_n(a_n), \mathcal{N})| \le |u_n(x) - u_n(a_n)| \le \frac{C_3}{\varepsilon_n\delta}|x - a_n|.$$

Hence, for every $x \in B_{C_4\varepsilon_n\delta^2}(a_n)$ with $C_4 := \inf\{1/(\delta_\mathcal{N} C_2); 1/C_3\}$, we have

$$\delta \le \operatorname{dist}(u_n(x), \mathcal{N}) \le 3\delta < \delta_\mathcal{N}.$$

Hence, we have if $n$ is large enough, using (1.4),

$$\frac{m_F}{2}C_4^2\pi\delta^6 \le \frac{m_F}{2}\int_{B_{C_4\varepsilon_n\delta^2}(a_n)} \frac{\operatorname{dist}(u_n, \mathcal{N})^2}{\varepsilon_n^2} \le \int_{\Omega\cap B_\rho(a)} \frac{F(u_n)}{\varepsilon_n^2},$$

which cannot hold by assumption if $n \in \mathbb{N}$ is large enough since the right-hand side goes to zero. $\square$

9.3. **Weak convergence of solutions.** The next result shows that, under some assumptions, the limit of weakly converging sequences of solutions to the Ginzburg–Landau equation are harmonic maps. For a related result when $\mathcal{N}$ is a compact manifold of dimension 1 we refer to [38].

**Theorem 9.6.** *Assume that $g \in W^{1/2,2}(\partial\Omega, \mathcal{N})$, that $(\varepsilon_n)_{n\in\mathbb{N}}$ is a sequence in $(0, +\infty)$ converging to $0$ and that for every $n \in \mathbb{N}$, $u_n \in W^{1,2}(\Omega, \mathbb{R}^\nu)$ is a solution to the Ginzburg–Landau equation* (1.5) *with $\operatorname{tr}_{\partial\Omega} u_n = g$. If $F \in \mathcal{C}^1(\mathcal{N}_\delta)$, for some $a \in \Omega$ and $\rho > 0$, we have*

*(i)* $(u_n|_{\Omega\cap B_\rho(a)})_{n\in\mathbb{N}}$ *converges weakly to some limit $u$ in $W^{1,2}(\Omega \cap B_\rho(a), \mathbb{R}^\nu)$,*
*(ii)* $\lim_{n\to\infty} \|\operatorname{dist}(u_n, \mathcal{N})\|_{L^\infty(\Omega\cap B_\rho(a))} = 0$,
*(iii)* $\lim_{n\to\infty} \int_{\Omega\cap B_\rho(a)} \frac{|D\Pi_{\mathcal{N}}(u_n)[\nabla F(u_n)]|}{\varepsilon_n^2} = 0$,

*then $u$ is a $\mathcal{N}$-valued harmonic map in $\Omega \cap B_\rho(a)$.*

For the classical Ginzburg–Landau, we have $D\Pi_{\mathcal{N}}[\nabla F(u_n)] = 0$ which implies (iii). We recover from Theorem 7.1, Theorem 9.3 and Theorem 9.6 that solutions to the classical Ginzburg–Landau satisfying an upper-bound of the form (7.1) converge to a harmonic map with values into $\mathbb{S}^1$ outside a finite set of singularities when $\varepsilon$ goes to zero [9, Theorem X.1].

When $F \in \mathcal{C}^3(\mathbb{R}^\nu)$, the condition (iii) in Theorem 9.6 follows from $\lim_{n\to\infty} \int_{\Omega\cap B_\rho(a)} \frac{F(u_n)}{\varepsilon_n^2} = 0$, in view of the next lemma:

**Lemma 9.7.** *Let $F \in \mathcal{C}^3(\mathcal{N}_{\delta_{\mathcal{N}}}, [0, +\infty))$. If $F^{-1}(\{0\}) = \mathcal{N}$ and $F$ satisfies* (1.4)*, then there exist constants $C \in (0, +\infty)$ and $\delta \in (0, \delta_{\mathcal{N}})$ such that for every $z \in \mathcal{N}_\delta$,*

$$|D\Pi_{\mathcal{N}}(z)[\nabla F(z)]| \le CF(z). \tag{9.14}$$

*Proof of Lemma 9.7.* By a second-order Taylor expansion of $DF(z)$, we have for $v \in \mathbb{R}^\nu$

$$\begin{aligned}DF(z)[D\Pi_{\mathcal{N}}(z)[v]] = DF(\Pi_{\mathcal{N}}(z))[D\Pi_{\mathcal{N}}(z)[v]] + D^2F(\Pi_{\mathcal{N}}(z))[z - \Pi_{\mathcal{N}}(z), D\Pi_{\mathcal{N}}(z)[v]]\\ + O(|v| \operatorname{dist}(z, \mathcal{N})^2).\end{aligned} \tag{9.15}$$

We first have for every $z \in \mathcal{N}_\delta$,

$$DF(\Pi_{\mathcal{N}}(z)) = 0, \tag{9.16}$$

so that the first term in the right-hand side of (9.15) vanishes. Differentiating (9.16), we get for $v, w \in \mathbb{R}^\nu$ and $z \in \mathcal{N}_\delta$,

$$D^2F(\Pi_{\mathcal{N}}(z))[w, D\Pi_{\mathcal{N}}(z)[v]] = 0, \tag{9.17}$$

so that the second term in the right-hand side of (9.15) also vanishes. We deduce from (9.15), (9.16) and (9.17), that

$$|D\Pi_{\mathcal{N}}(z)^*[\nabla F(z)]| \le C_1 \operatorname{dist}(z, \mathcal{N})^2.$$

Since $D\Pi_{\mathcal{N}}(z)$ is self-adjoint and $\nabla F(z) = 0$ when $z \in \mathcal{N}$, we have

$$|D\Pi_{\mathcal{N}}(z)^*[\nabla F(z)] - D\Pi_{\mathcal{N}}(z)[\nabla F(z)]| \le C_2 \operatorname{dist}(z, \mathcal{N})^2, \text{ for all } z \in \mathcal{N}_\delta \qquad \square$$

and the conclusion follows.

We begin the proof of Theorem 9.6 with the following geometrical identity for the nearest-point projection:

**Lemma 9.8.** *For every $y \in \mathcal{N}$, $h \in \mathbb{R}^\nu$ and $w \in T_y\mathcal{N}$, we have*

$$w \cdot D^2\Pi_{\mathcal{N}}(y)[h, h] = w \cdot D^2\Pi_{\mathcal{N}}(y)[h, D\Pi_{\mathcal{N}}(y)[h]] + D\Pi_{\mathcal{N}}(y)[h] \cdot D^2\Pi_{\mathcal{N}}(y)[h, w]. \tag{9.18}$$

*Proof.* Setting $h^\top := D\Pi_{\mathcal{N}}(y)[h]$ and $h^\perp := h - h^\top$, we have

$$\begin{aligned}w \cdot D^2\Pi_{\mathcal{N}}(y)[h, h] - w \cdot D^2\Pi_{\mathcal{N}}(y)[h, h^\top] - h^\top \cdot D^2\Pi_{\mathcal{N}}(y)[h, w]\\ = w \cdot D^2\Pi_{\mathcal{N}}(y)[h^\top, h^\perp] + w \cdot D^2\Pi_{\mathcal{N}}(y)[h^\perp, h^\perp]\\ - h^\top \cdot D^2\Pi_{\mathcal{N}}(y)[h^\top, w] - h^\top \cdot D^2\Pi_{\mathcal{N}}(y)[h^\perp, w].\end{aligned} \tag{9.19}$$

Since $h^\perp \in T_y^\perp\mathcal{N}$, and since $\Pi_{\mathcal{N}}(y + th^\perp) = \Pi_{\mathcal{N}}(y)$ for all $t$ small enough , we have, by differentiating twice:

$$D^2\Pi_{\mathcal{N}}(y)[h^\perp, h^\perp] = 0. \tag{9.20}$$

By the connection between the nearest-point projection and the second fundamental form [43, lemma 3.2], we have

$$w \cdot D^2\Pi_{\mathcal{N}}(y)[h^\perp, h^\top] = -h^\perp \cdot B_y(w, h^\top) = h^\top \cdot D^2\Pi_{\mathcal{N}}(y)[h^\perp, w]. \tag{9.21}$$

Finally, since $w \in T_y\mathcal{N}$, $h^\top \in T_y\mathcal{N}$ and by using that $D^2\Pi_{\mathcal{N}}(y) : T_y\mathcal{N} \otimes T_y\mathcal{N} \to T_y^\perp\mathcal{N}$ we have

$$h^\top \cdot D^2\Pi_{\mathcal{N}}(y)[h^\top, w] = 0. \tag{9.22}$$

In view of (9.20) ,(9.21) and (9.22), the right-hand side of (9.19) vanishes and the conclusion follows. □

**Lemma 9.9.** *For every $y \in \mathcal{N}_{\delta_{\mathcal{N}}}$, the map $\alpha_{\mathcal{N}}(y) := D\Pi_{\mathcal{N}}(y)D\Pi_{\mathcal{N}}(y)^* : T_{\Pi_{\mathcal{N}}(y)}\mathcal{N} \to T_{\Pi_{\mathcal{N}}(y)}\mathcal{N}$ is invertible. Moreover, if a map $u \in W^{2,1}_{\mathrm{loc}}(\Omega, \mathbb{R}^\nu)$ satisfies $\|\mathrm{dist}(u, \mathcal{N})\|_{L^\infty(\Omega)} < \delta_{\mathcal{N}}$, then we have*

$$\left|\alpha_{\mathcal{N}}(u)^{-1}D\Pi_{\mathcal{N}}(u)[\Delta u] - \mathrm{div}[\alpha_{\mathcal{N}}(u)^{-1}D(\Pi_{\mathcal{N}} \circ u)]\right| \le C|u - \Pi_{\mathcal{N}}(u)||Du|^2.$$

Here, we recall that $D\Pi_{\mathcal{N}}(y)^* : T_{\Pi_{\mathcal{N}}(y)}\mathcal{N} \to \mathbb{R}^\nu$ stands for the adjoint of $D\Pi_{\mathcal{N}}(y)$ which is defined by

$$D\Pi_{\mathcal{N}}(y)[v] \cdot w = v \cdot D\Pi_{\mathcal{N}}(y)^*[w] \quad \text{for all } v \in \mathbb{R}^\nu \text{ and } w \in T_{\Pi_{\mathcal{N}}(y)}\mathcal{N}.$$

Lemma 9.9 is a generalisation of the decomposition when $\mathcal{N} = \mathbb{S}^1$ of $u$ into its modulus and argument [8, (51)-52)], which is connected to the substitution in the Schrödinger equation to obtain Madelung equations, see e.g. [19] and references therein.

*Proof of Lemma 9.9.* First of all, we have that $\alpha_{\mathcal{N}}(y)$ is invertible since $D\Pi_{\mathcal{N}}(y)$ is onto.

Let $i \in \{1, 2\}$. We have on the one hand

$$\begin{aligned}\partial_i^2(\Pi_{\mathcal{N}} \circ u) &= \partial_i\left(\alpha_{\mathcal{N}}(u)\alpha_{\mathcal{N}}(u)^{-1}\partial_i(\Pi_{\mathcal{N}} \circ u)\right)\\ &= \alpha_{\mathcal{N}}(u)\partial_i\big(\alpha_{\mathcal{N}}(u)^{-1}\partial_i(\Pi_{\mathcal{N}} \circ u)\big) + \partial_i(\alpha_{\mathcal{N}}(u))\alpha_{\mathcal{N}}(u)^{-1}\partial_i(\Pi_{\mathcal{N}} \circ u),\end{aligned} \tag{9.23}$$

with

$$\partial_i(\alpha_{\mathcal{N}}(u)) = D^2\Pi_{\mathcal{N}}(u)[\partial_i u] \circ D\Pi_{\mathcal{N}}(u)^* + D\Pi_{\mathcal{N}}(u) \circ \big(D^2\Pi_{\mathcal{N}}(u)[\partial_i u]\big)^*. \tag{9.24}$$

On the other hand, we have

$$\partial_i^2(\Pi_{\mathcal{N}} \circ u) = \partial_i(D\Pi_{\mathcal{N}}(u)[\partial_i u]) = D^2\Pi_{\mathcal{N}}[\partial_i u, \partial_i u] + D\Pi_{\mathcal{N}}[\partial_i^2 u], \tag{9.25}$$

and therefore by (9.23), (9.24) and (9.25), we have

$$\begin{aligned}&D\Pi_{\mathcal{N}}(u)[\partial_i^2 u] - \alpha_{\mathcal{N}}(u)\partial_i\big(\alpha_{\mathcal{N}}(u)^{-1}\partial_i(\Pi_{\mathcal{N}} \circ u)\big)\\ &\quad= (D^2\Pi_{\mathcal{N}}(u)[\partial_i u] \circ D\Pi_{\mathcal{N}}(u)^* + D\Pi_{\mathcal{N}}(u) \circ D^2\Pi_{\mathcal{N}}(u)[\partial_i u]^*)[\alpha_{\mathcal{N}}(u)^{-1}\partial_i(\Pi_{\mathcal{N}} \circ u)]\\ &\qquad - D^2\Pi_{\mathcal{N}}(u)[\partial_i u, \partial_i u].\end{aligned} \tag{9.26}$$

Since the left-hand side of (9.26) lies in $T_{\Pi_{\mathcal{N}}(u)}\mathcal{N}$, it suffices to estimate the projection of the right-hand side of (9.26) on $T_{\Pi_{\mathcal{N}}(u)}\mathcal{N}$.

Since $D\Pi_{\mathcal{N}}(\Pi_{\mathcal{N}}(u))$ is the orthogonal projection onto the tangent space $T_{\Pi_{\mathcal{N}}(u)}\mathcal{N}$, we have that both $D\Pi_{\mathcal{N}}(\Pi_{\mathcal{N}}(u)) = D\Pi_{\mathcal{N}}(\Pi_{\mathcal{N}}(u))^*$ and $\alpha_{\mathcal{N}}(\Pi_{\mathcal{N}}(u))$ are the identity on $T_{\Pi_{\mathcal{N}}(u)}\mathcal{N}$. Hence, by using a Taylor expansion, we have for every $v \in T_{\Pi_{\mathcal{N}}(u)}\mathcal{N}$,

$$\begin{aligned}&v \cdot D^2\Pi_{\mathcal{N}}(u)[\partial_i u]\left[D\Pi_{\mathcal{N}}(u)^*\alpha_{\mathcal{N}}(u)^{-1}\partial_i(\Pi_{\mathcal{N}} \circ u)\right]\\ &\qquad= v \cdot D^2\Pi_{\mathcal{N}}\left(\Pi_{\mathcal{N}}(u)\right)[\partial_i u, D\Pi_{\mathcal{N}}\left(\Pi_{\mathcal{N}}(u)\right)[\partial_i u]] + O(|v||u - \Pi_{\mathcal{N}}(u)||\partial_i u|^2),\end{aligned} \tag{9.27}$$

$$\begin{aligned}&v \cdot D\Pi_{\mathcal{N}}(u) \circ D^2\Pi_{\mathcal{N}}(u)[\partial_i u]^*[\alpha_{\mathcal{N}}(u)^{-1}\partial_i(\Pi_{\mathcal{N}} \circ u)]\\ &\qquad= D^2\Pi_{\mathcal{N}}(u)[\partial_i u, D\Pi_{\mathcal{N}}(u)^*[v]] \cdot \alpha_{\mathcal{N}}(u)^{-1}\partial_i(\Pi_{\mathcal{N}} \circ u)\\ &\qquad\quad= D^2\Pi_{\mathcal{N}}(\Pi_{\mathcal{N}}(u))[\partial_i u, v] \cdot \partial_i(\Pi_{\mathcal{N}} \circ u) + O(|v||u - \Pi_{\mathcal{N}}(u)||\partial_i u|^2).\end{aligned} \tag{9.28}$$

and, in view of Lemma 9.8,

$$\begin{aligned}v \cdot D^2\Pi_{\mathcal{N}}(u)[\partial_i u, \partial_i u] &= v \cdot D^2\Pi_{\mathcal{N}}(\Pi_{\mathcal{N}}(u))[\partial_i u, \partial_i u] + O(|v||u - \Pi_{\mathcal{N}}(u)||\partial_i u|^2)\\ &= v \cdot D^2\Pi_{\mathcal{N}}\left(\Pi_{\mathcal{N}}(u)\right)\left[\partial_i u, D\Pi_{\mathcal{N}}\left(\Pi_{\mathcal{N}}(u)\right)[\partial_i u]\right]\\ &\quad + D^2\Pi_{\mathcal{N}}(\Pi_{\mathcal{N}}(u))[\partial_i u, v] \cdot D\Pi_{\mathcal{N}}(\Pi_{\mathcal{N}}(u))[\partial_i u]\\ &\qquad + O(|v||u - \Pi_{\mathcal{N}}(u)||\partial_i u|^2).\end{aligned} \tag{9.29}$$

Hence from (9.26), (9.27), (9.28) and (9.29) we arrive at

$$|\alpha_{\mathcal{N}}(u)^{-1}D\Pi_{\mathcal{N}}(u)[\partial_i^2 u] - \partial_i\big(\alpha_{\mathcal{N}}(u)^{-1}\partial_i(\Pi_{\mathcal{N}} \circ u)\big)| \le C_1|u - \Pi_{\mathcal{N}}(u)||\partial_i u|^2. \tag{9.30}$$

The conclusion then follows by the triangle inequality and summing (9.30) over $i \in \{1, 2\}$. □

*Proof of Theorem 9.6.* By classical regularity estimates, we have $u_n \in W^{2,p}(\Omega, )$. It follows from our assumption $\lim_{n\to\infty}\|\operatorname{dist}(u_n, \mathcal{N})\|_{L^\infty(\Omega\cap B_\rho(a))} = 0$, that for $n \in \mathbb{N}$ large enough we have $\|\operatorname{dist}(u_n, \mathcal{N})\|_{L^\infty(\Omega\cap B_\rho(a))} < \delta_{\mathcal{N}}$ so that we can define $v_n := \Pi_{\mathcal{N}} \circ u_n|_{\Omega\cap B_\rho(a)}$. By smoothness of $\Pi_{\mathcal{N}}$ and the assumption (i), we know that the sequence $(v_n)_{n\in\mathbb{N}}$ converges to $u$ weakly in $W^{1,2}(\Omega\cap B_\rho(a), \mathbb{R}^\nu)$. Moreover, we have

$$D\Pi_{\mathcal{N}}(v_n)[\Delta v_n] = f_n + g_n, \tag{9.31}$$

where, using the same notation $\alpha_{\mathcal{N}}(y) = D\Pi_{\mathcal{N}}(y)D\Pi_{\mathcal{N}}(y)^*$ as in Lemma 9.9,

$$f_n := D\Pi_{\mathcal{N}}(v_n)\left[\operatorname{div}\left(\left(\operatorname{id} - \alpha_{\mathcal{N}}(u_n)^{-1}\right)Dv_n\right)\right]$$

and

$$g_n = D\Pi_{\mathcal{N}}(v_n)\operatorname{div}\left(\alpha_{\mathcal{N}}(u_n)^{-1}Dv_n\right).$$

By weak convergence, $(Dv_n)_{n\in\mathbb{N}}$ is bounded in $L^2$. Using the fact that $\alpha_{\mathcal{N}}(y)$ depends smoothly on $y \in \mathcal{N}_{\delta_{\mathcal{N}}}$ and that $\alpha_{\mathcal{N}}(y) = \operatorname{id}_{T_y\mathcal{N}}$ when $y \in \mathcal{N}$, by find by our assumption (ii),

$$\lim_{n\to\infty}\|(\operatorname{id} - \alpha_{\mathcal{N}}(u_n)^{-1})Dv_n\|_{L^2(\Omega\cap B_\rho(a))} = 0,$$

and we deduce that

$$\|f_n\|_{H^{-1}(\Omega\cap B_\rho(a),\mathbb{R}^\nu)} \underset{n\in\mathbb{N}}{\longrightarrow} 0. \tag{9.32}$$

Now, we have from Lemma 9.9 and by smoothness of $D\Pi_{\mathcal{N}}$,

$$\|g_n\|_{L^1} \le C_1\left(\|\alpha_{\mathcal{N}}^{-1}(u_n)D\Pi_{\mathcal{N}}(u_n)\Delta u_n\|_{L^1} + \||u_n - \Pi_{\mathcal{N}}(u_n)||Du_n|^2\|_{L^1}\right)$$

and since $u_n$ satisfies the Ginzburg–Landau equation, we have by our assumption (iii), by Lemma 9.7 and by the assumption

$$\|D\Pi_{\mathcal{N}}(u_n)[\Delta u_n]\|_{L^1(\Omega\cap B_\rho(a))} \le \frac{1}{\varepsilon_n^2}\|D\Pi_{\mathcal{N}}(u_n)[\nabla F(u_n)]\|_{L^1(\Omega\cap B_\rho(a))} \underset{n\to\infty}{\longrightarrow} 0.$$

We have also $\||u_n - \Pi_{\mathcal{N}}(u_n)||Du_n|^2\|_{L^1} \to 0$ by the asumption (ii) and by boundedess of $(|Du_n|)_{n\in\mathbb{N}}$ in $L^2(\Omega \cap B_\rho(a))$. Hence

$$\|g_n\|_{L^1(\Omega\cap B_\rho(a),\mathbb{R}^\nu)} \underset{n\in\mathbb{N}}{\longrightarrow} 0. \tag{9.33}$$

Since $D\Pi_{\mathcal{N}}(v_n)$ is the orthogonal projection on $T_{v_n}\mathcal{N}$, the conclusion follows from (9.31), (9.32), (9.33) and the result about weak limits of Palais-Smale sequences for the harmonic maps equation in [7] (see also [28] and [47]). □

9.4. **Higher-order convergence of solutions.** Under regularity assumptions on the boundary, we improve the convergence away from singularities.

We assume in this section that the set $\Omega$ is a bounded open set with $\mathcal{C}^2$ boundary, that $F \in \mathcal{C}^3(\mathbb{R}^\nu, [0, +\infty))$ and that $F$ satisfies the non-degeneracy condition (2.10).

**Theorem 9.10.** *Let $g \in \mathcal{C}^2(\partial\Omega, \mathcal{N})$, $(\varepsilon_n)_{n\in\mathbb{N}}$ be a sequence in $(0, +\infty)$ converging to* 0 *and $(u_n)_{n\in\mathbb{N}}$ be a sequence of solutions to* (1.5) *with $u_n \in \mathcal{C}^2(\bar\Omega, \mathbb{R}^\nu)$ and $u_{n|\partial\Omega} = g$. If $F \in \mathcal{C}^2(\mathcal{N}_\delta)$, if for some $a \in \bar\Omega$ and $\rho \in (0, +\infty)$, we have*

*i) $(u_n)_{n\in\mathbb{N}}$ converges to some $\mathcal{N}$-valued harmonic map $u_*$ in $W^{1,2}(\Omega \cap B_\rho(a), \mathbb{R}^\nu)$,*
*ii) $\lim_{n\to\infty}\|\operatorname{dist}(u_n, \mathcal{N})\|_{L^\infty(\Omega\cap B_\rho(a))} = 0$,*
*iii) $\lim_{n\to\infty}\int_{\Omega\cap B_\rho(a)}\frac{F(u_n)}{\varepsilon_n^2} = 0$,*

*then* $(u_n)_{n\in\mathbb{N}}$ *is bounded in* $W^{2,p}_{\mathrm{loc}}(\bar\Omega \cap B_r(a))$ *for all* $p \in [1,+\infty)$ *and* $r \in (0,\rho)$.

In particular, it follows by the Morrey–Sobolev embedding that $(u_n)_{n\in\mathbb{N}}$ converges to $u_*$ in $C^{1,\alpha}(\bar\Omega \cap B_{\rho/2}(a))$ for all $0 < \alpha < 1$.

The first tool to prove Theorem 9.10 is the following proposition that was proved in [22] in dimension $n \ge 3$ and whose proof is the same for $n = 2$. It relies on the fact that when $\operatorname{dist}(u_n, \mathcal{N})$ is small, a Böchner-type formula holds: $-\Delta e_\varepsilon(u_n) \le C e_\varepsilon(u_n)^2$ if $u_n$ is a solution of (1.5) and where $e_\varepsilon(u) = \frac{|Du|^2}{2} + \frac{F(u)}{\varepsilon^2}$ and on boundary elliptic estimates on the gradient.

**Proposition 9.11** ([22, Proposition 3.1]). *Let* $\Omega \subset \mathbb{R}^2$ *be a* $\mathcal{C}^2$ *bounded domain and let* $g \in \mathcal{C}^2(\partial\Omega, \mathcal{N})$. *There exist* $\varepsilon_0, \eta_0 \in (0,+\infty)$ *and* $C = C(F,\Omega,g) \in (0,+\infty)$ *such that for every* $\varepsilon \in (0,\varepsilon_0)$, $\rho \in (0,1)$ *and* $a \in \overline{\Omega}$, *if* $u \in \mathcal{C}^2(\bar\Omega, \mathbb{R}^\nu)$ *is a solution of* (1.5) *with* $\operatorname{tr}_{\partial\Omega} u = g$, $\|\operatorname{dist}(u,\mathcal{N})\|_{L^\infty(\Omega\cap B_\rho(a))} < \delta_{\mathcal{N}}$ *and*

$$E := \int_{\Omega\cap B_\rho(a)} \frac{|Du|^2}{2} + \frac{F(u)}{\varepsilon^2} \le \eta_0, \tag{9.34}$$

*then*

$$\rho^2 \sup_{B_{\rho/2}(a)} \Big(\frac{|Du|^2}{2} + \frac{F(u)}{\varepsilon^2}\Big) \le C(E + \rho^2). \tag{9.35}$$

*Proof of Theorem 9.10.* By a covering argument, we can restrict our attention to the case $r = \frac{\rho}{4}$ with $\rho > 0$ sufficiently small so that

$$\int_{\Omega\cap B_\rho(a)} \frac{|Du_*|^2}{2} \le \eta_0/2,$$

with $\eta_0$ given by Proposition 9.11, and thus when $n \in \mathbb{N}$ is large enough

$$\int_{\Omega\cap B_\rho(a)} \frac{|Du_n|^2}{2} + \frac{F(u_n)}{\varepsilon_n^2} \le \eta_0.$$

It follows then, from Proposition 9.11, that

$$\sup_{n\in\mathbb{N}} \|Du_n\|_{L^\infty(B_{\rho/2}(a))} < +\infty. \tag{9.36}$$

Let $Q(y) := \operatorname{dist}_{\mathcal{N}}(y,\mathcal{N})^2$. A direct computation shows that

$$\Delta(Q(u_n)) = DQ(u_n)[\Delta u_n] + \sum_{i=1}^{2} D^2Q(u_n)[\partial_i u_n, \partial_i u_n]. \tag{9.37}$$

Since $u_n$ satisfies the Ginzburg–Landau equation (1.5) and by (2.10), we have

$$DQ(u_n)[\Delta u_n] = DQ(u_n)\Big[\frac{\nabla F(u_n)}{\varepsilon_n^2}\Big] = 2\frac{\nabla F(u_n)}{\varepsilon_n^2} \cdot (u_n - \Pi_{\mathcal{N}}(u_n)) \ge \frac{2m_F}{\varepsilon_n^2} \operatorname{dist}(u_n,\mathcal{N})^2.$$

Moreover, by the computation of the second derivatives of the squared distance given in Remark 2.4, using (2.2) and the inequality $\frac{1}{\sqrt{1-x}} \le 1 + x$ on $(0,\frac{1}{2})$, we have for every $z \in \mathcal{N}_{\delta_{\mathcal{N}}/2}$ and $v \in \mathbb{R}^\nu$,

$$D^2Q(z)[v,v] = 2|v|^2 - 2D\Pi_{\mathcal{N}}(z)[v]\cdot v \ge 2|v|^2 - \frac{2|v|^2}{\sqrt{1 - \frac{\operatorname{dist}(z,\mathcal{N})}{\delta_{\mathcal{N}}}}} \ge -\frac{2\operatorname{dist}(z,\mathcal{N})|v|^2}{\delta_{\mathcal{N}}}.$$

Hence, by (9.36), (9.37) and the two preceding estimates, we have for $n$ large enough,

$$\Delta(Q(u_n)) \ge \frac{2m_F}{\varepsilon_n^2}\operatorname{dist}(u_n,\mathcal{N})^2 - \frac{2\operatorname{dist}(z,\mathcal{N})|Du_n|^2}{\delta_{\mathcal{N}}} \ge \frac{C_2}{\varepsilon_n^2}Q(u_n) - C_3\sqrt{Q(u_n)}.$$

We have thus proved that the function $Q \circ u_n$ satisfies for $n$ large enough

$$\begin{cases} -\varepsilon_n^2\Delta(Q \circ u_n) + C_2\, Q \circ u_n \le C_3\varepsilon_n^2\sqrt{Q \circ u_n} & \text{in } B_{\rho/2}(a)\cap\Omega,\\ Q \circ u_n = 0 & \text{on } B_{\rho/2}(a)\cap\partial\Omega, \end{cases} \tag{9.38}$$

where the boundary condition holds because $u_n \in \mathcal{N}$ on $\partial\Omega$. As in [45, Lemma 6 and Lemma 7], we deduce from the maximum principle that

$$Q \circ u_n = \operatorname{dist}(u_n,\mathcal{N})^2 \le C_4\varepsilon_n^4 \quad \text{in } B_{\rho/4}(a)\cap\Omega.$$

Since $|\nabla F|^2 = 0$ on $\mathcal{N}$, by minimality, we have also $D(|\nabla F|^2) = 0$ on $\mathcal{N}$; as $F \in \mathcal{C}^3(\mathbb{R}^\nu)$, this means that there is a constant $C_5 \in (0, +\infty)$ with $|\nabla F(z)|^2 \le C_5 \operatorname{dist}(z, \mathcal{N})^2$ for every $z \in \mathcal{N}_{\delta_{\mathcal{N}}}$. Hence,

$$|\Delta u_n| = \frac{|\nabla F(u_n)|}{\varepsilon_n^2} \le \sqrt{C_4 C_5} \quad \text{in } B_{\rho/4}(a) \cap \Omega. \tag{9.39}$$

By elliptic estimates we obtain that $(u_n)_{n \in \mathbb{N}}$ is bounded in $W^{2,p}_{\mathrm{loc}}(B_{\rho/4}(a) \cap \overline{\Omega})$ for every $p \in [1, +\infty)$. □

The $\mathcal{C}^{1,\alpha}$ convergence is the best we can hope for if we consider convergence up to the boundary, since if we had $\mathcal{C}^2$ convergence up to the boundary we would have $\Delta u_* = 0$ on the boundary which is incompatible with $-\Delta u_* = B_{u_*}(\nabla u_*, \nabla u_*)$, where $B_u*$ is the second fundamental form of $\mathcal{N}$ at $u_*$, see [8, Remark 1] when $\mathcal{N} = \mathbb{S}^1$. However it is natural to address the question of higher convergence in the interior of $\Omega$ away from the singularities. Since this relies on a bootstrap argument such a result is not easy to obtain for general potential $F$ and should be rather addressed for specific $F$. We refer to [9] and [45] for results in this direction in the Ginzburg–Landau and Landau–de Gennes models.

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Antonin Monteil, University of Bristol, School of Mathematics, Fry building, Woodland Road, BS8 1UG Bristol, United Kingdom
*E-mail address*: antonin.monteil@bristol.ac.uk

Rémy Rodiac, Université Paris-Saclay, CNRS, Laboratoire de Mathématiques d'Orsay, 91405, Orsay, France,
*E-mail address*: remy.rodiac@universite-paris-saclay.fr

Jean Van Schaftingen, Université catholique de Louvain, Institut de Recherche en Mathématique et Physique, Chemin du cyclotron 2, L7.01.02, 1348 Louvain-la-Neuve, Belgium
*E-mail address*: jean.vanschaftingen@uclouvain.be